E. Barozzi

M. Emmer

E. H. A. González

Lagrange multipliers and variational methods for equilibrium problems of fluids

<http://www.numdam.org/item?id=RSMUP_1991__85__35_0>

Lagrange Multipliers and Variational Methods for Equilibrium Problems of Fluids.

E. BAROZZI - M. EMMER - E.H.A. GONZÁLEZ (*)

1. Introduction.

The equilibrium configurations of fluid masses have been studied in the last years in order to solve several problems related to surface tension phenomena. The point of view of the Calculus of Variations originates from the energy-minimizing character of the observed equilibrium configurations. By using a well known argument based on the principle of virtual work, one is lead to a variational formulation of the physical problems considered. Here, a certain functional $\mathcal{F}$ representing the global energy of the system under consideration has to be minimized, subject to some natural constraints, tipically concerning mass, angular momentum, center of mass and so on.

In general, the energy functional $\mathcal{F}$ consists of a *surface integral* and a *volume integral*, the first corresponding, e.g., to the forces acting on the interfaces between liquid and gas, while the second corresponds to body actions, such as gravity and kinetic forces:

$$(1.1) \qquad \mathcal{F}(E) = |\partial E|(V) + \int_E H(x)\,dx$$

Boundary conditions of Dirichlet, Neumann or mixed type can be asso-

(*) Indirizzo degli AA.: Dipartimento di Matematica, Università, 38050 Povo (Trento) – Ist. di Matematica e Fisica, via Vienna 2, 07100 Sassari – Dip. Metodi e Modelli Matematici per le Scienze Applicate, via Belzoni 7, 39100 Padova.

ciated with such a functional (see [2], [3]). Constraints like

$$(1.2) \qquad \int_E g_1 \, dx = V_1, \ldots, \int_E g_k \, dx = V_k,$$

$g_1, \ldots, g_k$ being given functions can also be considered. In this paper we address precisely this latter issue. Here E is a subset of $\mathbf{R}^n$, V is a fixed open subset of $\mathbf{R}^n$ (a container), $x = (y, z)$ with $y \in \mathbf{R}^{n-1}$ and $z \in \mathbf{R}$ denote an arbitrary point in $\mathbf{R}^n$, $|\partial E|(V)$ is the measure of the portion of the boundary of E contained in V, H is a given function. The classical definition of surface area is rather inadequate for treating all the types of problems we will discuss, mainly because it applies only to smooth or Lipschitz-continuous surfaces. The difficulties arising from the presence of a surface integral become even more evident when compared with the relatively simple treatment of the corresponding volume integral. This is generally well-defined on measurable sets and enjoys, in the simplest case, nice variational properties. Using the general class of surfaces of codimension 1 in $\mathbf{R}^n$ introduced by De Giorgi in the fifties [7], define the measure $|\partial E|(A)$ of the portion of the surface ∂E contained in the open set $A \subset \mathbf{R}^n$, as the only Radon measure in $\mathbf{R}^n$ such that

$$(1.3) \qquad |\partial E|(A) = \sup \left\{ \int_E \operatorname{div} \Phi dx, \ \ \Phi \in C^1(\mathbf{R}^n; \mathbf{R}^n), \ \ \|\Phi\|_\infty \leqslant 1 \right\}$$

for every open set $A \subset \mathbf{R}^n$. For an extensive treatment of the theory of Perimeter and functions of Bounded Variation (BV) see the volumes of Giusti [18] and of Massari and Miranda [30].

Let first recall some of the problems that have been studied through this approach.

A) *The sessile drop* for which the total energy of an incompressible fluid with constant density contained inside a fixed set V can be written (up to multiplicative constant) as

$$(1.4) \qquad \mathscr{F}(E) = |\partial E|(V) + \nu|\partial E|(\partial V) + \int_E z dx$$

The variational problem consists of minimizing the functional (1.4) in the family of all subsets of V which have a fixed volume V. This problem was studied by the first time by Gonzalez [19] in the case $V = \{(y, z) : z > 0\}$ (see also Gonzalez [20], Gonzalez and Tamanini [21]; a similar problem was studied by Giusti [17]). It is well known that for

$\nu > -1$ the sequences of admissible sets E_j such that $\mathcal{F}(E_j) \leqslant$ constant are compact in the $L^1_{loc}(\mathbf{R}^n)$-topology. Moreover, for $|\nu| \leqslant 1$, the functional $\mathcal{F}$ is lower semicontinuous (l.s.c.) in this topology. It is important to note that in most of the interesting interface problems like this the sets are restricted to have a prescribed volume. The main difficulty for proving existence of a minimum is that the $L^1_{loc}(\mathbf{R}^n)$-convergence does not preserve, in general, the volume constraint, i.e. measure of $E = |E| = V$.

B) *The pendent drop.* This problem was studied by Gonzalez, Massari and Tamanini[22]. They considered a drop of liquid hanging from the fixed horizontal reference plane $z = 0$. In this particular case, the container is $V = \{(y, z) : z < 0\}$, while the total energy is given by

$$(1.5) \qquad \mathcal{F}_g(E) = |\partial E|(V) + \nu|\partial E|(\partial V) + g \int_E z \, dx$$

g being a positive constant. As in the previous case, a volume constraint is imposed, namely $|E| = V$. For the sessile drop it was possible to prove existence of the minimum for the energy functional. For the pendent drop, however, the situation is different. Clearly in this case $\inf \mathcal{F}_g = -\infty$, so that we can only look for *local solutions*. E is a *local minimun* of the energy functional (1.5) if $|E| = V$, and there exists $T < 0$ and $\alpha \in (0, 1)$ such that $E \subset \{\alpha T < z < 0\}$ and, for every $F \subset \{T < z < 0\}$ with $|F| = V$ inequality $\mathcal{F}_g(E) \leqslant \mathcal{F}_g(F)$ holds. Existence of local minimum for sufficiently small $g > 0$ can be proved. A similar problem (following essentially the same approach) was considered by Giusti[16].

C) *The rotating drop.* Albano and Gonzalez[1] have studied the case of a liquid drop which rotates around a vertical axis z with constant angular velocity Ω, in the absence of gravity. In this situation the energy functional becomes

$$(1.6) \qquad \mathcal{F}_\Omega(E) = |\partial E|(\mathbf{R}^n) - \frac{1}{2}\Omega^2 \int_E |y|^2 \, dy dz$$

For the class of admissible sets, in addition to the volume condition $|E| = V$, a further condition is imposed on the center of mass of E. Note that no symmetry assumption for E is a priori requested. Also in this case existence of a local minimum for the energy functional for small $\Omega > 0$ can be proved. Here the definition of local minimum is the following: E is a *local minimum* of the energy functional (1.6) if there exists a sufficiently large R such that $E \subset\subset B_R$ (i.e. the closure $\overline{E}$ of E is a

compact subset of B_R) and, for every admissible $F \subset\subset B_R$ satisfying the above constraint, inequality $\mathcal{F}_\Omega(E) \leqslant \mathcal{F}_\Omega(F)$ holds. Here $B_R = = \{x \in \mathbf{R}^n : |x| < R\}$.

In this paper we present an approach to this type of problems based on the idea of eliminating the constraints by introducing a *Lagrange parameter*, λ, in the energy functional. The existence of Lagrange multipliers is not at all obvious; see the example presented by Barozzi and Gonzalez[3], where this kind of approach was introduced for the first time.

Our method works in all the problems we have considered. However, instead of applying the method to the general case, we shall present it by discussing the most significant problem. This is done primarily for a better understanding of all steps in the proof. In any case, the method can be applied in more general situations with very simple modifications.

For our purpose the most interesting example is the case of fluid masses rotating in space. Therefore this problem will be our model-problem. We shall first prove existence of the Lagrange multiplier (§ 2) and existence of local minima (§ 3). The regularity of the free boundary of a minimum value of the energy can finally be proved (§ 4).

2. Existence of Lagrange multipliers.

Let $E \subset \mathbf{R}^n$ be a measurable set. Denote with $|\partial E|(A)$ the perimeter of E in A, where A is an open set in $\mathbf{R}^n$, and by $x = (y, z)$, $y \in \mathbf{R}^{n-1}$, $z \in \mathbf{R}$, an arbitrary point in $\mathbf{R}^n$, $|E|$ standing for the Lebesgue measure of E. Consider the following functional

$$(2.1) \qquad \mathcal{F}_\Omega(E) = |\partial E|(\mathbf{R}^n) - \frac{1}{2}\Omega^2 \int_E |y|^2 \, dy\,dz$$

which represents the global energy of an incompressible fluid rotating around the z-axis with $\Omega \geqslant 0$ (the constant angular velocity). Here the first term represents the free surface energy, the second one the kinetic forces.

We study the energy functional (2.1) in the class of admissible sets E under the constraints

$$(2.2) \qquad \begin{cases} |E| = V \\ \int_E x_i \, dx = 0 \quad (i = 1, \ldots, n) \end{cases}$$

that is, among the sets E with prescribed volume and center of mass.

The energy functional being unbounded from below in such a class, $\inf \{\mathcal{F}_\Omega(E), \ E \text{ admissible}\} = -\infty$, we look for local minima for $\mathcal{F}_\Omega$ (see the introduction for the definition of a local minima). Denote by B_r the sphere centered in 0, of radius r, whose measure is V, by B_R a larger sphere also centered in 0, but with radius $R = 4r$. Set

$$(2.3) \qquad \mathcal{K}_\Omega(x) = \begin{cases} -\dfrac{1}{2}\Omega^2 |y|^2 & \text{inside } B_R \\ 0 & \text{outside } B_R, \end{cases}$$

$$(2.4) \qquad \mathcal{J}_\Omega(E) = |\partial E|(\mathbf{R}^n) + \int_E \mathcal{K}_\Omega \, dx.$$

REMARK 1.1. For every $E \subset B_R$ equality $\mathcal{F}_\Omega(E) = \mathcal{J}_\Omega(E)$ holds.

In order to introduce the Lagrange multipliers, we set

$$(2.5) \qquad \mathcal{L}_\lambda(E) = \lambda \left[\left| |E| - V \right| + \sum_{i=1}^{n} \left| \int_E x_i \, dx \right| \right]$$

We first prove the following

THEOREM 2.1. The functional

$$(2.6) \qquad \mathcal{J}_\Omega(E) + \mathcal{L}_\lambda(E)$$

has a (free) minimum $E_{\lambda,\Omega}$ for every $\lambda \geq 0$ and $\Omega \geq 0$. Moreover $E_{\lambda,0} = B_r$ for every $\lambda > \lambda_1$, where

$$\lambda_1 = \frac{|\partial B_r|(\mathbf{R}^n)}{|B_r|(\mathbf{R}^n)} = \frac{n}{r}$$

PROOF. It is easy to prove that the functional is bounded from below. In fact

$$(2.7) \qquad \mathcal{J}_\Omega(E) + \mathcal{L}_\lambda(E) \geq -\frac{1}{2}\Omega^2 R^2 |B_R|$$

for every $E \subset \mathbf{R}^n$. The only difficulty depends on the second term $\mathcal{L}_\lambda(E)$ of the functional (2.6); $\mathcal{L}_\lambda(E)$ is continuous in $L^1(\mathbf{R}^n)$ but not even l.s.c. in the $L^1_{loc}(\mathbf{R}^n)$-topology. However, the existence result can be proved by simple modifications of the method used by Barozzi for the Plateau problem in unbounded domains [2].

The second statement easily follows from the isoperimetric property of the sphere. ■

REMARK 2.2. The following inequality holds:

$$(2.8) \qquad \mathscr{T}_\Omega(E_{\lambda,\Omega}) + \mathscr{L}_\lambda(E_{\lambda,\Omega}) \leqslant \mathscr{T}_\Omega(B_r) + \mathscr{L}_\lambda(B_r) = \mathscr{T}_\Omega(B_r) = c_0$$

Moreover

$$(2.9) \qquad \left| \int_{E_{\lambda,\Omega}} \mathscr{H}_\Omega \, dx \right| \leqslant c_1,$$

both c_0 and c_1 being constants independent of λ. From (2.8), (2.9) we can conclude that

$$(2.10) \qquad \mathscr{L}_\lambda(E_{\lambda,\Omega}) \leqslant c_2$$

where c_2 is a constant depending on Ω but not on λ. For $\Omega \leqslant \Omega_0$ in (2.10) we can choose a universal constant.

It follows finally that

$$(2.11) \qquad \left| |E_{\lambda,\Omega}| - V \right| \leqslant \frac{c_2}{\lambda}$$

$$(2.12) \qquad \sum_{i=1}^{n} \left| \int_{E_{\lambda,\Omega}} x_i \, dx \right| \leqslant \frac{c_2}{\lambda},$$

and therefore

$$(2.13) \qquad \lim_{\lambda \to +\infty} |E_{\lambda,\Omega}| = V$$

$$(2.14) \qquad \lim_{\lambda \to +\infty} \int_{E_{\lambda,\Omega}} x_i \, dx = 0$$

Our aim is now to prove that there exists λ_2 such that

$$(2.15) \qquad |E_{\lambda,\Omega}| = V \qquad \forall \lambda \geqslant \lambda_2,$$

and

$$(2.16) \qquad \int_{E_{\lambda,\Omega}} x_i \, dx = 0 \qquad \forall \lambda \geqslant \lambda_2,$$

i.e.

$$(2.17) \qquad \mathscr{L}_\lambda(E_{\lambda,\Omega}) = 0 \qquad \forall \lambda \geqslant \lambda_2.$$

We prove (2.17) by contradiction. Suppose that $\mathscr{L}_\lambda(E_{\lambda,\Omega}) > 0$. Then $E_{\lambda,\Omega}$

can be deformed into another set, $G_{\lambda,\Omega}$, in such a way that

(2.18) $\mathcal{J}_\Omega(G_{\lambda,\Omega}) + \mathcal{L}_\lambda(G_{\lambda,\Omega}) < \mathcal{J}_\Omega(E_{\lambda,\Omega}) + \mathcal{L}_\lambda(E_{\lambda,\Omega})$,

(2.19) $\mathcal{J}_\Omega(G_{\lambda,\Omega}) - \mathcal{J}_\Omega(E_{\lambda,\Omega}) < \mathcal{L}_\lambda(E_{\lambda,\Omega}) - \mathcal{L}_\lambda(G_{\lambda,\Omega})$.

Loosely speaking, the idea is to deform $E_{\lambda,\Omega}$, decreasing the penalty term $\mathcal{L}_\lambda$ without altering too much $\mathcal{J}_\Omega$.

The main step is to prove the existence of a constant (independent of λ and Ω) and of a family of sets, $G_{\lambda,\Omega}$, satisfying the constraints and such that

(2.20) $|\partial G_{\lambda,\Omega}|(\mathbf{R}^n) - |\partial E_{\lambda,\Omega}|(\mathbf{R}^n) \leqslant c \cdot v$,

where

(2.21) $v = \big||E_{\lambda,\Omega}| - V\big|$.

To this aim prove first the following

LEMMA 2.1 (of deformation). For $\lambda \geqslant 0$ and $\Omega > 0$, there exist two constants p, V_0, $p < +\infty$, $V_0 > 0$, and a family of sets $N_{\lambda,\Omega} \subset E_{\lambda,\Omega}$ such that $\forall E \subset N_{\lambda,\Omega}$:

a) $|\partial N_{\lambda,\Omega}|(\mathbf{R}^n) \leqslant |\partial E|(\mathbf{R}^n) + \dfrac{p}{V_0}|N_{\lambda,\Omega} - E|$,

b) $|N_{\lambda,\Omega}| \geqslant \theta(p, V_0) > 0$,

where $\theta(p, V_0) = \left(\dfrac{1}{c(n)}\right)^n \left(\dfrac{V_0}{p}\right)^n$ and $c(n) = \dfrac{1}{n(\omega(n))^{1/n}}$ (see also [4], [33]).

PROOF. Let $V_0 > 0$, $p < +\infty$ such that, $\forall \lambda \geqslant 0$, $\forall \Omega > 0$

$$|E_{\lambda,\Omega}| \geqslant V_0, \qquad |\partial E_{\lambda,\Omega}| \leqslant p.$$

The existence of V_0 and p easily follows from the minimum property of $E_{\lambda,\Omega}$ and from the isoperimetric property of the sphere. To prove the existence of such a V_0, note that

(2.22) $n\omega(n) r^{n-1} + \displaystyle\int\limits_{E_{\lambda,\Omega}} \mathcal{H}_\Omega\, dx \leqslant \mathcal{J}_\Omega(E_{\lambda,\Omega}) + \mathcal{L}_\lambda(E_{\lambda,\Omega}) \leqslant$

$$\leqslant n\omega(n) r^{n-1} - \frac{1}{2}\Omega^2 \int\limits_{B_r} |y|^2\, dy dz$$

So

$$(2.23) \qquad \int_{E_{\lambda,\Omega}} \mathcal{H}_\Omega \, dx \leqslant -\frac{1}{2}\Omega^2 \int_{B_r} |y|^2 \, dy \, dz$$

Moreover, as

$$(2.24) \qquad -\frac{1}{2}\Omega^2 R^2 |E_{\lambda,\Omega} \cap B_R| \leqslant \int_{E_{\lambda,\Omega}} \mathcal{H}_\Omega \, dx \,,$$

we obtain, for $\forall \Omega > 0$, $\forall \lambda \geqslant 0$,

$$(2.25) \qquad |E_{\lambda,\Omega} \cap B_R| \geqslant \frac{\displaystyle\int_{B_R} |y|^2 \, dy \, dz}{R^2}$$

and finally

$$(2.26) \qquad |E_{\lambda,\Omega}| \geqslant V_0 \,.$$

The existence of p easily follows from the inequality

$$(2.27) \qquad \mathcal{T}_\Omega(E_{\lambda,\Omega}) + \mathcal{L}_\lambda(E_{\lambda,\Omega}) \leqslant \mathcal{T}_\Omega(B_r) + \mathcal{L}_\lambda(B_r) = \mathcal{T}_\Omega(B_r) \,.$$

Let now $N_{\lambda,\Omega}$ be a minimum of the functional

$$(2.28) \qquad \mathcal{G}(E) = |\partial E|(\mathbf{R}^n) - \frac{p}{V_0} |E|$$

with $E \subset E_{\lambda,\Omega}$.

Property $a)$ is an immediate consequence of the property of minimum of the set $N_{\lambda,\Omega}$. To prove $b)$, we note that, as $\mathcal{G}(N_{\lambda,\Omega}) \leqslant \mathcal{G}(\phi)$, it follows

$$(2.29) \qquad |\partial N_{\lambda,\Omega}|(\mathbf{R}^n) \leqslant \frac{p}{V_0} |N_{\lambda,\Omega}| < +\infty \,.$$

Using the isoperimetric property of the sphere, we then obtain

$$(2.30) \qquad |N_{\lambda,\Omega}|^{\frac{n-1}{n}} \leqslant c(n) \frac{p}{V_0} |N_{\lambda,\Omega}| \,,$$

i.e.

$$(2.31) \qquad |N_{\lambda,\Omega}| \geqslant \left(\frac{1}{c(n)}\right)^n \left(\frac{p}{V_0}\right)^n \,. \qquad \blacksquare$$

REMARK 2.3. It is easily seen that all sets $N_{\lambda,\varrho}$ posses a (generalized) mean curvature (Barozzi, Gonzalez and Tamanini[6]), bounded from above by the constant $\dfrac{p}{(n-1)V_0}$.

THEOREM 2.2. For Ω_0 there exists $\lambda_2 = \lambda_2(\Omega_0)$ such that

(2.32) $\qquad\qquad \mathcal{L}_\lambda(E_{\lambda,\varrho}) = 0 \quad \forall \lambda \geqslant \lambda_2, \ 0 \leqslant \Omega \leqslant \Omega_0 .$

Indeed, it suffices to take

$$\lambda_2 > \frac{p}{V_0} + \frac{1}{2}\Omega_0^2 + 1$$

PROOF: Let $v = \big||E_{\lambda,\varrho}| - V\big|$, $b = \sum\limits_{i=1}^{n} \left| \int\limits_{E_{\lambda,\varrho}} x_i\, dx \right|$. Suppose that $v + b > 0$.

Consider first the case $v > 0$.

If $V - |E_{\lambda,\varrho}| = v$ (that is if $E_{\lambda,\varrho}$ is «too small»), define $F_{\lambda,\varrho}$ as the union of $E_{\lambda,\varrho}$ with a convenient translation of $N_{\lambda,\varrho}$ in such a way that

(2.33) $\qquad\qquad\qquad V - |F_{\lambda,\varrho}| = \sigma$

(a suitable value for σ, $0 \leqslant \sigma < v$, will be chosen later).

If, instead, $|E_{\lambda,\varrho}| - V = v$ (that is if $E_{\lambda,\varrho}$ is «too big»), define $F_{\lambda,\varrho}$ as the difference between $E_{\lambda,\varrho}$ and a convenient translation of $N_{\lambda,\varrho}$, in such a way that we get again relation (2.33). Let now, $G_{\lambda,\varrho}^\sigma$ be the union of $F_{\lambda,\varrho}$ with two small spheres B_1 and B_2, with $|B_1| > 0$, $|B_2| > 0$, $B_1 \cap B_2 = \phi$, with the following properties:

$$|G_{\lambda,\varrho}^\sigma| = V, \quad \int\limits_{G_{\lambda,\varrho}^\sigma} x_i\, dx = 0 \quad (i = 1, ..., n).$$

(Note that it is possible to take B_1 and B_2 arbitrarily far away from the origin). Clearly $\mathcal{L}_\lambda(G_{\lambda,\varrho}^\sigma) = 0$ and, moreover,

(2.34) $\quad \mathcal{T}_\Omega(E_{\lambda,\varrho}) + \mathcal{L}_\lambda(E_{\lambda,\varrho}) - \mathcal{T}_\Omega(G_{\lambda,\varrho}^\sigma) \geqslant$

$$\geqslant -\frac{p}{V_0}(v - \sigma) - n(\omega(n))^{-\frac{1}{n}} 2^{\frac{1}{n}} \sigma^{\frac{n-1}{n}} - \frac{1}{2}\Omega^2 R^2(v - \sigma) + \lambda v + \lambda b$$

Choosing σ in such a way that $n(\omega(n))^{-\frac{1}{n}} 2^{\frac{1}{n}} \sigma^{\frac{n-1}{n}} < v$, we finally

obtain

(2.35) $\mathcal{T}_\Omega(E_{\lambda,\Omega}) + \mathcal{L}_\lambda(E_{\lambda,\Omega}) - \mathcal{T}_\Omega(G^\tau_{\lambda,\Omega}) \geqslant$

$$\geqslant \left(-\frac{p}{V_0} - \frac{1}{2}\Omega^2 R^2\right)v - v + \lambda v + \lambda b = \left(-\frac{p}{V_0} - \frac{1}{2}\Omega^2 R^2 - 1 + \lambda\right)v + \lambda b > 0.$$

Now, (2.35) is incompatible with the property of minimum of the set $E_{\lambda,\Omega}$. The case $v = 0$ (i.e. $b > 0$) is even easier to handle.   ∎

3. Existence of local minima.

We begin by proving the following

THEOREM 3.1. For $\lambda > \lambda^* = \max\{\lambda_1, \lambda_2\}$, $E_{\lambda,\Omega}$ converges to B_r in the $L^1(\mathbf{R}^n)$-topology, when Ω goes to 0^+. Moreover convergence is uniform with respect to λ, for $\lambda > \lambda^*$.

PROOF. The proof of the first part follows immediately using the same technique as in [1]. In fact, if G_k minimizes the functional $\mathcal{T}_{\Omega_k}(E)$ with the constraints (2.2), then, if Ω_k goes to 0^+ as $k \to +\infty$, we have, up to a subsequence, $G_k \to B_r$ in $L^1(\mathbf{R}^n)$ as $k \to +\infty$. Suppose now that the convergence

$$E_{\lambda,\Omega} \to B_r \quad \text{as} \quad \Omega \to 0^+$$

is not uniform in λ. Then there should be an $\varepsilon > 0$ such that, $\forall k \in \mathbf{N}$, it would be possible to find λ_k, $\Omega_k \leqslant \dfrac{1}{k}$ with

$$|E_{\lambda_k,\Omega_k} - B_r| + |B_r - E_{\lambda_k\Omega_k}| \geqslant \varepsilon$$

Then it would be impossible to extract from the sequence $G_k = E_{\lambda_k\Omega_k}$ of minima for $\mathcal{T}_{\Omega_k}(E)$ a subsequence which converges to B_r. This contradicts the first part of the theorem. This completes the proof.   ∎

We have now to prove that, at least for Ω sufficiently small, the set $E_{\lambda,\Omega}$ is actually a local minimum for the functional $\mathcal{T}_\Omega$ (i.e. that $E_{\lambda,\Omega} \subset\subset B_R$). To this aim, it will be useful the following technical result of Real Analysis:

LEMMA 3.1. Let $f \geqslant 0$ be a measurable function in the interval

[A, B] and such that

(3.1)
$$\int_t^B f(\tau)\,d\tau \leqslant f^\alpha(t) \quad \forall t \in [A, B)$$

where α is a constant, $\alpha > 1$. If, moreover,

(3.2)
$$w = \int_A^B f(\tau)\,d\tau < \frac{B - A}{3},$$

then there exists $t_0 \in (A, B)$ such that

(3.3)
$$\int_{t_0}^B f(\tau)\,d\tau = 0.$$

PROOF. Let $L = B - A$. From (3.1), (3.2) we get $w \leqslant f^\alpha(A)$. Let $t_1 \in \left(A, A + \dfrac{L}{3}\right)$ such that $\dfrac{L}{3} f(t_1) \leqslant w$, that is $f(t_1) \leqslant \dfrac{3w}{L}$. It follows that

(3.4)
$$\int_{t_1}^B f(\tau)\,d\tau \leqslant f^\alpha(t_1) \leqslant \left(\frac{3}{L}\right)^\alpha \cdot w^\alpha$$

Let $t_2 \in \left(t_1, t_1 + \dfrac{L}{3^2}\right)$ such that

(3.5)
$$\frac{L}{3^2} f(t_2) \leqslant \int_{t_1}^B f(\tau)\,d\tau \leqslant \left(\frac{3}{L}\right)^\alpha w^\alpha,$$

that is $f(t_2) \leqslant \dfrac{3^2}{L}\left(\dfrac{3}{L}\right)^\alpha w^\alpha$. Then

(3.6)
$$\int_{t_2}^B f(\tau)\,d\tau \leqslant f^\alpha(t_2) \leqslant \left(\frac{3^2}{L}\right)^\alpha \left(\frac{3}{L}\right)^{\alpha^2} w^{\alpha^2} = \frac{3^{2\alpha + \alpha^2}}{L^{\alpha + \alpha^2}} w^{\alpha^2}$$

Let now $t_3 \in \left(t_2, t_2 + \dfrac{L}{3^3}\right)$ such that

(3.7)
$$\frac{L}{3^3} f(t_3) \leqslant \int_{t_2}^B f(\tau)\,d\tau \leqslant \frac{3^{2\alpha + \alpha^2}}{L^{\alpha + \alpha^2}} w^{\alpha^2},$$

that is

$$(3.8) \qquad f(t_3) \leqslant \frac{3^{3+2\alpha+\alpha^2}}{L^{1+\alpha+\alpha^2}} \, w^{\alpha^2}.$$

Therefore,

$$(3.9) \qquad \int_{t_3}^{B} f(\tau)\,d\tau \leqslant f^\alpha(t_3) \leqslant \frac{3^{3+2\alpha^2+\alpha^3}}{L^{\alpha+\alpha^2+\alpha^3}} \, w^{\alpha^3},$$

and in general

$$(3.10) \qquad \int_{t_k}^{B} f(\tau)\,d\tau \leqslant \frac{3^{k\alpha+(k-1)\alpha^2+\ldots+2\alpha^{k-1}+\alpha^k}}{L^{\alpha+\alpha^2+\ldots+\alpha^k}} \, w^{\alpha^k}.$$

Now, if

$$(3.11) \qquad S_k = k\alpha + (k-1)\alpha^2 + \ldots + 2\alpha^{k-1} + \alpha^k,$$

it follows that

$$(3.12) \ S_k(\alpha-1) = \alpha^{k+1} + \alpha^k + \alpha^{k-1} + \ldots + \alpha^2 - k\alpha = \alpha\,\frac{\alpha^{k+1}-1}{\alpha-1} - k\alpha,$$

and finally

$$(3.13) \qquad S_k = \frac{\alpha^{k+2}-\alpha}{(\alpha-1)^2} - \frac{k\alpha}{\alpha-1}.$$

From (3.13) we have

$$(3.14) \qquad \int_{t_k}^{B} f(\tau)\,d\tau \leqslant \frac{3^{S_k}}{L^{(\alpha^{k+1}-1)/\alpha-1}} \, w^{\alpha^k}.$$

Taking $\dfrac{3w}{L} < 1$, (3.14) yields

$$(3.15) \qquad \lim_{k\to+\infty} \int_{t_k}^{B} f(\tau)\,d\tau = 0.$$

Thus, it suffices to take

$$(3.16) \quad t_0 = \lim_{k \to +\infty} t_k \leqslant A + \frac{L}{3} + \frac{L}{3^2} + \frac{L}{3^3} + \ldots =$$

$$= A + L \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k = A + \frac{L}{2}$$

in order to obtain

$$(3.17) \quad \int_{t_0}^{B} f(\tau)\, d\tau = \lim_{k \to +\infty} \int_{t_k}^{B} f(\tau)\, d\tau = 0.$$

∎

Using Lemma 3.1, we can prove the existence of local minima:

THEOREM 3.2. There exists $\Omega_0 > 0$ such that, for $\lambda > \lambda^* = \lambda^*(\Omega_0)$, the set $E_{\lambda,\Omega}$ is a rotating drop (i.e. $E_{\lambda,\Omega} \subset\subset B_R$ and $\mathcal{L}_\lambda(E_{\lambda,\Omega}) = 0$) $\forall \Omega \leqslant \Omega_0$.

PROOF. We know from Theorem 2.2 that $\mathcal{L}_\lambda(E_{\lambda,\Omega}) = 0$. So the only thing to prove is that $E_{\lambda,\Omega} \subset\subset B_R$. Let

$$v = v(\lambda,\Omega) = |E_{\lambda,\Omega} - B_R|$$

From Theorem 3.1 the convergence of v to 0 as Ω goes to 0^+ is uniform with respect to λ, for $\lambda > \lambda^*$.

For brevity in the following we write E instead of $E_{\lambda,\Omega}$.

Define, for $r \leqslant t \leqslant R$,

$$f(t) = |\partial(E - B_t)|(\partial B_t),$$

i.e. f is the «external trace» of $E_{\lambda,\Omega}$ on ∂B_t; see for example Giusti[18], Massari and Miranda[30].

Let $1 < \alpha < \dfrac{n}{n-1}$; if

$$(3.18) \quad \int_{t}^{R} f(\tau)\, d\tau \leqslant f^\alpha(t) \quad \forall t \in [r, R)$$

and if we choose $v < \dfrac{R-r}{3}$, then from Lemma 3.1 we obtain the existence of $t_0 \in (r, R)$ such that $\int_{t_0}^{R} f(\tau)\, d\tau = 0$, that is $|E - B_{t_0}| = 0$ and hence $E \subset B_{t_0} \subset\subset B_R$.

Suppose on the contrary the existence of $t_1 \in [r, R)$ such that

(3.19)
$$\int_{t_1}^{R} f(\tau)\, d\tau > f^\alpha(t_1),$$

that is

(3.20)
$$|E - B_{t_1}|^\gamma > f(t_1)$$

where $\gamma = \dfrac{1}{\alpha} \in \left(\dfrac{n-1}{n}, 1\right)$. From (3.20) and the isoperimetric inequality we get

(3.21)
$$|E - B_{t_1}|^{\frac{n-1}{n}} < c(n)[|E - B_{t_1}|^\gamma + |\partial E|(\mathbf{R}^n - \overline{B}_{t_1})]$$

and therefore

(3.22)
$$|\partial E|(\mathbf{R}^n - \overline{B}_{t_1}) - f(t_1) > \frac{1}{c(n)} |E - B_{t_1}|^{\frac{n-1}{n}} - 2|E - B_{t_1}|^\gamma.$$

From (3.22) we obtain that, if $|E - B_{t_1}|$ is small but strictly positive, we should have

(3.23)
$$|\partial E|(\mathbf{R}^n - \overline{B}_{t_1}) - f(t_1) > |E - B_{t_1}|^\gamma.$$

For the sake of brevity put $\varepsilon = |E - B_{t_1}|$. If $\varepsilon > 0$ we could substitute $E = E_{\lambda, \Omega}$ with the set $F = F_{\lambda, \Omega}$ obtained by a suitable homothety of $E \cap B_{t_1}$ so that

(3.24)
$$0 < V - |F| = \sigma < \varepsilon.$$

For the ratio $\mu > 1$ of the homothety the relation $|F| = \mu^n |E \cap B_{t_1}|$ holds, i.e. $V - \sigma = \mu^n (V - \varepsilon)$, from which

(3.25)
$$\mu = \left(1 + \frac{\varepsilon - \sigma}{V - \varepsilon}\right)^{\frac{1}{n}}.$$

Choosing $G = G^\sigma_{\lambda, \Omega}$ as in Theorem 2.2 we have $\mathcal{L}_\lambda(G) = 0$ and

(3.26)
$$\mathcal{T}_\Omega(E) - \mathcal{T}_\Omega(G) \geq |\partial(E \cap B_{t_1})|(\mathbf{R}^n) + |\partial E|(\mathbf{R}^n - \overline{B}_{t_1}) - f(t_1) -$$
$$- |\partial F|(\mathbf{R}^n) - n(\omega(n))^{-\frac{1}{n}} 2^{\frac{1}{n}} \sigma^{\frac{n-1}{n}}.$$

From (3.23), (3.25) and the equality $|\partial F|(\mathbf{R}^n) = \mu^{n-1} |\partial(E \cap B_{t_1})|(\mathbf{R}^n)$ we

obtain therefore

$$(3.27) \quad \mathcal{J}_\Omega(E) - \mathcal{J}_\Omega(G) > \left[1 - \left(1 + \frac{\varepsilon - \sigma}{V - \varepsilon}\right)^{\frac{n-1}{n}}\right] |\partial(E \cap B_{t_1})|(\mathbf{R}^n) +$$

$$+ \varepsilon^\gamma - n(\omega(n))^{-\frac{1}{n}} 2^{\frac{1}{n}} \sigma^{\frac{n-1}{n}}$$

By expanding in Taylor's series the coefficient of $|\partial(E \cap B_{t_1})|(\mathbf{R}^n)$, i.e.

$$1 - \left(1 + \frac{\varepsilon - \sigma}{V - \varepsilon}\right)^{\frac{n-1}{n}} = -\frac{n-1}{n} \frac{\varepsilon - \sigma}{V - \varepsilon} + \dots$$

we easily conclude from (3.27) that $\mathcal{J}_\Omega(E) - \mathcal{J}_\Omega(G) > 0$ for ε small but strictly positive and for a suitable choice of σ, $0 < \sigma < \varepsilon$. Since ε goes to 0 as Ω goes to 0^+ (uniformly with respect to λ, for $\lambda > \lambda^*$), it follows that it must be $\varepsilon = 0$ for small Ω. This concludes the proof. ∎

4. Regularity results.

In the previous sections we have seen that a local minimum for the functional $\mathcal{J}_\Omega(E)$ with $\Omega > 0$ and the constraints

$$(4.1) \quad \begin{cases} |E| = V \\ \int_E x_i \, dx = c_i, \quad i = 1, \dots, n, \end{cases}$$

where $V > 0$, $c = (c_1, \dots, c_n) \in \mathbf{R}^n$ are given, is a free minimum of the functional $\mathcal{J}_\Omega(E) + \mathcal{L}_\lambda(E)$ for $\lambda > \lambda^*$, where

$$(4.2) \quad \begin{cases} \lambda^* = \max\left\{\dfrac{n}{r}, \dfrac{p}{V_0} + \dfrac{1}{2}\Omega_0^2 R^2 + 1\right\} \\[2ex] n\omega(n) r^n = V, \quad V_0 = \dfrac{\displaystyle\int_{B_r} |y|^2 \, dy dz}{R^2} \end{cases}$$

(cfr. (2.25) and Theorem 2.2). Note that $V_0 = V_0(V)$ is a positive strictly increasing function of $V \in (0, \omega_n R^n)$. Also note that λ decreases when V increases and that λ goes to $+\infty$ as V goes to 0^+.

Due to the equivalence between the constrained problem and the penalty problem with the penalty term $\mathcal{L}_\lambda$, the regularity of the free

boundary ∂E of the minimum can be easily obtained. Moreover some regularity results for the minimum value of the energy functional can also be proved.

First we prove the regularity of the free boundary ∂E of a local minimum E for $\mathcal{J}_\Omega$ with the constraints (4.1). As we observed before, for $\lambda > \lambda^*$, the set E minimizes without any constraints the functional $\mathcal{J}_\Omega + \mathcal{L}_\lambda$.

Let $B_\rho(x) \subset\subset B_R(0)$, where $B_\rho(x)$ is a sphere centered in x and of radius ρ. Let L be a deformation inside $B_\rho(x)$ of E, that is

$$(4.3) \qquad L - B_p(x) = E - B_\rho(x) .$$

Then, from the trivial inequality

$$(4.4) \qquad \mathcal{J}_\Omega(E) + \mathcal{L}_\lambda(E) \leqslant \mathcal{J}_\Omega(L) + \mathcal{L}_\lambda(L) ,$$

we easily obtain

$$(4.5) \quad \left| |\partial E|(\mathbf{R}^n) - |\partial L|(\mathbf{R}^n) \right| \leqslant$$

$$\leqslant 2 \int_{B_\rho(x)} |\mathcal{H}_\Omega| dx + \lambda ||E| - |L|| + 2\lambda \sum_{i=1}^{n} \int_{B_\rho(x)} |x_i|\, dx \leqslant$$

$$\leqslant (\Omega^2 R^2 + \lambda + 2\lambda n R)\, \omega(n)\, \rho^n$$

i.e. there exists a constant M such that

$$(4.6) \qquad \left| |\partial E|(\mathbf{R}^n) - |\partial L|(\mathbf{R}^n) \right| \leqslant M\rho^n$$

holds for every set L satisfying (4.3). It is known (Massari[29], Tamanini[32]) that from an inequality of type (4.6) follows that ∂E is a $C^{1,\alpha}$-hypersurface, except, possibly, for a closed singular set whose Hausdorff dimension does not exceed $n - 8$.

Finally, we study a property of the minimum value for the energy functional $\mathcal{J}_\Omega$ (see also Barozzi and González[2] and Gurtin[25]). Let $\Psi(V, c, \Omega)$ be the minimum value of the functional $\mathcal{J}_\Omega$ with constant angular velocity Ω, with positive volume V, and center of mass c. Let δ and k be two positive constants. As we pointed out before, there exists $\lambda > 0$ such that for $V > \delta$ and $\Omega \in (0, k)$ the problem of minimizing $\mathcal{J}_\Omega$ under the constraints (4.1) is equivalent the problem for a free minimum for the functional $\mathcal{J}_\Omega + \mathcal{L}_\lambda$.

In order to study the regularity of the function $\Psi(V, c, \Omega)$ consider V_j, Ω_j, c_j ($j = 1, 2$) with

$$V_j > \delta, \qquad \Omega_j \in (0, k), \qquad c_j \in \mathbf{R}^n, \qquad j = 1, 2 .$$

Let E_1, E_2 be sets with measures V_1, V_2 and centers of mass in c_1, c_2 respectively, and such that

(4.7) $$\psi(V_j, c_j, \Omega_j) = \mathcal{J}_{\Omega_j}(E_j), \qquad j = 1, 2.$$

For λ as asserted above, we have

$$\psi(V_1, c_1, \Omega_1) = \mathcal{J}_{\Omega_1}(E_1) + \lambda\left(\left|\,|E_1| - V_1\right| + \sum_{i=1}^{n}\left|\int_{E_1}(x_i - c_{1_i})\,dx\right|\right) \leq$$

$$\leq \mathcal{J}_{\Omega_1}(E_2) + \lambda\left(\left|\,|E_2| - V_1\right| + \sum_{i=1}^{n}\left|\int_{E_2}(x_i - c_{1_i})\,dx\right|\right) =$$

$$= \psi(V_2, c_2, \Omega_2) + \int_{E_1}\mathcal{H}_{\Omega_1}\,dx - \int_{E_2}\mathcal{H}_{\Omega_2}\,dx + \lambda(|\,|E_2| - V_1| - |\,|E_2| - V_2|) +$$

$$+ \lambda\sum_{i=1}^{n}\left(\left|\int_{E_2}(x_i - c_{1_i})\,dx\right| - \left|\int_{E_2}(x_i - c_{2_i})\,dx\right|\right) \leq$$

$$\leq \psi(V_2, c_2, \Omega_2) + |\Omega_1 - \Omega_2|\frac{R^2}{2}|B_R| + \lambda|V_1 - V_2| + \lambda|B_R|\sum_{i=1}^{n}|c_{1_i} - c_{2_i}|$$

i.e.

(4.8) $$|\psi(V_1, c_1, \Omega_1) - \psi(V_2, c_2, \Omega_2)| \leq$$

$$\leq \frac{R^2}{2}|B_R|\,|\Omega_1 - \Omega_2| + \lambda|V_1 - V_2| + \lambda|B_R|\sum_{i=1}^{n}|c_{1_i} - c_{2_i}|$$

Thus for $\delta > 0$ and $k < +\infty$ ψ is a Lipschitz continuous function in $(\delta, +\infty) \times \mathbf{R}^n \times (0, k)$. Obviously, the Lipschitz constant goes to $+\infty$ as δ goes to 0^+ or k goes to $+\infty$.

REFERENCES

[1] S. ALBANO - E. H. A. GONZALEZ, *Rotating Drops*, Indiana Univ. Math. J., 32 (1983), 687-701.

[2] E. BAROZZI, *Il problema di Plateau in domini illimitati*, Rend. Sem. Mat. Univ. Padova, 70 (1983), 89-98.

[3] E. BAROZZI - E. H. A. GONZALEZ, *Least Area Problems with a Volume Constraint*, in [23], 33-53.

[4] E. BAROZZI - E. H. A. GONZALEZ, *Sul valore minimo del funzionale del-*

l'area con vincolo di volume, Dep. of Math., University of Lecce, report n. 7 (1984).

[5] E. BAROZZI - E. H. A. GONZALEZ, *Mean Curvature and Variational Problems*, Rend. Sem. Mat. Univers. Politecn. Torino, Fascicolo Speciale (1989), P.D.E. and Geometry (1988), 131-142.

[6] E. BAROZZI - E. H. A. GONZALEZ - I. TAMANINI, *The Mean Curvature of a Set of Finite Perimeter*, Proceedings of the Amer. Math. Soc., **99**, n. 2 (February 1987), 313-316.

[7] F. COLOMBINI - E. DE GIORGI - L. PICCININI, *Frontiere orientate di misura minima e questioni collegate*, Editrice Tecnico Scientifica, Pisa, 1972.

[8] P. CONCUS - R. FINN (eds.), *Variational Methods for Free Surface Interfaces*, Springer-Verlag, Berlin, 1987.

[9] G. CONGEDO - M. EMMER - E. H. A. GONZALEZ, *Rotating Drops in a Vessel*, Rend. Sem. Mat. Univ. Padova, **70** (1983), 167-186.

[10] M. EMMER, *Esistenza, unicità e regolarità nelle superfici di equilibrio nei capillari*, Ann. Univ. Ferrara, **18** (1973), 79-94.

[11] M. EMMER - E. H. A. GONZALEZ - I. TAMANINI, *A Variational Approach to the Study of Capillarity Phenomena*, in Proccedings of the Second International Colloquium on Drops and Bubbles (D. H. Le Croisette Ed.), NASA & Jet Propulsion Laboratory, Pasadena, CA, 1982, 374-379.

[12] M. EMMER - E. H. A. GONZALEZ - I. TAMANINI, *Sets of Finite Perimeter and Capillarity Phenomena*, in Free Boundary Problems: Theory and Applications (A. Fasano & M. Primicerio Eds.), Pitman Research Notes in Mathematics, **78** (1983), 29-38.

[13] M. EMMER, *A Review of 'Equilibrium Capillary Surfaces' by R. Finn*, Bull. Amer. Math. Soc., **16** (april 1987), 325-331.

[14] R. FINN, *Equilibrium Capillary Surfaces*, Springer Verlag, Berlin, 1986.

[15] C. GERHARDT, *On the Capillarity Problem with Constant Volume*, Ann. Sc. Norm. Sup. Pisa, 2, serie IV (1975), 303-320.

[16] E. GIUSTI, *The pendent water drop*, Boll. U.M.I. A (5) 17 (1980), 458-465.

[17] E. GIUSTI, *The Equilibrium Configuration of Liquid Drops*, J. Reine Angew. Math., **321** (1981), 53-63.

[18] E. GIUSTI, *Minimal Surfaces and Functions of Bounded Variation*, Birkäuser, Boston, 1984.

[19] E. H. A. GONZALEZ, *Sul problema della goccia appoggiata*, Rend. Sem. Mat. Univ. Padova, **55** (1976), 289-302.

[20] E. H. A. GONZALEZ, *Sulla regolarità della goccia appoggiata*, Rend. Sem. Mat. Univ. Padova, **58** (1977), 25-33.

[21] E. H. A. GONZALEZ - I. TAMANINI, *Convessità della goccia appoggiata*, Rend. Sem. Mat. Univ. Padova, **58** (1977), 35-43.

[22] E. H. A. GONZALEZ - U. MASSARI - I. TAMANINI, *Existence and Regularity for the Problem of a Pendent Liquid Drop*, Pacific J. Math., **88** (1980), 399-420.

[23] E. H. A. GONZALEZ - I. TAMANINI (eds.), *Variational Methods for*

Equilibrium Problems of Fluids, Astérisque **118**, Société Mathématique de France (1984).

[24] M. GRUTER, *Boundary Regularity for Solutions of a Partitioning Problem*, Arch. Rat. Mech. Anal., 97 (1987), 261-270.

[25] M. E. GURTIN, *On a Theory of Phase Transitions with Interfacial Energy*, Arch. Rat. Mech. Anal., 87 (1985), 187-212.

[26] M. E. GURTIN, *On Phase Transitions with Bulk, Interfacial, and Boundary Energy*, Arch. Rat. Mech. Anal., **96** (1986), 243-264.

[27] U. HORNUNG - H. MITTELMANN, *The Augmented Skeleton Method for Parametrizated Surfaces of Liquid Drops*, (to appear).

[28] U. HORNUNG - H. MITTELMANN, *The Skeleton Method for Capillary Surfaces with Volume Constraints*, (to appear).

[29] U. MASSARI, *Esistenza e regolarità delle ipersuperfici di curvatura media assegnata in R^n*, Arch. Rat. Mech. Anal., 55 (1974), 357-382.

[30] U. MASSARI - M. MIRANDA, *Minimal Surfaces of Codimension One*, North-Holland, Amsterdam, 1984.

[31] M. MIRANDA - S. ROBERTS, *Lagrange Multipliers and Geometric Measure Theory*, Rend. Sem. Mat. Univ. Padova, 74 (1985), 219-224.

[32] I. TAMANINI, *Boundaries of Caccioppoli sets with Hölder-continuous normal vector*, J. Reine Angew. Math., 334 (1982), 28-39.

[33] I. TAMANINI, *Variational Problems of Least Area Type with Constraints*, Ann. Univ. Ferrara, 39 (1988), 183-217.