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A note on hamiltonian 2-groups

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A Note on Hamiltonian 2-Groups.

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1. Introduction.

A nonabelian group all of whose subgroups are normal is called a Hamiltonian group. Hamiltonian groups were classified by Dedekind [3]: they are the direct product of a quaternion group of order 8 and a periodic abelian group with no elements of order 4. A Hamiltonian 2-group is thus the direct product of a quaternion group and an elementary abelian 2-group, and so is determined up to isomorphism by its order or by the cardinality of a minimal set of generators. We denote the Hamiltonian group of order 2^{m+2} by H_m (so that H_1 is the quaternion group of order 8). We will be interested in this paper in the ways in which a Hamiltonian 2-group can occur as a normal section of a 2-group.

The norm (or Kern) of a group was defined by Baer [1] in 1935 as the intersection of the normalisers of the subgroups of the group. Clearly every subgroup of the norm is normal in the norm, and so the norm is a Hamiltonian group if nonabelian. In [2] Baer showed that it is rare for the norm of a 2-group to be Hamiltonian: a 2-group has Hamiltonian norm if and only if it is itself Hamiltonian.

A subgroup dual in a sense to the norm can be defined as follows. For any group G, let

$$\beta(G) = \langle [S, G]: S \text{ is a non-normal subgroup of } G \rangle .$$

Clearly every subgroup of $G/\beta(G)$ is normal. Observe also that $\beta(G)$ behaves more or less well with respect to taking subgroups and factor groups. If S is a subgroup of G and N a normal subgroup of G, $\beta(S) \leqslant \beta(G)$ and $\beta(G/N) \leqslant \beta(G)N/N$.

Our first result is a dual to the result of Baer above.

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THEOREM 1. *Let G be a 2-group with $G/\beta(G)$ Hamiltonian. Then* $\beta(G)=1$.

A descending series can be defined in a group by iteration based on β. Set

$$\beta_0(G)=G$$

and then for $n\geq 1$,

$$\beta_n(G)=\beta(\beta_{n-1}(G)).$$

Note that $\beta_n(G)=1$ for some n if and only if G is soluble, since $\beta(G)\leq G'$, and $\beta_i(G)/\beta_{i+1}(G)$ is soluble for every i. For an odd prime p the β-series of a p-group will coincide with the derived series, since then $\beta_i(G)/\beta_{i+1}(G)$ is abelian. As a corollary to Theorem 1 we have that in a 2-group $\beta_i(G)/\beta_{i+1}(G)$ is Hamiltonian only if $\beta_{i+1}(G)=1$. Thus in a 2-group the β-series coincides with the derived series, except possibly at the last step if the group is soluble. For any positive integer m we show that there exist 2-groups of derived length exactly $m+2$ for which $\beta_m(G)=G^{(m)}$ is Hamiltonian. We can in fact give more precise information.

THEOREM 2. *Let m be a positive integer, and c a cardinal with $c\geq 2^m$. Then there is a 2-group $P_{c,m}$ with $P_{c,m}^{(m)}\cong H_c$.*

The quaternion group H_1 gives an example of a 2-group of derived length 2 with $\beta_0(H_1)=H_1$ Hamiltonian. However it is well known that H_1 cannot occur as a normal subgroup contained in the derived group for any 2-group. We will show by way of generalisation that a finite Hamiltonian 2-group cannot be normally embedded too deeply in the derived series of a 2-group.

THEOREM 3. *Let G be a 2-group containing a normal subgroup N isomorphic to H_n for some integer n, $n\geq 2$, and suppose that for some integer d, $G^{(d+1)}<N\leq G^{(d)}$. Then G has derived length at most $m+2$, where $2^m\leq n<2^{m+1}$.*

Note that it follows immediately from Theorem 2 that the bound in Theorem 3 is best possible.

2. Proof of Theorem 1.

We will need the following facts several times.

2.1. – Let G be a group with a subgroup $Q \cong H_1$, and a subgroup S such that $[S, Q] = S \cap Q = 1$. If S contains an element of order 4, then G contains an element x such that $\langle x \rangle$ is not normal in G, and $x^2 \notin S$.

The proof of 2.1 is easy and is omitted.

2.2. – Let G be a 2-group, with $G/\beta(G)$ Hamiltonian, and suppose $x \in G$ satisfies $x^2 \notin \beta(G)$. Then $\langle x \rangle$ is normal in G.

If $\langle x \rangle$ were not normal in G, we would have $[x, y] \in \beta(G)$ for all $y \in G$, and so $x\beta(G) \in \zeta(G/\beta(G))$, a contradiction.

To prove Theorem 1, we suppose that G is a 2-group for which $G/\beta(G)$ is nonabelian and $\beta(G) \neq 1$.

Firstly we show that the may assume G is finitely generated. Since $G/\beta(G)$ is nonabelian we can find x, $y \in G$ with $[x, y] \notin \beta(G)$; and since G is not Hamiltonian we can find g, $h \in G$ such that $g^h \notin \langle g \rangle$. Set $F = \langle x, y, g, h \rangle$. Then F is a finitely generated 2-group. Since $\beta(F) \leqslant \beta(G)$, $[x, y] \notin \beta(F)$, giving $F/\beta(F)$ nonabelian; and since $g^h \notin \langle g \rangle$, F is not Hamiltonian. So F is a counter-example and we may therefore suppose G to be finitely generated.

Next we show that we may assume $|\beta(G)| = 2$. To this end suppose that whenever H is a finite 2-group with $H/\beta(H)$ Hamiltonian, and $|\beta(H)| \leqslant 2$, then $\beta(H) = 1$.

Since G is finitely generated and $G/\beta(G)$ is nilpotent of class 2 and of exponent 4, $G/\beta(G)$ is finite. It follows that $\beta(G)$ is finitely generated. Choose N normal in G and maximal with respect to being a proper subgroup of $\beta(G)$. If $|\beta(G): N|$ were equal to 2 then, by the assumption above, G/N would be Hamiltonian. However $\beta(G)/N \leqslant \Phi(G/N)$ so G/N and $G/\beta(G)$ would have the same sized minimal generating set. Both are Hamiltonian, which is a contradiction. Therefore $|\beta(G): N| > 2$, and hence G/N is neither finite nor soluble. In particular G/N is not Hamiltonian.

Now $\beta(G)/N \geqslant \beta(G/N)$ so we must have that $\beta(G)/N = \beta(G/N)$. Write $F = G/N$. Then $\beta(F)$ is infinite and $F/\beta(F)$ is Hamiltonian.

Choose $x \in F$ with $x^2 \notin \beta(F)$. By 2.2, $\langle x \rangle$ is normal in F and then, since $\beta(F)$ contains no non-trivial subgroups normal in F, $\langle x \rangle \cap \beta(F) = 1$. $F/\beta(F)$ can be generated by elements $x\beta(F)$ with $x^2 \notin \beta(F)$ and thus $F = \beta(F)\,C_F(\beta(F))$ and $\beta(F) \cap C_F(\beta(F)) \leqslant \zeta(\beta(F)) = 1$. Therefore $C_F(\beta(F))$ is Hamiltonian and so contains a subgroup $Q \cong H_1$. If $\beta(F)$ contains an element of order 4 we have, by 2.1, that Fcontains an element x with $\langle x \rangle$ not normal in F and $x^2 \notin \beta(F)$, contradicting 2.2. Hence $\beta(F)$ is abelian contradicting the insolubility of F.

It remains to settle the case when G is finite with $|\beta(G)| = 2$.

Note that since $G/\beta(G)$ is Hamiltonian, we may write $G = QE$, with $\beta(G) = Q \cap E$, $Q/\beta(G) \cong H_1$, $E/\beta(G)$ elementary abelian, and Q, E normal in G.

Let $S = \langle x \rangle$ be a cyclic non-normal subgroup of G. Then by 2.2, $x^2 \in \beta(G)$. If $S \cap \beta(G) \neq 1$, $S/\beta(G)$ is normal in G, giving S normal in G, a contradiction. It follows that $|S| = 2$.

If $x \in Q$ satisfies $x^2 \notin \beta(G)$, then $x^4 \in \beta(G) \leqslant Q' \cap \zeta(Q) = 1$ since H_1 has trivial multiplicator. Hence x has order 4. Moreover $\langle x \rangle$ is normal in G. It follows that if $x\beta(G)$, $y\beta(G)$ generate $Q/\beta(G)$, $\langle x, y \rangle$ is a normal subgroup of G isomorphic to H_1 and $Q = \langle x, y \rangle \times \beta(G)$. Further $[\langle x, y \rangle, E] \leqslant \langle x, y \rangle \cap E \leqslant \langle x, y \rangle \cap \beta(G) = 1$. If E contains an element of order 4, then 2.1 tells us that G contains a non-normal cyclic subgroup $\langle u \rangle$ of order 4 with $u^2 \notin \beta(G)$, contradicting 2.2.

Thus E is elementary abelian and $G = \langle x, y \rangle \times E$ is Hamiltonian. This final contradiction completes the proof.

3. Proof of Theorem 2.

We start by defining two sequences of groups, K_n and L_n, for $n \geqslant 0$, as follows. Let C_2 denote a cyclic group of order 2. Then we set

$$K_0 = C_2, \qquad L_0 = H_1,$$

and if K_{n-1}, L_{n-1} have been defined for $n > 0$,

$$K_n = K_{n-1} \, wr C_2, \qquad L_n = L_{n-1} \, wr C_2.$$

We adopt the convention that K_{n-1} is identified with the «first» coordinate subgroup of the base group of K_n; and similarly for L_{n-1} in L_n. We also suppose that the top group of K_n is generated by x_n, $n \geqslant 0$, and that the top group of L_n is generated by y_n, $n \geqslant 1$.

The main step in the proof of Theorem 2 is to establish that

$$L_n^{(n)} \cong H_{2^n}. \tag{3.1}$$

We define subgroups B_n of K_n and T_n of L_n inductively as follows:

$$B_1 = K_0^{K_1}, \qquad T_1 = L_0^{L_1},$$

and then for $n>1$, $B_n = B_{n-1}^{K_n}$, $T_n = T_{n-1}^{L_n}$ (where $K_0^{K_1}$ denotes the normal closure of K_0 in K_1). Note that $K_n/B_n \cong L_n/T_n \cong K_{n-1}$ for $n>0$, and $B_n = B_{n-1} \times B_{n-1}^{x_n}$.

Clearly B_n is an elementary abelian normal subgroup of K_n. Further, B_n is self centralising. For if $C = C_{K_n}(B_n)$, $B_n \leqslant C$; and clearly $C \leqslant K_{n-1}^{K_n}$ and centralises B_{n-1}. By induction $C \leqslant B_{n-1}^{K_n} = B_n$; since $C_{K_1}(B_1) = B_1$ we are done.

We may regard B_n as a module for K_{n-1}. It is easy to see that T_n is a direct product of copies of H_1, and that T_n/T_n' is isomorphic, as module for K_{n-1}, to $B_n \oplus B_n$.

Set $S_n = \{x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} : \varepsilon_i = 0, 1,\ 1 \leqslant i \leqslant n\}$ and $R_n = \{y_1^{\varepsilon_1} \dots y_n^{\varepsilon_n} : \varepsilon_i = 0, 1,\ 1 \leqslant i \leqslant n\}$. We have then that B_n is the direct product of the subgroups K_0^s, $s \in S_n$. Now put $x = \prod_{s \in S_n} x_0^s$; then $1 \neq x \in \zeta(K_n)$.

Next we show that B_n is monolithic as $\mathbb{Z}_2 K_{n-1}$-module, $n \geqslant 1$. When $n=1$ this is immediate since B_1 is regular for K_0. Now suppose that $n>1$ and M is a minimal submodule of B_n. By induction B_{n-1} is monolithic, with monolith N say. Since B_n as module for the base group of K_n has socle $N \oplus N^{x_n}$, we have $M \leqslant N \oplus N^{x_n}$. Since M is then a non-zero proper submodule of the regular $\mathbb{Z}_2\langle x_n \rangle$-module $N \oplus N^{x_n}$, it follows that $[N, x_n] = M$. Thus M is unique, and B_n is monolithic, completing the induction.

Since B_n is self centralising, we have that $\zeta(K_n) = \sigma(B_n) = M = \langle x \rangle$ (where $\sigma(B_n)$ is the socle of B_n regarded as K_{n-1}-module).

We now consider the lower central series of K_n. For $n \geqslant 1$, we set

$$\overline{K}_{n-1} = \{(k, k^{x_n}) : k \in K_{n-1}\} \leqslant K_n,$$

and note that $\overline{K}_{n-1} \cong K_{n-1}$. We prove by induction on r and n that

$$\gamma_{2r}(K_n) = \gamma_r(\overline{K}_{n-1})\,\gamma_{r+1}(K_{n-1} \times K_{n-1}^{x_n}),$$

$$\gamma_{2r+1}(K_n) = \gamma_{r+1}(K_{n-1} \times K_{n-1}^{x_n}),$$

and $\gamma_r(K_n)/\gamma_{r+1}(K_n)$ is elementary abelian.

For $n=1$, K_1 is dihedral and the result is immediate. For $n>1$, we have, using the fact that K_{n-1}/K_{n-1}' has exponent 2 by induction,

$$K_n' = [K_{n-1}, x_n](K_{n-1}' \times (K_{n-1}^{x_n})') = \overline{K}_{n-1}(K_{n-1} \times K_{n-1}^{x_n})';$$

and then $K_n/K_n' \cong (K_{n-1}/K_{n-1}') \times \langle x_n \rangle$ is elementary, as required.

For $n>1$, if $r\geqslant 1$ and $\gamma_{2r}(K_n)=\gamma_r(\overline{K}_{n-1})\gamma_{r+1}(K_{n-1}\times K_{n-1}^{x_n})$, then

$$\gamma_{2r+1}(K_n)=[\gamma_r(\overline{K}_{n-1})\gamma_{r+1}(K_{n-1}\times K_{n-1}^{x_n}),(K_{n-1}\times K_{n-1}^{x_n})\langle x_n\rangle]=$$

$$=[\gamma_r(\overline{K}_{n-1}),K_{n-1}\times K_{n-1}^{x_n}]\gamma_{r+2}(K_{n-1}\times K_{n-1}^{x_n})\cdot$$

$$\cdot[\gamma_{r+1}(K_{n-1}\times K_{n-1}^{x_n}),x_n]=\gamma_{r+1}(K_{n-1}\times K_{n-1}^{x_n})$$

(using $[\gamma_r(\overline{K}_{n-1}),x_n]=1$). If $\gamma_{2r+1}(K_n)=\gamma_{r+1}(K_{n-1}\times K_{n-1}^{x_n})$, then

$$\gamma_{2r+2}(K_n)=$$

$$=[\gamma_{r+1}(K_{n-1}\times K_{n-1}^{x_n}),x_n][\gamma_{r+1}(K_{n-1}\times K_{n-1}^{x_n}),K_{n-1}\times K_{n-1}^{x_n}]=$$

$$=\gamma_{r+1}(\overline{K}_{n-1})\gamma_{r+2}(K_{n-1}\times K_{n-1}^{x_n})$$

(using that $\gamma_{r+1}(K_{n-1})/\gamma_{r+2}(K_{n-1})$ has exponent 2). Finally observe that $\gamma_{2r}(K_n)/\gamma_{2r+1}(K_n)$ is a subgroup, and $\gamma_{2r-1}(K_n)/\gamma_{2r}(K_n)$ is a quotient group, of

$$\gamma_r(K_{n-1}\times K_{n-1}^{x_n})/\gamma_{r+1}(K_{n-1}\times K_{n-1}^{x_n}),$$

which has esponent 2 by induction on n.

We now prove by induction on n that $K_n^{(n)}=\zeta(K_n)=\gamma_{2^n}(K_n)$, so that K_n has derived length $n+1$ exactly and nilpotency class 2^n exactly. For $n=1$ the result is immediate. Since $K_{n-1}\cong\overline{K}_{n-1}\leqslant K_n'$, we have K_n has derived length at least $n+1$ if K_{n-1} has derived length exactly n. On the other hand K_n clearly has derived length at most $n+1$, and hence has derived length exactly $n+1$. Thus $\gamma_{2^n}(K_n)\neq 1$. However $\gamma_{2^n+1}(K_n)=\gamma_{2^{n-1}+1}(K_{n-1}\times K_{n-1}^{x_n})=1$ by induction, and hence K_n has class exactly 2^n. Moreover $1\neq K_n^{(n)}\leqslant\gamma_{2^n}(K_n)\leqslant\zeta(K_n)$, and so $K_n^{(n)}=$ $=\gamma_{2^n}(K_n)=\zeta(K_n)$ since the order of $\zeta(K_n)$ is two.

We now turn to the question of identifying $L_n^{(n)}$ in L_n. Set

$$Q_n=\left\{\prod_{r\in R_n}h^r : h\in L_0\right\}.$$

Then we claim $L_n^{(n)}=Q_nT_n'$.

Suppose that u_0,v_0 generate L_0, and set $u=\prod_{r\in R_n}u_0^r$, $v=\prod_{r\in R_n}v_0^r$. Then $Q_n=\langle u,v\rangle$. Since $T_n/T_n'\cong B_n\oplus B_n$ as K_{n-1}-module, it follows that

$$(L_n/T_n')^{(n)}\leqslant\langle uT_n',vT_n'\rangle=Q_nT_n'/T_n',$$

and hence $L_n^{(n)}\leqslant Q_nT_n'$.

If $L_n^{(n)}T_n'<Q_nT_n'$, then there is an onto homomorphism of L_n onto K_n

whose kernel contains $L_n^{(n)} T_n'$, contradicting the fact that $K_n^{(n)} \neq 1$. Thus $L_n^{(n)} T_n' = Q_n T_n'$. But then

$$L_n^{(n)} \geqslant [T_n, L_n^{(n)}] = [T_n, L_n^{(n)} T_n'] = [T_n, Q_n T_n'] = [T_n, Q_n] = T_n',$$

and so $L_n^{(n)} = Q_n T_n'$.

Since $Q_n \cong H_1$ and T_n' is elementary abelian of order 2^n and central in T_n, we have that $L_n^{(n)}$ is Hamiltonian and isomorphic to H_{2^n}, establishing 3.1.

We can now prove Theorem 2. Let m be a positive integer, $m \geqslant 2$, and suppose c is a cardinal with $2^m \leqslant c$. Set $d = c - 2^m$, and then define

$$P_{c,m} = L_m \times K_m^d.$$

We then have $P_m^{(m)} = L_m^{(m)} \times (K_m^{(m)})^d \cong H_{2^m} \times C_2^d \cong H_{2^m + d} = H_c$, as required.

4. Proof of Theorem 3.

Let n be an integer, $n \geqslant 2$, and define m by $2^m \leqslant n < 2^{m+1}$.

If A is a 2-subgroup of $GL(n, 2)$, we claim that A has derived length at most m if $n = 2^m$, and at most $m + 1$ otherwise, and that if V is the natural module for $GL(n, 2)$, $[V, A^{(m)}]$ has dimension at most $n - 2^m$. It is enough to prove these claims for a Sylow 2-subgroup of $GL(n, 2)$, and for this they follow as easy corollaries of Huppert [4, Satz 3.16.2].

Now let G be a 2-group containing a normal subgroup N isomorphic to H_n for some integer n, $n \geqslant 2$, and suppose that for some integer d, $G^{(d+1)} < N \leqslant G^{(d)}$. Set $Z = \zeta(N)$: Z is elementary abelian of rank n. Let Y be a normal subgroup of G satisfying $Z < Y < N$ (so that $|Y/Z| = 2$), and set $K = G^{(m)}$. It will be enough to prove that $[Y, K, K] = 1$. For then, since $[K, Y, K] = [Y, K, K]$, the three subgroup lemma gives $[K', Y] = [K, K, Y] = [Y, K, Y] = 1$; that is $[G^{(m+1)}, Y] = 1$. Now $m \geqslant d$ or else $[Y, N] = 1$, a contradiction. But then $G^{(m+1)} \leqslant Z$, and G has derived length at most $m + 2$, as required.

To prove $[Y, K, K] = 1$, consider the action of G on Y'/N', which has dimension n as vector space over $\mathbb{Z}_2$. Thus $\dim [Y/N', K]$ is at most $n - 2^m$. Since $[Y, K] \leqslant Z$, it follows that $[Y, K]$ is a module for G of dimension at most $n - 2^m + 1 \leqslant 2^m$, and so $[Y, K, K] = 1$, as required.

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