

RENDICONTI  
*del*  
SEMINARIO MATEMATICO  
*della*  
UNIVERSITÀ DI PADOVA

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*Rendiconti del Seminario Matematico della Università di Padova*,  
tome 85 (1991), p. 79-103

[http://www.numdam.org/item?id=RSMUP\\_1991\\_\\_85\\_\\_79\\_0](http://www.numdam.org/item?id=RSMUP_1991__85__79_0)

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## On Integral Currents with Constant Mean Curvature.

FRANK DUZAAR - MARTIN FUCH(\*)

### 1. Introduction.

The present paper continues the analysis of the existence problem for oriented  $m$ -dimensional submanifolds  $\Sigma$  of Euclidean space  $\mathbf{R}^{m+k}$  with prescribed mean curvature vector  $H$  and fixed boundary  $\Gamma = \partial\Sigma$  which we started in [DF1, 2]. Our aim here is to give a positive answer to following question which has not been settled in the general framework of [DF1]:

*Suppose that  $\Gamma$  is compact and the curvature vector is independent of the base point in  $\mathbf{R}^{m+k}$ , i.e. only depending on the oriented  $m$ -planes in  $\mathbf{R}^{m+k}$ . Does there exist a (generalized) solution with compact closure and boundary  $\Gamma$ ?*

In order to get a variational formulation of the problem we make use of some basic tools from *Geometric Measure Theory*: assuming that  $H \in \Lambda^m(\mathbf{R}^{m+k}, \mathbf{R}^{m+k})$  has the property

$$H(v_1 \wedge \dots \wedge v_m) \in \text{Span}[v_1, \dots, v_m]^\perp$$

and that  $\Gamma$  is represented by the boundary  $\partial T_0$  of some integral  $m$ -current  $T_0$  with compact support we are then looking for an integral  $m$ -current  $T$  with compact support,  $\partial T = \partial T_0$ , and the property

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$$(T = \underline{\tau}(M, \theta, T), \mu_T = \mathcal{H}^m \llcorner \theta)$$

$$(1) \quad \int_M \operatorname{div}_M X \, d\mu_T + \int_M X \cdot H(T) \, d\mu_T = 0$$

for all vectorfields  $X \in C_0^1(\mathbf{R}^{m+k}, \mathbf{R}^{m+k})$ ,  $\operatorname{spt}(X) \cap \operatorname{spt}(\partial T_0) = \emptyset$ .

For  $T$  smooth it is easy to check that (1) is equivalent to the fact that  $T$  has constant mean curvature vector  $H(T)$  and following the discussion in [S, 16.5] we say that a current  $T$  with (1) and  $\partial T = \partial T_0$  is a generalized solution of the mean curvature problem. Equation (1) now corresponds to the functional

$$\mathbf{F}(T) = \mathbf{M}(T) + \mathbf{V}_h(T, T_0)$$

where  $h$  is the curvature form associated to  $H$  and  $\mathbf{V}_h(T, T_0)$  denotes the  $h$ -volume enclosed by  $T$  and  $T_0$  (compare section 2).  $\mathbf{F}(T)$  makes sense for all integral currents (of finite mass) and in [DF1, Theorem 3.1] we showed that  $\mathbf{F}$  can be minimized even for nonconstant  $H$  if one of the following conditions holds:

all admissible currents have compact support

$$(2) \quad \text{in some prescribed compact set } K$$

or

$$(3) \quad \sup_{|z| \geq R} |h_z| \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Clearly (3) is inadequate for the case of constant curvature  $h_z \equiv h$ . In the situation of (2) we proved in [DF1, Section 6] that a minimizer  $T$  is a solution of (1) provided

$$\operatorname{spt}(\partial T_0) \subset \operatorname{Int}(K)$$

and

$$(4) \quad |h_p| < \min_{i=1, \dots, m+k-1} \kappa_i(p) \quad \text{for all } p \in \partial K,$$

where  $\kappa_1, \dots, \kappa_{m+k-1}$  denote the principal curvatures of  $\partial K$  with respect to the inward normal of  $\partial K$ .

So if we start with a boundary  $\partial T_0$  contained in a rather large ball  $B_R(0)$  condition (4) implies the bound  $|h| < R^{-1}$ . In the present paper we try to prove existence without assuming a smallness condition relating  $|h|$  and the diameter of  $\operatorname{spt}(\partial T_0)$ .

Our approach to the existence of solutions has been inspired by the work of [B], [BG], and [GMT]: for  $j \in \mathbf{N}$  (sufficiently large) we consider

the auxiliary problem of minimizing  $F$  among all currents  $T$  with support in the ball  $\overline{B_j}$ . If  $T_j$  denotes a solution whose existence follows along the lines of [DF1, Theorem 3.1] we then prove the existence of a bounded sequence  $\{r_j\}$  of radii  $r_j$  with the following properties:

$$(5) \quad \partial(T_j \llcorner B_{r_j} + S_j) = \partial T_0, \quad \mathbf{M}(S_j) \leq \frac{1}{j}$$

for a suitable sequence of  $m$ -currents  $S_j$  supported on  $\partial B_{r_j}$ .

By adding a properly chosen  $m$ -sphere  $\partial Y_j$  we may compensate the  $h$ -volume enclosed by  $T_j$  and  $T_j \llcorner B_{r_j} + S_j$ . Then Almgren's *Optimal Isoperimetric Theorem* [AF] combined with (5) implies

$$(6) \quad \mathbf{F}(T_j^*) \leq \mathbf{F}(T_j) + \frac{2}{j}$$

for the currents  $T_j^* := T_j \llcorner B_{r_j} + S_j + \partial Y_j$  which by construction have compact supports in some compact set independent of  $j$ . From this it follows rather easily that a subsequence of  $\{T_j^*\}$  converges to the desired solution of (1).

The same technique also applies to the problem of minimizing area for fixed boundary and constant  $h$ -volume  $V_h(T, T_0)$ . The details will be given in section 3.

In section 4 we discuss the variational equations, especially it is shown that the mass minimizers with a volume constraint are generalized surfaces with constant mean curvature vector. A final chapter is devoted to the study of oriented tangent cones which exist at all points  $x \in \text{spt}(T) - \text{spt}(\partial T_0)$  and moreover minimize area with respect to their boundaries. This extends some results obtained in [DF2].

## 2. Notations.

Suppose that  $m \geq 2, k \geq 1$  are integers and let  $H: \Lambda_m(\mathbb{R}^{m+k}) \rightarrow \mathbb{R}^{m+k}$  denote a nontrivial linear mapping with the additional property

$$H(v_1 \wedge \dots \wedge v_m) \in \text{Span}[v_1, \dots, v_m]^\perp$$

for all  $v_1, \dots, v_m \in \mathbb{R}^{m+k}$ . Associated with  $H$  is the nontrivial curvature form

$$h(v_0 \wedge \dots \wedge v_m) := v_0 \cdot H(v_1 \wedge \dots \wedge v_m).$$

$h$  is contained in the space  $\Lambda^{m+1}(\mathbb{R}^{m+k})$ . Let

$$0 < \lambda := \sup \{h(w_0 \wedge \dots \wedge w_m) : w_i \in \mathbb{R}^{m+k}, w_i \cdot w_j = \delta_{ij}\}.$$

Having fixed  $v_0, \dots, v_m \in \mathbf{R}^{m+k}$ ,  $v_i \cdot v_j = \delta_{ij}$ , such that

$$\lambda = h(v_0 \wedge \dots \wedge v_m)$$

we define

$$E := \text{Span}[v_0, \dots, v_m], \quad \mathbf{E} = v_0 \wedge \dots \wedge v_m.$$

Let us now briefly recall some notations from *Geometric Measure Theory*—the standard references are [F] and [S]: a current  $T \in \mathcal{O}_n(\mathbf{R}^{m+k})$ ,  $n \in \mathbf{N}$ , is said to be an *integer multiplicity rectifiable  $n$ -current* if

$$T(\omega) = \int_M \langle \omega, \zeta \rangle \theta \, d\mathcal{H}^n, \quad \omega \in \mathcal{O}^n(\mathbf{R}^{m+k}),$$

where  $M$  is an  $\mathcal{H}^n$ -measurable countably  $n$ -rectifiable subset of  $\mathbf{R}^{m+k}$ ,

$$\theta: M \rightarrow \mathbf{N}$$

denotes a locally  $\mathcal{H}^n$ -integrable function, and

$$\zeta: M \rightarrow \Lambda_n(\mathbf{R}^{m+k})$$

is a  $\mathcal{H}^n$ -measurable function such that

$$\zeta(x) = \tau_1(x) \wedge \dots \wedge \tau_n(x)$$

for  $\mathcal{H}^n$ -almost all  $x \in M$  where  $\tau_1(x), \dots, \tau_n(x)$  denotes an orthonormal basis for the approximate tangent space  $T_x M$  (compare [S, 27.1]). If in addition the boundary  $\partial T$  is an integer multiplicity rectifiable  $(n-1)$ -current, then  $T$  is in the class of all *integral  $n$ -currents*.

We abbreviate

$$\mathbf{I}_n := \{T \in \mathcal{O}_n(\mathbf{R}^{m+k}): T \text{ is an integral current, } \mathbf{M}(T) + \mathbf{M}(\partial T) < \infty\}.$$

DEFINITION. For  $S, T \in \mathbf{I}_m$  with  $\partial S = \partial T$  we define the  $h$ -volume enclosed by  $S$  and  $T$  as the quantity

$$\mathbf{V}_h(S, T) := Q(h)$$

for some  $Q \in \mathbf{I}_{m+1}$  with  $\partial Q = S - T$ .

The existence of  $Q \in \mathbf{I}_{m+1}$  with boundary  $S - T$  follows directly from the *Deformation Theorem* (see [F, 4.2.9, 4.2.10]) and since  $Q$  is representable by integration and of finite mass the expression  $Q(h)$  makes sense. Now if  $Q' \in \mathbf{I}_{m+1}$  denotes a second current with boundary  $S - T$ ,

then  $Q - Q'$  is closed and according to [F, 4.2.10] we may write

$$Q - Q' = \partial L$$

with  $L \in \mathbf{I}_{m+2}$ . Now consider a sequence  $\eta_n \in C_0^\infty(\mathbf{R}^{m+k})$  such that  $\eta_n = 1$  on  $B_n(0)$ ,  $\eta_n \equiv 0$  on  $\mathbf{R}^{m+k} - B_{n+1}(0)$ ,  $|\nabla \eta_n| \leq 2$ , and  $0 \leq \eta_n \leq 1$ . Then

$$(Q - Q')(h) = \lim_{n \rightarrow \infty} (Q - Q')(\eta_n h) = \lim_{n \rightarrow \infty} L(d(\eta_n h)) = \lim_{n \rightarrow \infty} L(d\eta_n \wedge h) = 0$$

since

$$|L(d\eta_n \wedge h)| \leq c \mathbf{M}_{\mathbf{R}^{m+k} - \overline{B_n(0)}}(L) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

where  $c > 0$  denotes a suitable constant independent of  $n$ .

From this we deduce that  $\mathbf{V}_h(S, T)$  is well defined. For later purposes we state an optimal mass estimate for mass minimizing currents  $Q$ :

**ISOPERIMETRIC THEOREM.** ([AF]) *Suppose  $Q \in \mathbf{I}_{m+1}$  minimizes mass with respect to the boundary  $\partial Q$ . Then*

$$\mathbf{M}(Q) \leq \gamma_m \mathbf{M}(\partial Q)^{1+1/m},$$

where

$$\gamma_m := (m + 1)^{-1-1/m} \alpha_{m+1}^{-1/m}$$

denotes the isoperimetric constant. Equality holds if and only if  $Q$  is an  $(m + 1)$ -ball in  $\mathbf{R}^{m+k}$ .

Finally we introduce certain oriented  $(m + 1)$ -balls in  $\mathbf{I}_{m+1}$ :

**DEFINITION.** *For  $t > 0$  and  $x \in \{-1, 1\}$  we let*

$$Y_{t,x} := \underline{\tau}(E \cap B_t(0), 1, xE) \in \mathbf{I}_{m+1}. \quad \square$$

Clearly we have

$$\mathbf{V}_h(\partial Y_{t,x}, 0) = Y_{t,x}(h) = \alpha_{m+1} x \lambda t^{m+1}$$

and

$$|Y_{t,x}(h)| = \lambda \gamma_m \mathbf{M}(\partial Y_{t,x})^{1+1/m}.$$

### 3. Existence results.

If all currents under consideration have supports in a fixed compact set  $K \subset \mathbb{R}^{m+k}$  then the existence of minimizers of the functional

$$F(T) := \mathbf{M}(T) + \mathbf{V}_h(T, T_0)$$

in suitable subclasses of  $\mathbf{I}_m(T_0)$  given) can be proved very easily, we refer the reader to [DF1, Theorem 3.1]. We pass now to derive existence theorems in a more general situation.

**THEOREM 3.1.** *Assume that  $T_0 \in \mathbf{I}_m$  has compact support and let*

$$\mathcal{C}_{R, T_0} := \{T \in \mathbf{I}_m : \partial T = \partial T_0, \mathbf{M}(T) \leq R\}$$

for a given  $R > 0$ . Then the problem

$$F(T) := \mathbf{M}(T) + \mathbf{V}_h(T, T_0) \rightsquigarrow \text{Min in } \mathcal{C}_{R, T_0}$$

admits a solution with compact support.

Concerning the related problem of minimizing mass subject to a volume constraint we have

**THEOREM 3.2.** *Assume that  $T_0 \in \mathbf{I}_m$  has compact support and let*

$$\mathcal{C} := \{T \in \mathbf{I}_m : \partial T = \partial T_0, \mathbf{V}_h(T, T_0) = c\}$$

for a given number  $c \in \mathbb{R}$ . Then the problem

$$\mathbf{M}(\cdot) \rightsquigarrow \text{Min in } \mathcal{C}$$

has a solution with compact support.

**REMARK.** In section 4 we discuss conditions under which the above minimizers are solutions of the constant curvature equation.

**PROOF OF THEOREM 3.2.** We first observe that  $\mathcal{C}$  contains a current  $S_0$  with compact support, for example

$$S_0 := T_0 + \partial Y_{t,x} \in \mathcal{C}$$

provided we choose  $t > 0, x \in \{-1, 1\}$  to satisfy

$$\mathbf{V}_h(\partial Y_{t,x}, 0) = c,$$

that is  $x = \text{sign}(c)$  and  $t = (|c|\alpha_{m+1}^{-1}\lambda^{-1})^{1/(m+1)}$ .

For  $j \geq 1$  such that

$$\text{spt}(T_0) \cup \text{spt}(S_0) \Subset B_j,$$

where  $B_j := \{z \in \mathbf{R}^{m+k} : |z| < j\}$ , the problem

$$\mathbf{M}(\cdot) \rightsquigarrow \text{Min} \quad \text{in } \mathcal{C}_j := \mathcal{C} \cap \{T \in \mathbf{I}_m : \text{spt}(T) \subset \overline{B_j}\}$$

admits a solution  $T_j$  whose existence follows along the lines of the proof of [DF1, Theorem 3.1] by considering a minimizing sequence. Clearly

$$(1) \quad M(T_{j+1}) \leq M(T_j) \leq M(S_0).$$

Let  $Q_j$  denote a mass minimizing current for the boundary  $T_j - T_0$ . As we shall prove below there exist numbers  $0 < \rho < \sigma < \infty$  such that

$$(2) \quad \text{spt}(T_0) \cup \text{spt}(S_0) \subset B_\rho$$

and a sequence  $\rho \leq r_j \leq \sigma$  of radii such that the slices

$$\langle Q_j, |\cdot|, r_j^+ \rangle, \quad \langle Q_j, |\cdot|, r_j^- \rangle \in \mathcal{O}_m(\mathbf{R}^{m+k})$$

coincide and that the common value denoted by  $\langle Q_j, |\cdot|, r_j \rangle$  is an integer multiplicity current satisfying

$$(3) \quad \mathbf{M}(\langle Q_j, |\cdot|, r_j \rangle) \leq \frac{1}{j}.$$

Now let  $T_j^* := T_j \llcorner B_{r_j} + \langle Q_j, |\cdot|, r_j \rangle \in \mathbf{I}_m$ . Then

$$\begin{aligned} \partial T_j^* &= \partial(T_j \llcorner B_{r_j}) + \partial(\langle Q_j, |\cdot|, r_j \rangle - (\partial Q_j) \llcorner B_{r_j}) \\ &= \partial(T_j \llcorner B_{r_j}) - \partial(T_j \llcorner B_{r_j} - T_0) = \partial T_0, \end{aligned}$$

and

$$\text{spt}(T_j^*) \subset \overline{B_{r_j}} \subset \overline{B_\sigma},$$

but  $T_j^*$  does not necessarily belong to  $\mathcal{C}_j$ . In order to compensate the change of volume we choose  $t_j > 0, x_j \in \{-1, 1\}$  to satisfy

$$(4) \quad \mathbf{V}_h(\partial Y_j, 0) = -\mathbf{V}_h(T_j^*, T_j), \quad Y_j := Y_{t_j, x_j}.$$



From (1), (4) we infer the bound ( $Q_j^* \in \mathbf{I}_{m+1}$  denotes a mass minimizer for the boundary  $T_j - T_j^*$ )

$$\begin{aligned} \alpha_{m+1} \lambda t_j^{m+1} &= |\mathbf{V}_h(T_j^*, T_j)| \leq \gamma_m \lambda \mathbf{M}(Q_j^*) \leq \\ &\leq \gamma_m \lambda \mathbf{M}(\partial Q_j^*)^{1+1/m} = \gamma_m \lambda \mathbf{M}(T_j - T_j^*)^{1+1/m} \stackrel{(3)}{\leq} \gamma_m \lambda (1 + \mathbf{M}(S_0))^{1+1/m} \end{aligned}$$

so that

$$\text{spt}(Y_j) \subset \overline{B_{\sigma^*}}$$

for some real number  $\sigma^* > \sigma$ . Therefore the currents

$$S_j := T_j^* + \partial Y_j$$

are in  $\mathcal{C}_j$  (provided  $j \geq \sigma^*$ ) and have supports in the ball  $\overline{B_{\sigma^*}}$ .

The minimality of  $T_j$  implies:

$$\begin{aligned} \mathbf{M}(T_j) &\leq \mathbf{M}(S_j) \leq \mathbf{M}(T_j^*) + \mathbf{M}(\partial Y_j) \stackrel{(4)}{=} \mathbf{M}(T_j^*) + \\ &+ (\lambda^{-1} \gamma_m^{-1} |\mathbf{V}_h(T_j^*, T_j)|)^{m/(m+1)} \leq \mathbf{M}(T_j^*) + (\gamma_m^{-1} \mathbf{M}(Q_j^*))^{m/(m+1)} \leq \mathbf{M}(T_j^*) + \\ &+ \mathbf{M}(T_j - T_j^*) \leq \mathbf{M}(T_j \lfloor B_{r_j}) + \mathbf{M}(T_j - T_j \lfloor B_{r_j}) + \\ &\quad + 2\mathbf{M}(\langle Q_j, \cdot |, r_j \rangle) \stackrel{(3)}{\leq} \mathbf{M}(T_j) + \frac{2}{j}, \end{aligned}$$

where we have used the optimal version of the Isoperimetric Theorem.

Recalling the mass bounds (1), (3) as well as

$$\mathbf{M}(\partial Y_j) = (\lambda^{-1} \gamma_m^{-1} |\mathbf{V}_h(T_j^*, T_j)|)^{m/(m+1)} \leq \mathbf{M}(S_0) + 1$$

we may assume (after passing to a subsequence)

$$S_j \rightarrow S \quad \text{in } \mathcal{O}_m(\mathbf{R}^{m+k})$$

for some  $S \in \mathcal{C}$  with  $\text{spt}(S) \subset \overline{B_{\sigma^*}}$ . By construction we have for all  $U \in \mathcal{C}$  with compact support and  $j$  large enough (depending on  $U$ )

$$\mathbf{M}(S_j) - \frac{2}{j} \leq \mathbf{M}(T_j) \leq \mathbf{M}(U),$$

hence

$$\mathbf{M}(S) \leq \mathbf{M}(U)$$

by the lower semicontinuity of the mass.

If  $U \in \mathcal{C}$  is arbitrary we let  $U_r$  denote the projection of  $U$  onto the ball  $\overline{B}_r$ , that is  $U_r := f_{\#} U$  where  $f: \mathbb{R}^{m+k} \rightarrow \overline{B}_r$  is the nearest point retraction from  $\mathbb{R}^{m+k}$  onto  $\overline{B}_r$ . We then define  $t > 0$  and  $x \in \{-1, 1\}$  to satisfy

$$U_r + \partial Y_{t,x} \in \mathcal{C}.$$

Then  $\mathbf{M}(S) \leq \mathbf{M}(U_r) + \mathbf{M}(\partial Y_{t,x})$ . Obviously  $\mathbf{M}(U_r) \leq \mathbf{M}(U)$  and

$$\begin{aligned} \mathbf{M}(\partial Y_{t,x}) &= (\lambda^{-1} \gamma_m^{-1} |\mathbf{V}_h(U_r, U)|)^{m/(m+1)} \leq \mathbf{M}(U - U_r) \leq \\ &\leq 2\mathbf{M}_{\mathbb{R}^{m+k} - B_r}(U) \xrightarrow{r \rightarrow \infty} 0 \end{aligned}$$

which gives the minimizing property of  $S \in \mathcal{C}$ .  $\square$

It remains to prove the existence of suitable slices  $\langle Q_j, |\cdot|, r_j \rangle$  satisfying (3). Following the ideas of [B] and [BG] we first show

**LEMMA 3.1.** *Suppose that  $j_0 < j_1$  are integers where  $j_0$  has to satisfy*

$$j_0 > \left( \frac{\gamma_m}{\alpha_{m+1}} \right)^{1/(m+1)} A^{1/m}, \quad \text{spt}(S_0) \cup \text{spt}(T_0) \subset \overline{B}_{j_0},$$

where  $A := \mathbf{M}(T_0) + \mathbf{M}(S_0)$ . For  $j > j_0$  define the set  $\Lambda$  as the set of all  $t \in [j_0, j_1]$  with  $\mathbf{M}(Q_j \lfloor \partial B_t) = 0$  and  $\mathbf{M}(T_j \lfloor \partial B_t) = 0$  such that the slices  $\langle Q_j, |\cdot|, t^+ \rangle$ ,  $\langle Q_j, |\cdot|, t^- \rangle$  represent the same integer multiplicity current  $\langle Q_j, |\cdot|, t \rangle$  of finite mass. Then the following statements hold:

- (i)  $\mathcal{L}^1([j_0, j_1] - \Lambda) = 0$ .
- (ii) For arbitrary  $t_1 < t_2 < t_3$  in  $\Lambda$  we have

$$\begin{aligned} \min \{ \mathbf{M}_{B_{t_3} - \overline{B}_{t_2}}(Q_j), \mathbf{M}_{B_{t_2} - \overline{B}_{t_1}}(Q_j) \} &\leq \\ &\leq \gamma_m \left( \frac{3}{1 - 2^{-1/(m+1)}} \right)^{1+1/m} \left[ \max_{i=1}^3 \mathbf{M}(\langle Q_j, |\cdot|, t_i \rangle) \right]^{1+1/m} \end{aligned}$$

**PROOF OF LEMMA 3.1.** Since  $Q_j$  and  $T_j$  have finite mass we clearly have

$$\mu_{T_j}(\partial B_t) = 0 = \mu_{Q_j}(\partial B_t)$$

for  $\mathcal{L}^1$ -almost all  $t$ . The remaining claims in (i) follow from general slicing theory (e.g. [S, 28]).

Now, we introduce the currents

$$\hat{T}_j := T_j \llcorner B_{t_1} + \langle Q_j, |\cdot|, t_1 \rangle + T_j \llcorner (\mathbf{R}^{m+k} - \overline{B_{t_3}}) - \langle Q_j, |\cdot|, t_3 \rangle,$$

$$\hat{Q}_j := Q_j \llcorner B_{t_1} + Q_j \llcorner (\mathbf{R}^{m+k} - \overline{B_{t_3}}),$$

and get

$$\begin{aligned} \partial \hat{Q}_j &= \partial(Q_j \llcorner B_{t_1}) + \partial(Q_j \llcorner (\mathbf{R}^{m+k} - \overline{B_{t_3}})) = \\ &= \langle Q_j, |\cdot|, t_1 \rangle + \partial Q_j \llcorner B_{t_1} + (\partial Q_j) \llcorner (\mathbf{R}^{m+k} - \overline{B_{t_3}}) - \langle Q_j, |\cdot|, t_3 \rangle = \\ &= \langle Q_j, |\cdot|, t_1 \rangle + T_j \llcorner B_{t_1} - T_0 + T_j \llcorner (\mathbf{R}^{m+k} - \overline{B_{t_3}}) - \langle Q_j, |\cdot|, t_3 \rangle = \hat{T}_j - T_0. \end{aligned}$$

Hence  $\partial \hat{T}_j = \partial T_0$ . This gives  $\hat{T}_j \in \mathbf{I}_m$  with boundary  $\partial T_0$ . On the other hand we have

$$\mathbf{V}_h(\hat{T}_j, T_0) = \mathbf{V}_h(\hat{T}_j - T_j + T_j, T_0) = \mathbf{V}_h(\hat{T}_j, T_j) + c$$

so that we have to consider the currents

$$\tilde{T}_j := \hat{T}_j + \partial Y_{s_j, x_j}$$

with  $(m+1)$ -balls  $Y_{s_j, x_j}$  satisfying

$$Y_{s_j, x_j}(h) = -\mathbf{V}_h(\hat{T}_j, T_j).$$

Observing

$$\partial(\hat{Q}_j - Q_j) = \hat{T}_j - T_j$$

we deduce

$$\begin{aligned} |Y_{s_j, x_j}(h)| &= |(\hat{Q}_j - Q_j)(h)| \leq \\ &\leq \lambda \mathbf{M}(\hat{Q}_j - Q_j) = \lambda \mathbf{M}_{B_{t_3} - \overline{B_{t_1}}}(Q_j) \leq \lambda \mathbf{M}(Q_j) \leq \\ &\leq \gamma_m \lambda \mathbf{M}(T_j - T_0)^{1+1/m} \stackrel{(1)}{\leq} \gamma_m \lambda (\mathbf{M}(T_0) + \mathbf{M}(S_0))^{1+1/m} = \gamma_m \lambda A^{1+1/m} \end{aligned}$$

and in conclusion

$$|s_j| \leq \left( \frac{\gamma_m}{\alpha_{m+1}} \right)^{1/(m+1)} A^{1/m}.$$

From this we infer

$$\text{spt}(\tilde{T}_j) \subset \overline{B_{j_0}}$$

so that  $\tilde{T}_j$  is in the class  $\mathcal{C}_j$ . The resulting inequality

$$\mathbf{M}(T_j) \leq \mathbf{M}(\tilde{T}_j)$$

can easily be rewritten as

$$\mathbf{M}_{B_{t_3} - \overline{B_{t_1}}}(T_j) \leq \mathbf{M}(\langle Q_j, |\cdot|, t_1 \rangle) + \mathbf{M}(\langle Q_j, |\cdot|, t_3 \rangle) + \mathbf{M}(\partial Y_{s_j, x_j})$$

with

$$\mathbf{M}(\partial Y_{s_j, x_j}) = (\gamma_m^{-1} \lambda^{-1} |\mathbf{V}_h(\partial Y_{s_j, x_j}, 0)|)^{m/(m+1)} \leq (\gamma_m^{-1} \mathbf{M}_{B_{t_3} - \overline{B_{t_1}}}(Q_j))^{m/(m+1)}$$

and we arrive at

$$(5) \quad \mathbf{M}_{B_{t_3} - \overline{B_{t_1}}}(T_j) \leq \\ \leq \mathbf{M}(\langle Q_j, |\cdot|, t_1 \rangle) + \mathbf{M}(\langle Q_j, |\cdot|, t_3 \rangle) + (\gamma_m^{-1} \mathbf{M}_{B_{t_3} - \overline{B_{t_1}}}(Q_j))^{m/(m+1)}.$$

Using the Isoperimetric Theorem, we easily check that

$$\mathbf{M}_{B_{t_2} - \overline{B_{t_1}}}(Q_j) \leq \gamma_m \mathbf{M}(\partial(Q_j \llcorner (B_{t_2} - \overline{B_{t_1}})))^{m/(m+1)}$$

which implies

$$(6) \quad (\gamma_m^{-1} \mathbf{M}_{B_{t_2} - \overline{B_{t_1}}}(Q_j))^{m/(m+1)} \leq \\ \leq \mathbf{M}_{B_{t_2} - \overline{B_{t_1}}}(T_j) + \mathbf{M}(\langle Q_j, |\cdot|, t_1 \rangle) + \mathbf{M}(\langle Q_j, |\cdot|, t_2 \rangle)$$

and

$$(7) \quad (\gamma_m^{-1} \mathbf{M}_{B_{t_3} - \overline{B_{t_2}}}(Q_j))^{m/(m+1)} \leq \\ \leq \mathbf{M}_{B_{t_3} - \overline{B_{t_2}}}(T_j) + \mathbf{M}(\langle Q_j, |\cdot|, t_2 \rangle) + \mathbf{M}(\langle Q_j, |\cdot|, t_3 \rangle).$$

Combining (5), (6), (7) we then have

$$(8) \quad 2 \sum_{i=1}^3 \mathbf{M}(\langle Q_j, |\cdot|, t_i \rangle) \geq \\ \geq \gamma_m^{-m/(m+1)} \{ \mathbf{M}_{B_{t_2} - \overline{B_{t_1}}}(Q_j)^{m/(m+1)} + \mathbf{M}_{B_{t_3} - \overline{B_{t_2}}}(Q_j)^{m/(m+1)} - \mathbf{M}_{B_{t_3} - \overline{B_{t_1}}}(Q_j)^{m/(m+1)} \}.$$

Inequality (8) corresponds to the estimate [B, (7)] and as demonstrated there we arrive at (ii) of our Lemma.  $\square$

The existence of numbers  $\rho$  and  $\sigma$  and of a sequence  $\{r_j\}$  of radii  $r_j \in [\rho, \sigma]$  with the property (2) follows from (ii) of Lemma 3.1 along the lines of step 1 in [BG]. For the reader's convenience we give a brief outline of the arguments.

Define  $j_0 \in \mathbb{N}$  according to Lemma 3.1,  $j_1 \in \mathbb{N}$  will be fixed at the end of the proof and suppose that  $j > j_1$  is arbitrary. In the sequel we will omit the index  $j$ . We introduce the following quantities:

$$\rho_0 := j_0, \quad \sigma_0 := j_1, \quad l_0 := j_1 - j_0, \quad \delta_0 := \frac{l_0}{6}.$$

In order to define sequences  $\{\rho_n\}$ ,  $\{\sigma_n\}$  by induction we let for a given  $\rho_n < \sigma_n$

$$l_n := \sigma_n - \rho_n, \quad v_n := \mathbf{M}_{B_{\rho_n} - \overline{B_{\rho_n}}}(Q), \quad \delta_n := \frac{l_n}{6}$$

and apply Lemma 3.1. Hence there exist numbers

$$t_n^1 \in ]\rho_n, \rho_n + \delta_n[, \quad t_n^2 \in \left] \rho_n + \frac{l_n}{2} - \frac{\delta_n}{2}, \rho_n + \frac{l_n}{2} + \frac{\delta_n}{2} \right[, \quad t_n^3 \in ]\sigma_n - \delta_n, \sigma_n[$$

such that  $\mathbf{M}(T \lfloor \partial B_{t_n^i}) = 0 = \mathbf{M}(Q \lfloor \partial B_{t_n^i})$  for  $i = 1, 2, 3$  and such that the slices  $\langle Q, |\cdot|, t_n^i \rangle \in \mathcal{O}_m(\mathbb{R}^{m+k})$  are integer multiplicity currents of finite mass. Quoting [S, 28.5 (i)] we can further arrange

$$(9) \quad \mathbf{M}(\langle Q, |\cdot|, t_n^i \rangle) \leq 2 \frac{v_n}{\delta_n}.$$

Next we define  $v_n^i := \mathbf{M}(Q \lfloor B_{t_n^{i+1}} - \overline{B_{t_n^i}})$ ,  $i = 1, 2$ , and  $v_{n+1} := \min(v_n^1, v_n^2)$  and choose

$$]\rho_{n+1}, \sigma_{n+1}[ := \begin{cases} ]t_n^1, t_n^2[ & \text{if } v_n^1 < v_n^2, \\ ]t_n^2, t_n^3[ & \text{if } v_n^1 \geq v_n^2 \end{cases}$$

Moreover we define

$$u_n := \max_{i=1}^3 \mathbf{M}(\langle Q, |\cdot|, t_n^i \rangle).$$

Then, from (9) we infer that

$$(10) \quad u_n \leq 2 \frac{v_n}{\delta_n}$$

and Lemma 3.1, (ii) implies

$$(11) \quad v_{n+1} \leq c(m) u_n^{1+1/m}$$

with a constant  $c(m) \geq 1$  depending only on  $m$ . Combining (10), (11) and observing  $4^{-n} \delta_0 \leq \delta_n \leq (3/4)^n \delta_0$  we arrive at ( $q := 1 + 1/m$ )

$$u_n \leq \left[ 6\gamma_m (2c(m) 4^m)^{m+1} \frac{(\mathbf{M}(S_0) + \mathbf{M}(T_0))^q}{j_1 - j_0} \right]^{q^n} \leq 2^{-q^n}$$

provided we choose  $j_1 := j_1(m, S_0, T_0)$  such that

$$\left[ 6\gamma_m (2c(m) 4^m)^{m+1} \frac{(\mathbf{M}(S_0) + \mathbf{M}(T_0))^q}{j_1 - j_0} \right] \leq \frac{1}{2}.$$

Clearly this implies

$$u_n = \max_{i=1}^3 \mathbf{M}(\langle Q, |\cdot|, t_n^i \rangle) \xrightarrow{n \rightarrow \infty} 0$$

and by construction we have  $l_n \leq (3/4)^n l_0 \rightarrow 0$ , hence  $\rho_n \uparrow r, \sigma_n \downarrow r$  for some radius  $r \in [j_0, j_1]$ , and we also conclude that

$$\lim_{n \rightarrow \infty} \mathbf{M}(\langle Q, |\cdot|, \rho_n \rangle) = \lim_{n \rightarrow \infty} \mathbf{M}(\langle Q, |\cdot|, \sigma_n \rangle) = 0$$

which clearly implies the desired property (2).  $\square$

**PROOF OF THEOREM 3.1.** For  $j \in \mathbb{N}$  we now let

$$\mathcal{C}_j := \{T \in \mathbf{I}_m : \text{spt}(T) \subset \overline{B_j}\} \cap \mathcal{C}_{R, T_0}.$$

Since we assume  $S_0 \in \mathcal{C}_{R, T_0}$  for some  $S_0 \in \mathbf{I}_m$  and since  $\text{spt}(T_0)$  is compact, e.g.  $\text{spt}(T_0) \subset \overline{B_r}$  for some  $r > 0$ , we deduce

$$\mathcal{C}_j \neq \emptyset$$

for  $j \geq r$  by projecting  $S_0$  onto the ball  $\overline{B_r}$ . Again, let  $T_j$  denote an  $\mathbf{F}$ -minimizer in  $\mathcal{C}_j$  (compare [DF1, Theorem 3.1]) and define  $S_j$  precisely as in the proof of Theorem 3.2. (In order to justify this step we have to check that the argument in Lemma 3.1 can be extended to the situation here: fix some  $j \in \mathbb{N}$  and observe that (8) is true under the assumption  $\mathbf{M}(T_j) \leq \mathbf{M}(\tilde{T}_j)$ . If  $\mathbf{M}(\tilde{T}_j) \leq \mathbf{M}(T_j)$  then clearly  $\mathbf{M}(\tilde{T}_j) \leq R$ , hence  $\tilde{T}_j \in \{S \in \mathbf{I}^m : \text{spt}(S) \subset \overline{B_j}\} \cap \mathcal{C}_{R, T_0}$  and in conclusion  $\mathbf{F}(T_j) \leq \mathbf{F}(\tilde{T}_j)$  so that we again arrive at  $\mathbf{M}(T_j) \leq \mathbf{M}(\tilde{T}_j)$  and the proof of Lemma 3.1 can be

completed as before.) Then it is easy to check that

$$(12) \quad \mathbf{M}(S_j) \leq \mathbf{M}(T_j) + \frac{2}{j}, \quad \mathbf{F}(S_j) \leq \mathbf{F}(T_j) + \frac{2}{j}.$$

Note that  $S_j$  does not necessarily belong to  $\mathcal{C}_j$  but the limit  $S$  of a subsequence is in  $\mathcal{C}_{R,T_0}$ . From (12) we infer

$$\mathbf{F}(S) \leq \mathbf{F}(U)$$

for  $U \in \mathcal{C}_{R,T_0}$  with compact support and by pushing an arbitrary current  $U \in \mathcal{C}_{R,T_0}$  onto a large ball  $\bar{B}_t$  and letting  $t \rightarrow \infty$  we get the  $F$ -minimizing property of  $S$ .  $\square$

#### 4. Variational Equations.

In this section we briefly discuss conditions under which the minimizers constructed in Theorem 3.1, 3.2 induce varifolds of bounded first variation. The resulting variational equations show that the minimizers represent generalized surfaces of constant mean curvature.

**THEOREM 4.1.** *Suppose that  $T_0 \in \mathbf{I}_m$  has compact support and define*

$$R := (2m + 1) \mathbf{M}(\tilde{T})$$

where  $\tilde{T} \in \mathbf{I}_m$  is mass minimizing for the boundary  $\partial T_0$ . Moreover, assume

$$(1) \quad \lambda < \gamma_m^{-1} 2^{-1/m} m(m+1)^{-1-1/m} \mathbf{M}(\tilde{T})^{-1/m}$$

and let  $T$  denote the  $\mathbf{F}$ -minimizing current in  $\mathcal{C}_{R,T_0}$  introduced in Theorem 3.1. Then we have ( $T = \underline{\tau}(M, \theta, T), \mu_T := \partial \mathcal{C}^m \llcorner \theta$ ):

$$(2) \quad \int_M [\operatorname{div}_M X + X \cdot H(T)] d\mu_T = 0$$

for all  $X \in C_0^1(\mathbf{R}^{m+k}, \mathbf{R}^{m+k})$ ,  $\operatorname{spt}(X) \cap \operatorname{spt}(\partial T_0) = \emptyset$ .

**COROLLARY 4.1.** *Let  $\mathcal{R} := \{x \in \mathbf{R}^{m+k} - \operatorname{spt}(\partial T_0) : \operatorname{spt}(T) \cap B_\rho(x) \text{ is for some ball } B_\rho(x) \text{ an oriented } m\text{-dimensional submanifold of class } C^1 \text{ with mean curvature vector } H(T)\}$ . Then  $\mathcal{R} \cap \operatorname{spt}(T)$  is dense in  $\operatorname{spt}(T) - \operatorname{spt}(\partial T_0)$ . Moreover, there exists a positive number  $\delta$*

such that

$$\mu_T(B_\rho(x)) \leq (1 + \delta) \alpha_m \rho^m$$

for some ball  $B_\rho(x)$  implies  $x \in \mathcal{R}$ .

REMARK. If  $T = \llbracket \Sigma \rrbracket$  for a smooth oriented  $m$ -dimensional manifold  $\Sigma$  then (2) immediately implies that the mean curvature vector  $H(x)$  of  $\Sigma$  at  $x$  is given by the quantity  $H(T(x))$ . On the other hand equation (2) shows that the varifold  $\mathbf{V} = v(M, \theta)$  associated with the minimizing current  $T$  is of bounded first variation. Hence we may adopt the terminology of [S, 16.5] by the way generalizing the classical concept of mean curvature: from this point of view equation (2) tells us that the current  $T$  is in a suitable generalized sense a solution of the problem of finding an oriented  $m$ -dimensional manifold with fixed boundary  $\partial T_0$  and prescribed constant mean curvature form.

PROOF OF THEOREM 4.1. Combining (1) and the choice of  $R$  it is easy to check (using the Isoperimetric Theorem) that  $T$  is an interior minimizer, i.e.

$$\mathbf{M}(T) < R,$$

we refer the reader to [DF1, Theorem 6.1]; in this case (2) follows directly along the lines of [DF1, Theorem 4.1].  $\square$

Corollary 4.1 is a consequence of the Allard Regularity Theorem, compare [AW] or [DF1, Theorem 5.1].

THEOREM 4.2. Assume that the hypothesis of Theorem 3.2 hold and let  $T = \tau(M, \theta, T)$  denote the mass minimizing current constructed in Theorem 3.2. Then there exists a real number  $\mu$  such that the current  $T$  is of constant mean curvature  $\mu H$  on  $\mathbf{R}^{m+k} - \text{spt}(\partial T_0)$ , i.e.

$$(3) \quad \int_M \text{div}_M X \, d\mu_T + \mu \int_M X \cdot H(T) \, d\mu_T = 0$$

for all  $X \in C_0^1(\mathbf{R}^{m+k}, \mathbf{R}^{m+k})$  with  $\text{spt}(X) \cap \text{spt}(\partial T_0) = \emptyset$ .

COROLLARY 4.2. The Allard Regularity Theorem holds for points in  $\text{spt}(T) - \text{spt}(\partial T_0)$ .

PROOF OF THEOREM 4.2. We first assume that there exists a vector-



field  $Y \in C_0^1(\mathbb{R}^{m+k}, \mathbb{R}^{m+k})$ ,  $\text{spt}(Y) \cap \text{spt}(\partial T_0) = \emptyset$ , such that

$$\int_M Y \cdot H(\mathbf{T}) d\mu_T \neq 0.$$

For  $X \in C_0^1(\mathbb{R}^{m+k}, \mathbb{R}^{m+k})$  with  $\text{spt}(X) \subset \mathbb{R}^{m+k} - \text{spt}(\partial T_0)$  we define

$$\phi(s, z) := z + sX(z), \quad \psi(t, z) := z + tY(z), \quad \varphi(s, t, z) := z + sX(z) + tY(z)$$

and

$$T_{s,t} := \varphi(s, t, \cdot)_\# T.$$

We clearly have

$$\partial T_{s,t} = \partial T_0, \quad g(s, t) := \mathbf{V}_h(T_{s,t}, T_0) = \mathbf{V}_h(T_{s,t}, T_0) + c,$$

and (using the first variation formula for the  $h$ -volume obtained in [DF1, section 4])

$$\begin{aligned} \frac{\partial}{\partial t} g(0, 0) &= \frac{d}{dt} \Big|_{t=0} \mathbf{V}_h(T_{0,t}, T) = \frac{d}{dt} \Big|_{t=0} \mathbf{V}_h(\psi(t, \cdot)_\# T, T) = \\ &= \int_M Y \cdot H(\mathbf{T}) d\mu_T \neq 0. \end{aligned}$$

Observing that  $g(0, 0) = c$  we deduce the existence of a curve  $\sigma(s)$  such that  $\sigma(0) = 0$  and  $g(s, \sigma(s)) = c$ . This implies

$$T_{s, \sigma(s)} = \varphi(s, \sigma(s), \cdot)_\# T \in \mathcal{C}$$

and by the first variation formula for the mass we arrive at

$$\begin{aligned} 0 &= \frac{d}{ds} \Big|_{s=0} \mathbf{M}(T_{s, \sigma(s)}) = \int_M \text{div}_M \left( \frac{\partial}{\partial s} \Big|_{s=0} \varphi(s, \sigma(s), \cdot) \right) d\mu_T = \\ &= \int_M \text{div}_M X d\mu_T + \sigma'(0) \int_M \text{div}_M Y d\mu_T. \end{aligned}$$

In view of

$$\begin{aligned} 0 &= \frac{d}{ds} \Big|_{s=0} g(s, \sigma(s)) = \frac{\partial}{\partial s} g(0, 0) + \sigma'(0) \frac{\partial}{\partial t} g(0, 0) = \\ &= \frac{d}{ds} \Big|_{s=0} \mathbf{V}_h(T_{s,0}, T) + \sigma'(0) \int_M Y \cdot H(\mathbf{T}) d\mu_T, \end{aligned}$$

and

$$\mathbf{V}_h(T_{s,0}, T) = \mathbf{V}_h(\phi(s, \cdot)_\# T, T) = \phi_\#([\!(0, s)\!] \times T)(h)$$

we find

$$\sigma'(0) = - \int_M X \cdot H(T) d\mu_T / \left( \int_M Y \cdot H(T) d\mu_T \right)$$

which proves equation (3) with  $\mu = - \int_M \operatorname{div}_M Y d\mu_T / \left( \int_M Y \cdot H(T) d\mu_T \right)$ .

Now, if

$$\int_M Y \cdot H(T) d\mu_T = 0$$

for all  $Y \in C_0^1(\mathbf{R}^{m+k}, \mathbf{R}^{m+k})$  with  $\operatorname{spt}(Y) \cap \operatorname{spt}(\partial T_0) = \emptyset$ , we clearly have

$$H(T) = 0 \quad \mu_T\text{-almost everywhere on } \mathbf{R}^{m+k} - \operatorname{spt}(\partial T_0).$$

In this case we proceed as follows: with  $X$  and  $\phi$  as above we let

$$f(t) := \mathbf{V}_h(\phi_\#(t, \cdot) T, T)$$

denote the change of volume. As usual the volume change is compensated by adding a suitable  $m$ -sphere  $\partial Y_{\rho,x}$  where  $\rho = \rho(t)$ ,  $x = x(t)$  satisfy

$$Y_{\rho,x}(h) = -f(t).$$

Hence  $\phi_\#(t, \cdot) T + \partial Y_{\rho,x}$  is an admissible current and for  $t > 0$  we deduce

$$(4) \quad 0 \leq \frac{1}{t} [\mathbf{M}(\phi_\#(t, \cdot) T) - \mathbf{M}(T)] + \frac{1}{t} \mathbf{M}(\partial Y_{\rho,x}).$$

Observing that

$$\mathbf{M}(\partial Y_{\rho,x}) = (\lambda^{-1} \gamma_m^{-1} |Y_{\rho,x}(h)|)^{m/(m+1)} = (\lambda^{-1} \gamma_m^{-1} |f(t)|)^{m/(m+1)}$$

and

$$f(0) = f'(0) = 0$$

we arrive at (recall  $m \geq 2$ )

$$\frac{1}{t} \mathbf{M}(\partial Y_{\rho, x}) = (\lambda^{-1} \gamma_m^{-1})^{m/(m+1)} \frac{1}{t} |f(t)|^{m/(m+1)} \xrightarrow{t \downarrow 0} 0$$

and (4) turns into

$$0 \leq \int_M \operatorname{div}_M X \, d\mu_T.$$

Since  $X$  was arbitrary we have proved (3) with  $\mu = 0$ .  $\square$

### 5. Existence of area minimizing tangent cones.

We here show that the minimizers defined in Theorem 3.1 and Theorem 3.2 have tangent cones which minimize mass with respect to their boundaries. This extends some of the result obtained in [DF2].

**THEOREM 5.1.** *Assume that the hypothesis of Theorem 4.1 or Theorem 4.2 hold. In the situation of Theorem 4.2 we suppose in addition that*

$$\mu_T(\{z \in \mathbf{R}^{m+k} - \operatorname{spt}(\partial T_0) : |H(\mathbf{T}(z))| > 0\}) > 0.$$

Let  $x \in \operatorname{spt}(T) - \operatorname{spt}(\partial T)$  and a sequence  $\{\lambda_i\}$ ,  $\lambda_i \downarrow 0$  be given. Then there is a subsequence  $\{\lambda_i^*\}$  of  $\{\lambda_i\}$  such that

$$T_i := \eta_{x, \lambda_i^*} T \rightarrow C \quad \text{in } \mathcal{O}_m(\mathbf{R}^{m+k}),$$

where  $C$  is integer multiplicity. We have:

(i)  $\partial C = 0$ ,  $\Theta^m(\mu_T, x) = \Theta^m(\mu_C, 0)$  and  $\mu_{T_i} \rightarrow \mu_C$  in the sense of Radon measures.

(ii)  $C$  is a cone, i.e.  $\eta_{0, t\#} C = C$  for all  $t > 0$ .

(iii)  $C$  is locally area minimizing, i.e. we have

$$\mathbf{M}_U(C) \leq \mathbf{M}_U(C + X)$$

for any open set  $U \in \mathbf{R}^{m+k}$  and any  $X \in \mathbf{I}_m$  with  $\operatorname{spt}(X) \subset U$  and  $\partial X = 0$ .

**REMARK 1.** Here we use the notation

$$\eta_{x, t}(y) := t^{-1}(y - x)$$

for  $x, y \in \mathbf{R}^{m+k}$  and  $t > 0$ .

REMARK 2. Assertion 1 of Theorem 5.1 especially includes the fact that the  $m$ -dimensional density

$$\Theta^m(\mu_T, x) = \lim_{\rho \downarrow 0} \frac{\mu_T(B_\rho(x))}{\alpha_m \rho^m}$$

exists for all points  $x \in \text{spt}(T) - \text{spt}(\partial T_0)$ .

REMARK 3. After the proof of Theorem 5.1 we will discuss natural conditions guaranteing the validity of

$$\mu_T(\{z \in \mathbf{R}^{m+k} - \text{spt}(\partial T_0) : |H(T(z))| > 0\}) > 0$$

for the minimizing current  $T$  defined in Theorems 3.2, 3.4.

PROOF OF THEOREM 5.1. From [DF2] we deduce that Theorem 5.1 is valid for  $\mathbf{F}$ -minimizing currents  $T$  as described in Theorem 4.1. Now suppose that we are in the case of Theorem 4.2. We may assume that

$$x = 0 \in \text{spt}(T) - \text{spt}(\partial T_0).$$

Let  $T_j := \eta_{\lambda_j} \# T$  with  $\eta_t := \eta_{0,t}$  and fix  $r > 0$ . Then

$$(1) \quad \sup_{j \in \mathbf{N}} \{\mathbf{M}_{B_r(0)}(T_j) + \mathbf{M}_{B_r(0)}(\partial T_j)\} < \infty.$$

This follows from the facts that  $\partial T_j = 0$  on  $B_r(0)$  (recall  $0 \notin \text{spt}(\partial T)$ ) and

$$\mathbf{M}_{B_r(0)}(T_j) = \lambda_j^{-m} \mu_T(B_{\lambda_j r}(0)) \xrightarrow{j \rightarrow \infty} \alpha_m r^m \Theta^m(\mu_T, 0) < \infty.$$

The existence of  $\Theta^m(\mu_T, 0)$  can be deduced from Theorem 4.2 and [S, 17.8]. According to (1) and the compactness theorem for integer multiplicity currents [S, 27.3] we can select a subsequence of  $T_j$  (again denoted by  $T_j$ ) which converges in  $\mathcal{O}_m(\mathbf{R}^{m+k})$  to an integer multiplicity current  $C$  with  $\partial C = 0$ .

In order to prove the minimizing property of  $C$  we proceed similarly as in [S, Proof of 34.5] and [DF2]: given a compact set  $K \subset \mathbf{R}^{m+k}$  and a smooth function  $\phi: \mathbf{R}^{m+k} \rightarrow [0, 1]$  with compact support and with  $\phi \equiv 1$  in a neighborhood of  $K$  we define for  $0 \leq t \leq 1$

$$W_t := \{z \in \mathbf{R}^{m+k} : \phi(z) > t\}.$$

Quoting [S, 31,2] we may write

$$(2) \quad C - T_j = \partial R_j + S_j, \quad \mathbf{M}_{W_0}(R_j) + \mathbf{M}_{W_0}(S_j) \rightarrow 0,$$

for suitable integer multiplicity currents  $R_j$  and  $S_j$ . By elementary slicing theory we can find  $0 < \alpha < 1$  and an integer multiplicity current  $P_j$  such that

$$(3) \quad \partial(R_j \llcorner W_\alpha) = (\partial R_j) \llcorner W_\alpha + P_j,$$

with

$$\text{spt}(P_j) \subset \partial W_\alpha, \quad \mathbf{M}(P_j) \rightarrow 0,$$

and in addition we may also assume that

$$(4) \quad \mathbf{M}(T_j \llcorner \partial W_\alpha) = 0, \quad \mathbf{M}(C \llcorner \partial W_\alpha) = 0.$$

From (2) and (3) we deduce

$$(5) \quad C \llcorner W_\alpha = T_j \llcorner W_\alpha + \partial \tilde{R}_j + \tilde{S}_j$$

where

$$\tilde{R}_j := R_j \llcorner W_\alpha, \quad \tilde{S}_j := S_j \llcorner W_\alpha - P_j,$$

and

$$(6) \quad \mathbf{M}(\tilde{R}_j) + \mathbf{M}(\tilde{S}_j) \rightarrow 0.$$

Next let  $X \in \mathcal{O}_m(\mathbf{R}^{m+k})$  denote a given integer multiplicity current satisfying  $\partial X = 0$  and  $\text{spt}(X) \subset K$ . Abbreviating  $Z_j := X + \partial \tilde{R}_j$  we have  $\text{spt}(Z_j) \subset \overline{W}_\alpha$  and

$$(7) \quad \mathbf{M}_{W_\alpha}(C + X) \stackrel{(5)}{=} \mathbf{M}_{W_\alpha}(T_j + X + \partial \tilde{R}_j + \tilde{S}_j) \geq \mathbf{M}_{W_\alpha}(T_j + Z_j) - \mathbf{M}(\tilde{S}_j).$$

For arbitrary  $t < \alpha$  we find ( $W_t^j := \lambda_j W_t$ )

$$\mathbf{M}_{W_t}(T_j + Z_j) = \lambda_j^{-m} \mathbf{M}_{W_t^j}(T + \eta_{\lambda_j^{-1} \#} Z_j)$$

and

$$\mathbf{M}_{W_t^j}(T + \eta_{\lambda_j^{-1} \#} Z_j) - \mathbf{M}_{W_t^j}(T) = \mathbf{M}(T + \eta_{\lambda_j^{-1} \#} Z_j) - \mathbf{M}(T).$$

Suppose now that there exists a sequence of positive numbers  $\varepsilon_j \downarrow 0$  with the property

$$(8) \quad \mathbf{M}(T + \eta_{\lambda_j^{-1} \#} Z_j) - \mathbf{M}(T) \geq -\varepsilon_j \lambda_j^m.$$

From (8) we then deduce

$$\mathbf{M}_{W_t}(T_j + Z_j) - \mathbf{M}_{W_t}(T_j) \geq -\varepsilon_j$$

which gives after passing to the limit  $t \uparrow \alpha$

$$(9) \quad \mathbf{M}_{W_\alpha}(T_j + Z_j) - \mathbf{M}_{W_\alpha}(T_j) \geq -\varepsilon_j - \mathbf{M}(P_j).$$

Here we have used

$$\lim_{t \uparrow \alpha} \mathbf{M}_{W_t}(T_j) = \mu_{T_j}(\overline{W}_\alpha) \stackrel{(4)}{=} \mathbf{M}_{W_\alpha}(T_j),$$

and

$$\begin{aligned} \lim_{t \uparrow \alpha} \mathbf{M}_{W_t}(T_j + Z_j) &= \mu_{T_j + Z_j}(\overline{W}_\alpha) = \\ &= \mu_{T_j + Z_j}(W_\alpha) + \mu_{T_j + X + \partial \tilde{R}_j}(\partial W_\alpha) = \mathbf{M}_{W_\alpha}(T_j + Z_j) + \mathbf{M}(P_j). \end{aligned}$$

Remembering (5) we then deduce from (9)

$$\mathbf{M}_{W_\alpha}(C + X) \geq \mathbf{M}_{W_\alpha}(T_j) - \varepsilon_j - \mathbf{M}(P_j) - \mathbf{M}(\tilde{S}_j),$$

therefore by (6) and the lower semicontinuity of the mass

$$\mathbf{M}_{W_\alpha}(C + X) \geq \mathbf{M}_{W_\alpha}(C)$$

which proves the minimizing property of  $C$ .

So it remains to prove the desired inequality (8) for which we have to make use of the minimizing property of  $T$ . For simplicity we abbreviate

$$\hat{Z}_j := \eta_{\lambda_j^{-1} \#} Z_j.$$

From our hypothesis we get the existence of a vectorfield  $Y \in C_0^1(\mathbf{R}^{m+k}, \mathbf{R}^{m+k})$ ,  $\text{spt}(Y) \cap \text{spt}(\partial T_0) = \emptyset$  such that

$$\alpha := \int_M Y \cdot H(T) d\mu_T > 0.$$

Defining  $U_s := \phi(s, \cdot) \# T$ ,  $\phi(s, z) := z + sY(z)$  we want to calculate  $s_j$  such that

$$(10) \quad \mathbf{V}_h(U_{s_j} + \hat{Z}_j, T_0) = c$$

is valid. Clearly (10) is equivalent to the equation

$$(11) \quad \mathbf{V}_h(U_{s_j}, T) = \mathbf{V}_h(-\hat{Z}_j, 0) := \xi_j.$$

The function

$$f(s) := \mathbf{V}_h(U_s, T)$$

has the properties  $f(0) = 0$ ,  $f'(0) = \alpha > 0$ , especially we find  $\delta > 0$  such that  $f' \geq \alpha/2$  on  $[-\delta, \delta]$  which implies

$$f([-\delta, \delta]) \supset \left[-\frac{\alpha}{2}\delta, \frac{\alpha}{2}\delta\right].$$

On the other hand it is easy to check that

$$|\xi_j| \leq A\lambda_j^{m+1}$$

holds for some absolute constant  $0 < A < \infty$  so that we may assume

$$\xi_j \in \left[-\frac{\alpha}{2}\delta, \frac{\alpha}{2}\delta\right], \quad \forall j \in \mathbb{N}.$$

Hence there exists a unique  $s_j \in [-\delta, \delta]$  with  $f(s_j) = \xi_j$  and one easily proves that

$$(12) \quad |s_j| \leq \frac{2}{\alpha} A\lambda_j^{m+1}.$$

The currents  $U_j := U_{s_j}$  now satisfy (11), hence

$$\mathbf{M}(T) \leq \mathbf{M}(U_j + \hat{Z}_j),$$

and we are now in the position to prove (8):

$$\begin{aligned} \mathbf{M}(T + \hat{Z}_j) - \mathbf{M}(T) &\geq \mathbf{M}(T + \hat{Z}_j) - \mathbf{M}(U_j + \hat{Z}_j) = \\ &= \mathbf{M}(T) - \mathbf{M}(U_j) = \frac{1}{s_j} [\mathbf{M}(T) - \mathbf{M}(\phi(s_j, \cdot)_\# T)] s_j \end{aligned}$$

and since

$$\frac{1}{s_j} [\mathbf{M}(T) - \mathbf{M}(\phi(s_j, \cdot)_\# T)] \xrightarrow{j \rightarrow \infty} - \int_M \operatorname{div}_M Y d\mu_T$$

we arrive at

$$\mathbf{M}(T + \hat{Z}_j) - \mathbf{M}(T) \geq -\Lambda |s_j|$$

for some  $\Lambda > 0$  and all  $j \gg 1$ . Recalling (12) we have established (8).

In order to be precise it remains to justify the identity

$$\mathbf{M}(T + \hat{Z}_j) - \mathbf{M}(U_j + \hat{Z}_j) = \mathbf{M}(T) - \mathbf{M}(U_j)$$

which is valid provided we can arrange

$$(13) \quad \text{spt}(\hat{Z}_j) \cap \text{spt}(Y) = \emptyset$$

for  $j \gg 1$ . To this purpose we first observe that  $\text{spt}(\hat{Z}_j) \subset \lambda_j \bar{W}_0 \rightarrow \{0\}$  so that the assumption

$$0 = \mu_T(\{z \in \mathbf{R}^{m+k} - (\text{spt}(\partial T_0) \cup \lambda_j \bar{W}_0): |H(T(z))| > 0\})$$

for all  $j \geq 1$  would imply

$$\mu_T(\{z \in \mathbf{R}^{m+k} - \text{spt}(\partial T_0): |H(T(z))| > 0\}) = 0,$$

contradicting our assumption. Hence there exists  $j_1 \geq 1$  with the property

$$0 < \mu_T(\{z \in \mathbf{R}^{m+k} - (\text{spt}(\partial T_0) \cup \lambda_j \bar{W}_0): |H(T(z))| > 0\})$$

for all  $j \geq j_1$  and we may find an open set  $V \subset \mathbf{R}^{m+k} - \text{spt}(\partial T_0)$ ,  $V \cap \lambda_j \bar{W}_0 = \emptyset$  for  $j \geq j_1$  such that  $|H(T(z))| > 0$   $\mu_T$ -a.e. on  $V$ . Hence there exists a  $C^1$ -vectorfield  $Y^*$  supported in  $V$  with the property

$$\int_M Y^* \cdot H(T) d\mu_T > 0$$

and we may replace  $Y$  by  $Y^*$  if necessary. This implies that (13) is satisfied.

Finally, the remaining assertions from Theorem 5.1, i.e. that  $C$  is an oriented cone and that  $\mu_{T_j} \rightarrow \mu_C$  in the sense of Radon measures, follows as in [S, Proof of 34.5, 19.3]. This completes the proof of Theorem 5.1.  $\square$

**PROPOSITION 5.1.** *Suppose that the hypothesis of Theorem 3.2 hold and let  $T$  denote a solution of*

$$\mathbf{M}(\cdot) \rightsquigarrow \text{Min} \quad \text{in } \mathcal{C}$$

*introduced in Theorem 3.2. Moreover, assume  $\partial T_0 = \theta_0 \llbracket \Sigma \rrbracket$  for some  $\theta_0 \in \mathbf{N}$  and a compact, oriented  $(m-1)$ -dimensional submanifold*



$\Sigma$  of  $\mathbb{R}^{m+k}$ . Then we have

$$(14) \quad \mu_T(\{z \in \mathbb{R}^{m+k} - \text{spt}(\partial T_0) : |H(T(z))| > 0\}) > 0$$

if one of the following conditions holds:

(i)  $T_0(H^i) = (\partial T_0)(\omega^i) \neq 0$  for at least one  $i \in \{1, \dots, m+k\}$ , where  $\omega^i$  is defined by

$$\omega_x^i(v_1 \wedge \dots \wedge v_{m-1}) := \frac{1}{m} H^i(x \wedge v_1 \wedge \dots \wedge v_{m-1}).$$

(ii)  $0 \times \partial T_0 \notin \mathcal{C}$ , i.e.

$$c \neq \mathbf{V}_h(0 \times \partial T_0, T_0) = \frac{-1}{m+1} \int_{\mathbb{R}^{m+k}} x \cdot H(T_0(x)) d\mu_{T_0}.$$

**REMARK.** Condition (ii) says that  $T_0$  and the cone  $0 \times \partial T_0$  over  $\partial T_0$  enclose an  $h$ -volume different from  $c$ .

**PROOF.** (i) From  $d\omega^i = H^i$  we infer

$$T(H^i) = \partial T(\omega^i) = (\partial T_0)(\omega^i) = T_0(H^i),$$

hence

$$0 \neq T(H^i) = \int_{\mathbb{R}^{m+k}} H^i(T) d\mu_T = \int_{\mathbb{R}^{m+k} - \text{spt}(\partial T_0)} H^i(T) d\mu_T$$

where the last identity follows from the fact (remember that  $T = \underline{\tau}(M, \Theta, T)$ ,  $\mu_T = \mathcal{I}C^m \llcorner \Theta$ )

$$\mu_T(\text{spt}(\partial T_0)) = (\mathcal{I}C^m \llcorner \Theta)(\Sigma) = 0.$$

This gives

$$\int_{\mathbb{R}^{m+k} - \text{spt}(\partial T_0)} |H^i(T)| d\mu_T > 0$$

which clearly proves (14).

(ii) Using

$$\partial\{0 \times (T - T_0)\} = T - T_0$$

we first observe

$$(0 \times T)(h) = c + (0 \times T_0)(h).$$

Since

$$0 \ast S(h) = \int_0^1 \int_{\mathbb{R}^{m+k}} t^m \langle h, x \wedge S \rangle d\mu_S = \frac{1}{m+1} \int_{\mathbb{R}^{m+k}} x \cdot H(S) d\mu_S$$

for any  $S \in \mathbf{I}_m$  and

$$\mathbf{V}_h(0 \ast \partial T_0, T_0) = -(0 \ast T_0)(h)$$

our assumption (ii) implies

$$0 \neq (0 \ast T)(h) = \frac{1}{m+1} \int_{\mathbb{R}^{m+k}} x \cdot H(T) d\mu_T = \frac{1}{m+1} \int_{\mathbb{R}^{m+k} - \text{spt}(\partial T_0)} x \cdot H(T) d\mu_T$$

from which (14) easily follows.  $\square$

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Manoscritto pervenuto in redazione il 9 maggio 1990.