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Topologically Semiperfect Rings.

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Introduction.

This paper deals with a generalization of the concept of semiperfectness to linearly topologized rings. We define first what projective covers are and, when this definition has been given, we call «semiperfect» those linearly topologized rings such that all *finitely generated discrete* modules over them have projective covers.

Two lines of research are possible: one in the direction that considers projective covers as topological modules and the second one when we prescribe to all projective covers of finitely generated modules to be discrete.

As an example of the first kind of generalization we can consider the ring $\boldsymbol{Z}$ of integers endowed with its *natural topology* ν. Thus a finitely generated discrete module over $\boldsymbol{Z}_\nu$ is just a finite torsion group and so it can be decomposed as a direct sum of finite indecomposable p-groups. If now G is a finite (cyclic) indecomposable p-group, then there exists a continuous epimorphism $f: J_p \to G$, where J_p is the (compact) group of p-adic integers. This group has the following projectivity property: for any epimorphism $\alpha: M \to N$ of discrete torsion groups and any continuous morphism $\beta: J_p \to N$, there is a continuous morphism $\gamma: J_p \to M$ with $\alpha \circ \gamma = \beta$ (this could be proved easily by using Pontrjagin duality). Hence we can regard (J_p, f) as a «projective cover» of G, with respect to all discrete modules over $\boldsymbol{Z}_\nu$; if $G_1, \ldots, G_n$ are indecomposable torsion groups, then the «projective cover» of their direct sum is the (topological) product of the projective covers of the summands.

It is precisely this concept that we study: we take a right linearly topologized ring R_τ, the category Mod-R_τ of discrete modules over R_τ

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and, if $M \in \text{Mod-}R_\tau$, we define a *τ-projective cover* of M as a linearly topologized module P over R_τ with a continuous epimorphism from P onto M with inessential kernel (in a topological sense which we precise in Section 1) and we want P to have a «projectivity» property like the one described above, with respect to the category $\text{Mod-}R_\tau$.

It turns out that there exist rings which are not semiperfect, but admit «projective covers» of discrete finitely generated modules: an example was given above, while another one will be in Section 2. We call such rings *t-semiperfect*. Of course, semiperfect rings are t-semiperfect when endowed with the discrete topology. It is true also that t-semiperfectness is preserved under weakening the topology: if a ring is t-semiperfect, then it is t-semiperfect when endowed with any (right linear) topology coarser than the given one. It can also be shown that t-semiperfectness is preserved under factoring the ring modulo a two-sided ideal (giving the factor ring the quotient topology). On the other hand a discrete t-semiperfect ring is nothing else than a semiperfect ring, as is shown in Theorem 2.11.

A well-behaved class of t-semiperfect rings is the one consisting of all (right linearly topologized) rings such that the «projective covers» of discrete finitely generated modules are discrete; for such rings, which we call *d-semiperfect*, the theory is very similar to the classical theory of semiperfect rings: namely we can associate to a d-semiperfect ring a *basic ring* which is also d-semiperfect and is, modulo its topological Jacobson radical (defined in Section 1) a product of (possibly infinitely many) division rings. Moreover the basic ring is *similar* to the given ring in the sense that the categories of discrete right modules over the two rings are equivalent [4]. It follows also that the projective covers of discrete finitely generated modules over the basic ring are indeed projective as abstract modules.

Section 1 of the paper deals with the preliminaries, namely the definition of a topological radical of a linearly topologized module and of a topological analogous of the Jacobson radical of a ring. The concept of inessential submodule of a linearly topologized module is given, and some of the properties linking radicals and inessential submodules are proved. It should be noted that this theory is perfectly analogous to the classical theory (i.e. the discrete case) and that most of the classical results can be generalized.

In Section 2 we define the concepts of projective cover of a (discrete) module and of t-semiperfect ring. We prove next that a sufficient condition for a ring to be t-semiperfect is that all simple discrete modules over it have a projective cover. Thus the discrete finitely generated modules over a t-semiperfect ring are semisimple modulo the radical, a generalization of the well known fact holding for semiperfect rings. We end the section by giving an example of a t-semiperfect ring which is not semiperfect (though we have already given such an example, namely the ring of integers with the natural topology).

In Section 3 we study d-semiperfect rings and develop a theory of the basic ring: the main results are that the basic ring is similar to the given ring and that it is, modulo the radical, a product of division rings.

Finally, in Section 4 we prove that every linearly compact ring is t-semiperfect: the proof of this fact relies on a duality theorem by Menini and Orsatti [7]. This theorem clarifies the examples we give in the paper: in fact the rings we proved to be t-semiperfect (or their completion) are linearly compact.

The rings we consider are always associative with 1, and modules are unital. We shall deal only with linear topologies both on rings and on modules, that is topologies having a basis of neighbourhoods of zero (or, briefly, a local basis) consisting of right ideals or submodules respectively. We shall denote by $\mathcal{F}(M_R)$ the family of all open submodules of the topological module M_R. While topologies on rings are not assumed to be Hausdorff, all topologies on modules will: this means that in every topological module the intersection of all open submodules is zero. However non Hausdorff rings are not very important: we include them only to state the results in larger generality. Indeed, in Sections 3 and 4 we use rings that are not only Hausdorff, but also complete (it is clear that we cannot distinguish a topological ring from its Hausdorff completion only by the discrete modules over it).

The notation Mod-R_τ denotes the category of all discrete modules over the (right linearly) topologized ring R_τ, while LT-R_τ is the category of all Hausdorff linearly topologized modules over R_τ. Analogous notations are used for categories of left modules. We use the convention of writing module morphisms on the opposite side to the scalars.

1. Radicals.

1.1 DEFINITION. Let M_R be a linearly topologized module:

(1) the (Jacobson) *t-radical* of M_R is the intersection of all maximal open submodules of M_R and is denoted by $\mathrm{rad}_t(M_R)$;

(2) if X is a submodule of M_R, then X is said to be *inessential* if, for all open submodules V of M_R, $X + V = M$ implies that $V = M$; we shall denote this by $X \leqslant_{\text{iness}} M_R$.

1.2 REMARKS. (*a*) If M_R is discrete, the notions we have defined above coincide with the usual ones.

(*b*) If we denote by «rad» the classical radical, i.e. the intersection of all maximal submodules, it is plain that $\mathrm{rad}(M_R) \leq \mathrm{rad}_t(M_R)$, for every linearly topologized module M_R. Moreover any inessential submodule of an abstract module M_R is inessential, no matter what topology we endow M with. Sometimes the two notions of radical coincide (cf. Theorem 1.11).

(*c*) The definition of inessential submodule could have been given by using closed submodules instead of open ones, since every *proper* closed submodule is contained in a *proper* open submodule.

(*d*) Another definition of inessentiality could be as follows: a submodule X of a linearly topologized module M_R is inessential if and only if, for every submodule Y of M_R, $X + Y = M$ implies that Y is dense in M. Indeed a submodule Y of M_R is dense if and only if $Y + V = M$ for any open submodule V of M_R.

1.3 PROPOSITION. *If X is inessential in the linearly topologized module M_R, then the closure $\overline{X}$ of X in M is also inessential in M.*

PROOF. It is well known that $\overline{X}$ is the intersection of all submodules of M of the form $X + V$, as V runs through the family $\mathcal{F}(M)$ of all open submodules of M_R. ■

1.4 PROPOSITION. *The t-radical of the linearly topologized module M_R coincides with the sum of all inessential submodules of M_R.*

PROOF. Obviously, $X \leq_{\mathrm{iness}} M$ implies that X is contained in $\mathrm{rad}_t(M_R)$.

For the converse, let $x \in \mathrm{rad}_t(M_R)$ and let $V \in \mathcal{F}(M)$ be an open submodule of M with $xR + V = M$. If V is a proper submodule, then $x \notin V$, hence we can take a submodule W of M which is maximal with respect to containing V and not containing x. Then W is open (since it contains V) and is maximal in M: indeed, if $W' > W$, then $W' \ni x$ and so $W' \geq xR + W \geq xR + V = M$. But x was chosen in $\mathrm{rad}_t(M)$ and so $x \in W$, a contradiction. Hence V is not proper. ■

Let us recall that a linearly topologized module is said to be *topologically finitely generated* (t.f.g. for short) if, for any open submodule V of M, the factor module M/V is finitely generated [4].

1.5 PROPOSITION. *The t-radical of a t.f.g. module is inessential.*

PROOF. If a module is t.f.g., then each one of its proper open submodules is contained in a maximal (open) submodule. ■

1.6. DEFINITION. Let R_τ be a right linearly topologized ring: then $\mathrm{rad}_t(R_\tau)$ is a closed two-sided ideal of R_τ, since it is the intersection of the annihilators of all simple modules in Mod-R_τ. We shall use the special notation $J(R_\tau)$ to denote the t-radical of R_τ, whereas $J(R)$ will denote the usual Jacobson radical of R (i.e. the t-radical of R endowed with the discrete topology). If R_τ is an indiscrete ring, then $J(R_\tau) = R$.

We shall say that an element t of a right linearly topologized ring R_τ is *right t-invertible* if the right ideal rR is dense in R_τ. Note that every invertible element is also t-invertible; if r is discrete, the two notions coincide. On the other hand, the following conditions are equivalent: 1) R_τ is indiscrete; 2) 0 is t-invertible; 3) every element of R is t-invertible.

The following property of the t-radical is well known in the discrete case (e.g. [1, Theorem 15.3]).

1.7 PROPOSITION. *If R_τ is a right linearly topologized ring, then*

$$J(R_\tau) = \{x \in R \mid \forall r \in R,\ 1 - xr \text{ is right } t\text{-invertible}\}.$$

PROOF. If $x \notin J(R_\tau)$, then there exists an open maximal right ideal V of R_τ such that $x \notin V$; thus we can find $r \in R$ and $y \in V$ with $1 = xr + y$, so that $(1 - xr)R = yR \leqslant V$. Hence $1 - xr$ is not right t-invertible.

Conversely, let $x \in J(R_\tau)$ and let $r \in R$. Assume that $(1 - xr)R$ is not dense in R_τ; then there is an open maximal ideal V of R_τ with $(1 - xr)R \leqslant V$. But now $xr \in J(R_\tau)$ and so $xr \in V$; hence $1 = (1 - xr) + xr \in V$, a contradiction. ■

If a topological ring R_τ is both left and right linearly topologized, we shall call it *two-sided linearly topologized.* In this case we can define two t-radicals of R_τ: we shall prove that these two radicals are in fact equal (see Corollary 1.10).

1.8 LEMMA. *Let R_τ be a two-sided linearly topologized ring. Then R_τ has a local basis consisting of two-sided ideals.*

PROOF. Let V be an open right ideal of R_τ: then V contains an open left ideal W. It is easy to see now that $W \subseteq I = \mathrm{Ann}_R(R/V)$, and this is the largest two-sided ideal of R contained in V. Hence I is open. ■

The next proposition characterizes the (right) t-radical of a two-sided linearly topologized ring.

1.9 PROPOSITION. *Let R_τ be a two-sided linearly topologized ring; if I is a two-sided ideal of R, we denote by $\pi_I\colon R \to R/I$ the canonical projection. Consider the right t-radical $J(R_\tau)$ of R_τ. Then, for all $x \in R$, $x \in J(R_\tau)$ if and only if $\pi_I(x) \in J(R/I)$, for all open two-sided ideals I of R_τ.*

PROOF. If $x \in J(R_\tau)$ and I is an open two-sided ideal of R_τ, then, for every $r \in R$, one has $(1-xr)R + I = R$ and so $\pi_I(1-xr)\overline{R} = \overline{R}$, where $\overline{R} = R/I$ is the factor ring. Hence $\pi_I(1-xr)$ is right invertible in $\overline{R}$ and $\pi_I(x) \in J(R/I)$.

Conversely, let $x \notin J(R_\tau)$: there exist $r \in R$ and open right ideal V of R_τ such that $(1-xr)R + V \neq R$. If I is an open two-sided ideal of R_τ with $I \leqslant V$, it is $(1-xr)R + V \neq R$ and therefore $\pi_I(1-xr)\overline{R} \neq \overline{R}$, so that $\pi_I(x) \notin J(R/I)$. ■

1.10 COROLLARY. *If R_τ is a two-sided linearly topologized ring, then the intersection of all open maximal right ideals of R_τ coincides with the intersection of all open left maximal ideals.*

The characterization of the t-radical of a ring given in 1.7 enables us to prove easily the result that follows [5]. Recall that a right linearly compact ring is a right linearly topologized ring R_τ such that, for any downward directed family $(I_\lambda)_{\lambda \in \Lambda}$ of closed ideals of R_τ, the canonical morphism of R into the inverse limit in Mod-R $\varprojlim_{\lambda \in \Lambda} R/I_\lambda$ is surjective. If R_τ is linearly compact, then every continuous epimorphic image of R_τ is complete.

1.11 THEOREM. *If R_τ is a right linearly compact ring, then $J(R_\tau) = J(R)$.*

PROOF. It is sufficient to prove that $J(R_\tau) \subseteq J(R)$. If $r \in R$ is right t-invertible, then xR is dense in R. But the morphism $r \mapsto xr$ of R into xR is continuous and so xR is complete (in the relative topology), hence closed. Thus any right t-invertible element of R is right invertible and we are done. ■

We conclude the section with some technical results. The first one is the generalization of Nakayama's lemma (or, better, Azumaya's lemma).

We shall fix a right linearly topologized ring R_τ.

1.12 PROPOSITION. *Let M_R be a linearly topologized module over R_τ. Set $J = J(R_\tau)$. If M is t.f.g. and $MJ = M$, then $M = 0$.*

PROOF. First, let M be discrete, hence finitely generated. Choose a minimal set $F = \{x_1, \dots, x_n\}$ of generators of M_R. Since $M = MJ$, we have

$$x_n = \sum_{i=1}^{n} x_i r_i,$$

with $r_i \in J$ ($i = 1, \dots, n$). But now the right ideal $(1 - r_n)R$ is dense and $\mathrm{Ann}_R(x_n)$ is open in R_τ, so that $\mathrm{Ann}_R(x_n) + (1 + r_n)R = R$. Therefore we can write $1 = \alpha + (1 - r_n)\beta$, for some $\alpha \in \mathrm{Ann}_R(x_n)$ and $\beta \in R$. Hence

$$x_n = x_n(\alpha + (1 - r_n)\beta) = \sum_{i=1}^{n-1} x_i r_i \beta,$$

contradicting to the minimality of F.

Finally, let M be an arbitrary t.f.g. module. If V is an open submodule of M_R and we put $\overline{M} = M/V$, we have $\overline{M}J = (MJ + V)/V = \overline{M}$ and so $\overline{M} = 0$ and so $M = 0$. ∎

1.13 LEMMA. *Let $M, N \in$ LT-R_τ and let $f\colon M \to N$ be a continuous morphism. Let X and Y be submodules of M_R.*

(i) *If $X \leqslant Y$ and $Y \leqslant_{\text{iness}} M$, then $X \leqslant_{\text{iness}} M$.*

(ii) *If X and Y are inessential in M, then $X + Y \leqslant_{\text{iness}} M$.*

(iii) *If $X \leqslant_{\text{iness}} Y$, then $X \leqslant_{\text{iness}} M$.*

(iv) *If X is inessential in M, then $f(X) \leqslant_{\text{iness}} N$.*

(v) $f(\mathrm{rad}_t(M_R)) \leqslant \mathrm{rad}_t(N_R)$.

(vi) *If the module $M \times N$ is endowed with the product topology, then* $\mathrm{rad}_t(M \times N) = \mathrm{rad}_t(M) \times \mathrm{rad}_t(N)$.

PROOF. (i) and (ii) are obvious.

(iii) If V is an open submodule of M, it follows from $X + V = M$ that $Y = M \cap Y = (X + V) \cap Y = X + (V \cap Y)$ and so $V \cap Y = Y$. But then $X \leqslant Y \leqslant V$ and $V = M$.

(iv) Assume first that f is surjective and that V is an open submodule of N with $f(X)+V=N$; then, as it is easy to see, $X+f^{-1}(V)=$ $=M$, so that $f^{-1}(V)=M$ and $V=f(M)=N$. For the general case, we can factorize f through the image and apply (ii).

(v) Follows from (iv) and Proposition 1.4.

(vi) The inclusion $\mathrm{rad}_t(M\times N)\subseteq\mathrm{rad}_t(M)\times\mathrm{rad}_t(N)$ follows from (v). The reverse one also follows from (v), since the canonical embedding $M\to M\times N$ and $N\to M\times N$ yield $\mathrm{rad}_t(M)\subseteq\mathrm{rad}_t(M\times N)$ and $\mathrm{rad}_t(N)\subseteq\mathrm{rad}_t(M\times N)$. ∎

2. Projective covers

In all this section R_τ will denote a fixed right linearly topologized ring. We shall denote by $\mathcal{F}_\tau$ the filter of open right ideals of R_τ and put

$$\Gamma_R=\bigoplus_{I\in\mathcal{F}_\tau}R/I$$

with the discrete topology; Γ_R is a generator of the category Mod-R_τ.

2.1 Definition. Let $M\in$ Mod-R_τ be a discrete module; a topological module $P\in$ LT-R_τ is called *M-projective* if, for any epimorphism $f\colon M\to N$ with $N\in$ Mod-R_τ and any continuous morphism $g\colon P\to N$, there exists a continuous morphism $h\colon P\to M$ such that $gh=f$.

We shall say that $P\in$ LT-R_τ is *τ-projective* if it is M-projective for all $M\in$ Mod-R_τ. It is clear that the completion of a τ-projective module is also τ-projective.

2.2 Proposition. *Let $P\in$ LT-R_τ. Then P is τ-projective if and only if it is $\Gamma^{(X)}$-projective, for every set X.*

If P is t.f.g., then P is τ-projective if and only if it is R/I-projective, for every $I\in\mathcal{F}_\tau$.

Proof. Lemma 4.7 of [4] (which is a generalization of Proposition 16.12 of [1]) asserts that the class of all modules $M\in$ Mod-R_τ such that P is M-projective, is closed under epimorphic images and, when P is t.f.g., also under direct sums. ∎

2.3 Definition. Let $M\in$ Mod-R_τ; a *τ-projective cover* of M is a pair (P,p), where $P\in$ LT-R_τ is a τ-projective module and $p\colon P\to M$ is a continuous epimorphism, with inessential kernel.

It is not possible, in general, to prove the uniqueness (up to topological isomorphisms) of τ-projective covers; in fact, any dense submodule of a τ-projective cover, with the restriction morphism, is also a τ-projective cover. Nevertheless, under certain additional hypotheses, uniqueness holds. An example is given by the following.

2.4 PROPOSITION. *Let $M \in$ Mod-R_τ and let (P, p) and (Q, q) be τ-projective covers of M. If P is discrete, then also Q is discrete and there exists an isomorphism $f\colon Q \to P$ such that $pf = q$.*

PROOF. Since Q is τ-projective, there exists a continuous morphism $f\colon Q \to P$ with $pf = q$; then $\ker f$ is an open submodule of Q. If $\ker f \neq 0$, there is an open submodule V of Q properly contained in $\ker f$; moreover $Q/\ker f \cong P$ is τ-projective and so there exists a morphism $g\colon Q/\ker f \to Q/V$ such that πg is the identity on $Q/\ker f$ ($\pi\colon Q/V \to Q/\ker f$ is the canonical projection). If we put $\operatorname{Im} g = X/V$, with $X \geq V$, we have

$$\frac{Q}{V} = \ker \pi \oplus \operatorname{Im} g = \frac{\ker f}{V} \oplus \frac{X}{V}.$$

Therefore $X + \ker f = Q$ and, from $\ker q \leq_{\text{iness}} Q$, $\ker f \leq \ker q$ and $V \leq \ker f$, it follows that $X = Q$, a contradiction. Hence $\ker f = 0$ and Q is discrete. It is now trivial to verify that $P = \operatorname{Im} f + \ker p$, so that $\operatorname{Im} f = P$. ■

The following two results are technical, but useful in the rest of the section.

2.5 LEMMA. *If $M \in$ Mod-R_τ is finitely generated and (P, p) is a τ-projective cover of M, then P_R has a finitely generated dense submodule. In particular P_R is topologically finitely generated.*

PROOF. The claim follows easily from the fact that $\ker p$ is inessential in P and from Remark 1.2(*d*). ■

2.6 LEMMA. *Let (P, p) and (Q, q) be τ-projective covers of $M, N \in$ $\in$ Mod-R_τ respectively. Then $(P \times Q, (p, q))$ is a τ-projective cover of $M \oplus N$.*

PROOF. This is an immediate corollary of the facts that $P \times Q$ is τ-projective and that (p, q) is surjective and of Lemma 1.13. ■

Let us recall that a module $G \in$ LT-R_τ is called a *τ-generator* if, for every non zero morphism $f\colon M \to N$ in Mod-R_τ, there exists a continuous morphism $g: G \to M$ such that $fg \neq 0$ [4, 2.4(iv)]. We can define analogously the notion of a family of τ-generators: a family $(G_\lambda)_{\lambda \in \Lambda}$ of modules in LT-R_τ is called a *family of τ-generators* if, for every non zero morphism $f: M \to N$ *in* Mod-R_τ, there exist $\lambda \in \Lambda$ and a morphism $g: G_\lambda \to M$ with $fg \neq 0$.

2.7 LEMMA. *Let $(G_\lambda)_{\lambda \in \Lambda}$ be a family of modules in* LT-R_τ. *The following conditions are equivalent*:

(*a*) *$(G_\lambda)_{\lambda \in \Lambda}$ is a family of τ-generators*;

(*b*) *$G = \prod_{\lambda \in \Lambda} G_\lambda$ (with the product topology) is a τ-generator*;

(*c*) *for every open proper right ideals V and I of R_τ with $I \leq V$, there exist $\lambda \in \Lambda$ and a morphism $g: G_\lambda \to R/I$ such that $\operatorname{Im} g \not\leq V/I$.*

PROOF. $(a) \Rightarrow (b)$ is obvious, recalling that the projections π_λ: $G \to G_\lambda$ are continuous.

$(b) \Rightarrow (a)$ Let $f: M \to N$ be a non zero morphism in Mod-R_τ and let $g: G \to M$ be such that $fg \neq 0$. For every $\lambda \in \Lambda$ the injection i_λ: $G_\lambda \to G$ is continuous and $H = \sum_{\lambda \in \Lambda} \operatorname{Im} i_\lambda$ is dense in G. Then it must be $f(gi_\lambda) \neq 0$ for some $\lambda \in \Lambda$.

$(a) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (a)$ If $f: M \to N$ is a non zero morphism in Mod-R_τ and $x \in M$ is such that $f(x) \neq 0$, put $I = \operatorname{Ann}_R(x)$ and consider the injective morphism $j: R/V \to M$ defined by $j(1 + I) = x$. If $\ker fj = V/I$, it is $V \neq R$ and so there exist $\lambda \in \Lambda$ and a morphism $g: G_\lambda \to R/I$ such that $\operatorname{Im} g \not\leq V/I$; now $f(jg) \neq 0$. ■

2.8 LEMMA. *Assume that $(S_\vartheta)_{\vartheta \in \Theta}$ is a system of representatives of all simple modules in* Mod-R_τ *and that, for every $\vartheta \in \Theta$, $(P_\vartheta, p_\vartheta)$ is a τ-projective cover of S_ϑ. Then $(P_\vartheta)_{\vartheta \in \Theta}$ is a family of τ-generators.*

PROOF. If I and V are open right ideals of R_τ with $I \leq V \neq R$, there is an open maximal ideal W of R_τ such that $V \leq W$. If now j: $S_\vartheta \to R/W$ is an isomorphism, then the epimorphism jp_ϑ: $P_\vartheta \to R/W$ can be lifted to a morphism $g: P_\vartheta \to R/I$. Obviously, $\operatorname{Im} g \not\leq V/I$. ■

2.9 THEOREM. *Let R_τ be a right linearly topologized ring. The following conditions are equivalent*:

(*a*) *every finitely generated module in Mod-R_τ has a τ-projective cover*;

(*b*) *every simple module in Mod-R_τ has a τ-projective cover.*

PROOF. Assume (*b*) and let $M \in \text{Mod-}R_\tau$ be finitely generated. Then, by Proposition 1.5, $\text{rad}(M) = \text{rad}_t(M)$ is inessential and $\overline{M} = M/\text{rad}(M)$ is finitely generated. By Lemma 2.8 there are a finite number of τ-projective covers of simple modules $P_1, \ldots, P_n$ and a continuous epimorphism

$$f: P = \prod_{i=1}^{n} P_i \to M.$$

Now $f(\text{rad}_t(P)) \leq \text{rad}_t(M)$ (1.13(iv)) and so f induces an epimorphism $\bar{f}: P/\text{rad}_t(P) \to M/\text{rad}_t(M)$; but

$$\text{rad}_t(P) = \prod_{i=1}^{n} \text{rad}_t(P_i)$$

and $P_i/\text{rad}_t(P_i)$ is simple. Hence $M/\text{rad}_t(M)$ is semisimple and finitely generated, so, by Lemma 2.6, it has a τ-projective cover (Q, q). If we lift $q: Q \to M/\text{rad}_t(M)$ to a morphism $q': Q \to M$, then it is clear that (Q, q') is a τ-projective cover of M. ■

2.10 DEFINITION. A right linearly topologized ring R_τ is called *t*-semiperfect if it satisfies the conditions of Theorem 2.9; it will be called *d*-semiperfect if it is *t*-semiperfect and, moreover, the τ-projective covers of finitely generated modules in Mod-R_τ are discrete (the proof of 2.9 shows that it is sufficient to verify that the τ-projective covers of all simple modules in Mod-R_τ are discrete).

Observe that a ring endowed with the discrete topology is *d*-semiperfect if and only if it is semiperfect in the usual sense: we shall use the word *semiperfect* without prefixes only with this meaning. More than this is true: namely a discrete ring is *t*-semiperfect if and only if it is semiperfect, as we shall see in the next theorem.

2.11 THEOREM. *A discrete t-semiperfect ring is d-semiperfect.*

PROOF. Let R_d be a discrete *t*-semiperfect ring: then $R/J(R)$ is semisimple, say

$$R/J(R) = \bigoplus_{i=1}^{n} S_i,$$

as is shown in the proof of Theorem 2.9. But then a *d*-projective cover of R is the product

$$P = \prod_{i=1}^{n} P_i,$$

where P_i is a d-projective cover of S_i. But it is obvious that R is a d-projective cover of itself and so $P \cong R$ is discrete by 2.4, yielding that the d-projective covers of simple modules are all discrete. ∎

We want now to study the behaviour of a t-semiperfect ring under weakening the topology and under factoring modulo two-sided ideals.

2.12 PROPOSITION. *Let R_τ be a t-semiperfect ring and let σ be a right linear topology on R, coarser that τ. Then R_σ is t-semiperfect.*

PROOF. Let $M \in$ Mod-R_σ be finitely generated and let (P, p) be a τ-projective cover of M (it exists because, by hypothesis, Mod-$R_\sigma \subseteq$ Mod-R_τ). Consider on P the topology having as a local basis the kernels of all continuous morphisms $P \to N$, as N runs through Mod-R_σ and let L be the kernel of this topology. It is clear that the factor module $P^\sigma = P/L$, with the quotient topology, is in LT-R_σ and, since $L \leqslant \ker p$, p induces a continuous epimorphism p^σ: $P^\sigma \to M$. It is immediate to verify that (P^σ, p^σ) is a σ-projective cover of M. ∎

2.13 REMARK. Let R_τ be a d-semiperfect ring and let σ be a right linear topology on R coarser than τ. Under this assumptions, R_σ is not necessarily d-semiperfect, as the following example shows.

EXAMPLE. Let J_p be the ring of p-adic integers, for a prime p. Then J_p is local, hence semiperfect. If σ is the p-adic topology on J_p, it is clear that a projective cover of J_p/pJ_p is still J_p but endowed with the p-adic topology, so that (J_p, σ) is not d-semiperfect, by 2.4.

2.14 THEOREM. *Let R_τ be a right linearly topologized ring and I be a two-sided ideal of R_τ. Denote by R_τ/I the factor ring endowed with the quotient topology τ/I. If R_τ is t-semiperfect (resp. d-semiperfect) then R_τ/I is t-semiperfect (resp. d-semiperfect).*

PROOF. Let M be a discrete finitely generated module over R_τ/I: we can regard M as a discrete finitely generated module over R_τ and so we can take a τ-projective cover (P, p) of M.

Consider now the module $\tilde{P} = P/\overline{PI}$ endowed with the quotient topology (of course it will be discrete if P is). Since obviously $\ker p \geqslant PI$, p induces a continuous epimorphism of R/I-modules, with inessential kernel, from $\tilde{P}$ onto M and it is easy to see that $\tilde{P}$ is a τ/I-projective object in LT-(R_τ/I). ∎

We shall now give an example of a d-semiperfect ring which is not semiperfect.

2.15 PROPOSITION. *Let R_τ be the product of the fields with p elements, as p varies in the set of all primes, with the product topology of the discrete topologies. Then R_τ is d-semiperfect but not semiperfect.*

PROOF. R is not semiperfect because $J(R)=0$ and R is not artinian.

Denote by $\boldsymbol{P}$ the set of all primes and by $\boldsymbol{F}_p$ the field with p elements. The simple modules in Mod-R_τ are, up to isomorphisms, those of the form

$$\frac{R_\tau}{\prod_{p\neq q}\boldsymbol{F}_p}$$

for $q\in\boldsymbol{P}$: indeed, if $S\in$ Mod-R_τ is simple, there is a finite subset Φ of $\boldsymbol{P}$ such that

$$\mathrm{Ann}_R(S)\supseteq\prod_{p\notin\Phi}\boldsymbol{F}_p$$

and so S is a simple module over $\prod_{p\in\Phi}\boldsymbol{F}_p$, so that it is isomorphic, under an R-linear morphism, to $\boldsymbol{F}_q$, for some $q\in\boldsymbol{P}$.

It is also obvious that S_R is τ-projective, since we can check this on finitely generated modules only. ■

3. D-semiperfect rings.

All ring topologies considered in this section will be right linear and Hausdorff.

We shall use the notations of [4]: in particular, if M and N are linearly topologized modules over the topological ring R_τ and A is a ring that acts on M on the left by continuous morphisms, $\mathrm{Chom}_R^{\mathfrak{u}}(M,N)$ will denote the right A-module of all continuous morphisms from M into N, endowed with the topology of uniform convergence. In particular $\mathrm{Chom}_R^{\mathfrak{u}}(M,M)$ is a right linearly topologized ring. We refer the reader to [4] for the definition of a *τ-progenerator*, recalling that τ-progenerators play the same role in the theory of equivalences between categories of discrete modules over topological rings as progenerators in classical Morita equivalence. We say that two right linearly topologized rings R_τ and A_σ are *similar* if the categories Mod-R_τ and Mod-A_σ are equivalent. The main theorem in [4] establishes then that there exists a τ-progenerator P_R such that the functor $\mathrm{Chom}_R(P, -)$ is an equivalence between those two categories; moreover $A_\sigma \cong \mathrm{Chom}_R^{\mathfrak{u}}(P,P)$ canonically.

3.1 LEMMA. *Let R_τ be a right linearly topologized ring, P_R a τ-progenerator, $A_\sigma = \mathrm{Chom}_R^{\mathfrak{u}}(P,P)$. The following conditions are equivalent:*

(*a*) $f \in J(A_\sigma)$;

(*b*) $\mathrm{Im}\, f \leqslant \mathrm{rad}_t(P_R)$;

(*c*) $\mathrm{Im}\, f$ *is inessential in* P.

PROOF. $(c) \Leftrightarrow (b)$ is clear, since every τ-progenerator is t.f.g.

$(a) \Rightarrow (b)$ The open right ideals of A_σ are all those of the form $\mathfrak{I}(V) = \{f \in \mathrm{Chom}_R(P,P) | \mathrm{Im}\, f \leqslant V\}$, as V runs through the open submodules of P_R. Now, if $f \in J(A_\sigma)$, it is $f \in \mathfrak{I}(V)$, for all open maximal submodules V of P_R. In particular $f \in \mathrm{rad}_t(P_R)$.

$(c) \Rightarrow (a)$ If $f \notin J(A_\sigma)$, then $f \notin \mathfrak{I}(V)$, for some open maximal submodule V of P_R. Therefore $\mathrm{Im} f \not\leqslant V$ and so $\mathrm{Im} f + V = P$. Hence $\mathrm{Im} f$ is not inessential in P_R. ■

In particular, if we put $\tilde{P} = P/\mathrm{rad}_t(P)$ with the quotient topology, we obtain a ring morphism $A \to \mathrm{Chom}_R(\tilde{P}, \tilde{P})$, with kernel $J(A_\sigma)$. We shall denote by ψ: $A/J(A_\sigma) \to \mathrm{Chom}_R(\tilde{P}, \tilde{P})$ the induced morphism.

3.2 PROPOSITION. *Let P_R be a τ-progenerator over the right linearly topologized ring R_τ and set $A_\sigma = \mathrm{Chom}_R^{\mathfrak{u}}(P,P)$. Then the induced morphism $\psi: A/J(A_\sigma) \to \mathrm{Chom}_R^{\mathfrak{u}}(\tilde{P}, \tilde{P})$ is continuous with dense image.*

PROOF. The ring morphism ψ: $A/J(A_\sigma) \to \mathrm{Chom}_R^{\mathfrak{u}}(\tilde{P}, \tilde{P})$ is injective by 3.1. A right open ideal of $\mathrm{Chom}_R^{\mathfrak{u}}(\tilde{P}, \tilde{P})$ contains an ideal which has the form $\mathfrak{I}(V/\mathrm{rad}_t(P))$, where V is an open submodule of P_R containing $\mathrm{rad}_t(P)$. Then $\psi^{-1}(\mathfrak{I}(V/\mathrm{rad}_t(P))) = \mathfrak{I}(V)/J(A_\sigma)$ and ψ is continuous.

We come now to the density: let g: $\tilde{P} \to \tilde{P}$ be a continuous morphism and let V, as before, be an open submodule of P_R containing $\mathrm{rad}_t(P)$. Consider the canonical projections α: $P \to \tilde{P}$ and β: $\tilde{P} \to P/V$. Since P_R is quasi-projective, there exists a continuous endomorphisms f: $P \to P$

making commutative the following diagram:

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{f} & \downarrow{\scriptstyle \beta g\alpha} & & \\ P & \xrightarrow{\beta\alpha} & P/V & \longrightarrow & 0 \end{array}$$

and $\psi(f+J(A_\sigma)) - g \in \Im((V/\mathrm{rad}_t(P))$. ■

3.3 REMARK. Under the same hypotheses as in 3.2, let us assume that there is a bijective correspondence between the open submodules V of P_R containing $\mathrm{rad}_t(P_R)$ and the open right ideals of A_σ containing $J(A_\sigma)$ in the sense that $\Im(V)$ contains $J(A_\sigma)$ if and only if V contains $\mathrm{rad}_t(P_R)$. Then the morphism ψ is a topological embedding, because, as one can easily verify, we have

$$\psi(\Im(V)/J(A_\sigma)) = \mathrm{Im}\,\psi \cap \Im(V/\mathrm{rad}_t(P)).$$

3.4. Let R_τ be a (right) d-semiperfect ring. We shall associate to R_τ another ring, called the *basic ring* of R_τ, which is analogous to the usual basic ring of the theory of semiperfect rings.

We shall obtain this ring by using a particular τ-progenerator, which we shall call the *canonical τ-progenerator*. In particular the basic ring will be similar to the ring R_τ (in the sense of [4]) and will be d-semiperfect.

We need some preliminary lemmas.

3.5 LEMMA. *Let R_τ be a right linearly topologized ring and let P and M be objects of* LT-R_τ. *Put* $\Sigma_P(M) = \sum \{\mathrm{Im}\, f \mid f \in \mathrm{Chom}_R(P, M)\}$. *The following conditions are equivalent:*

(*a*) *$\Sigma_P(M)$ is dense in M;*

(*b*) *for any proper open submodule W of M_R, there exists a continuous morphism $f\colon P \to M$ such that* $\mathrm{Im}\, f \nsubseteq W$.

PROOF. (*a*) ⇒ (*b*) Let $x \in M \setminus W$; by the density of $\Sigma_P(M)$ in M, it follows that $(x+W) \cap \Sigma_P(M) \neq \emptyset$ and so there exist $f_1, \dots, f_n \in$ $\in \mathrm{Chom}_R(P, M)$ and $y_1, \dots, y_n \in P$ such that

$$\sum_{i=1}^{n} f_i(y_i) = x + W.$$

In particular $f_i(y_i) \notin W$ for at least one i.

$(b) \Rightarrow (a)$ The closure of $\Sigma_P(M)$ in M is the intersection of all open submodules of M_R containing $\Sigma_P(M)$. Condition (b) just says that no proper open submodule of M has this property. ■

If R_τ is a right linearly topologized ring and M is a topological right module over R_τ, then we denote by $t_\tau(M)$ the submodule of M_R

$$t_\tau(M) = \{x \in M | \operatorname{Ann}_R(x) \text{ is open in } R_\tau\}.$$

This is the usual pretorsion radical associated to the topology τ.

3.6 Lemma. *Let $(M_\lambda)_{\lambda \in \Lambda}$ be a family of discrete right modules over the right linearly topologized ring R_τ and let*

$$M = \prod_{\lambda \in \Lambda} M_\lambda$$

with the product topology. For each open submodule V of M, $t_\tau(V)$ is dense in V.

Proof. If F is a subset of Λ and $x \in M$, we denote by x^F the element of M such that

$$\pi_\lambda(x^F) = \begin{cases} 0 & \text{if } \lambda \in F \\ \pi_\lambda(x) & \text{if } \lambda \notin F \end{cases}$$

where $\pi_\lambda \colon M \to M_\lambda$ is the projection.

A local basis of M is given by the submodules of the form

$$\mathcal{W}(F) = \{x \in M | \pi_\lambda(x) = 0, \ \ \forall \lambda \in F\}$$

as F runs through the finite subsets of Λ; equivalently, it is

$$\mathcal{W}(F) = \{x \in M | x^{\Lambda \setminus F} = 0\}.$$

If V is an open submodule of M, then $V \geq \mathcal{W}(F)$ for some finite subset F of Λ. Now, if $F \subseteq G \subseteq \Lambda$, it is $x^G \in \mathcal{W}(\mathcal{F}) \leq \mathcal{V}$ and so $x - x^G = x^{\Lambda \setminus G} \in V$, for all $x \in V$. If G is in turn finite, the element $x^{\Lambda \setminus G}$ has only a finite number of non zero components and so $x^{\Lambda \setminus G} \in t_\tau(V)$.

Let us denote by $\Lambda(F)$ the directed set of all finite subsets of Λ containing F: it is obvious that, for all $x \in V$, we have

$$x = \lim_{G \in \Lambda(F)} x^{\Lambda \setminus G}. \quad ■$$

For the rest of the section, R_τ will denote a d-semiperfect right linearly topologized ring, $(S_\vartheta)_{\vartheta\in\Theta}$ will be a system of representatives of all non isomorphic simple modules in Mod-R_τ and, for each $\vartheta\in\Theta$, $(P_\vartheta, p_\vartheta)$ will be a τ-projective cover of S_ϑ (obviously P_ϑ is discrete).

3.7 THEOREM. *The module $\Pi = \prod_{\vartheta\in\Theta} P_\vartheta$, endowed with the product topology of the discrete topologies, is a τ-progenerator.*

PROOF. We shall keep the notations introduced in the proof of 3.6.

(1) Π_R is t.f.g. since every P_ϑ is finitely generated (2.5).

(2) Π_R is a *τ-generator* by 2.7 and 2.8.

(3) We now prove that Π_R is τ-projective. Let $g\colon M \to N$ be an epimorphism in Mod-R_τ and $f\colon \Pi \to M$ be a continuous morphism. Then there exists a finite subset F of Θ such that $\mathcal{W}(F) \subseteq \ker f$: in particular $f(x^{\Theta\setminus F}) = 0$, for all $x\in\Pi$. Moreover f induces a morphism $\bar{f}\colon P_F = \bigoplus_{\vartheta\in F} P_\vartheta \to N$.

Now P_F is τ-projective and so there is $h\colon P_F \to M$ with $gh = \bar{f}$. If, for $\eta\in F$, $j_\eta\colon P_\eta \to P_F$ is the inclusion and we identify P_F with a submodule of Π_R in the obvious way, we can define $h'\colon \Pi_R \to M_R$ by

$$h'\colon (x_\vartheta)_{\vartheta\in\Theta} \mapsto \sum_{\vartheta\in F} hj_\vartheta(x_\vartheta).$$

Hence, if $x = (x_\vartheta)_{\vartheta\in\Theta}\in\Pi$, it is

$$gh'(x) = \sum_{\vartheta\in F} ghj_\vartheta(x_\vartheta) = \sum_{\vartheta\in F} fj_\vartheta(x_\vartheta) =$$

$$= f(x^F) = f(x^F) + f(x^{\Theta\setminus F}) = f(x^F + x^{\Theta\setminus F}) = f(x)$$

and it is clear that h' is continuous.

(4) We want now to prove that Π_R *is* (*topologically*) *quasi-projective*. Let $V \supseteq \mathcal{W}(F)$ be an open submodule of Π_R (with $F\subseteq\Theta$ finite) and assume we are given a continuous morphism $f\colon \Pi \to \Pi/V$. The projection $\pi\colon \Pi \to \Pi/V$ induces

$$\bar{\pi}\colon P_F = \bigoplus_{\vartheta\in F} P_\vartheta \to \Pi/V$$

and so, by (3), there exists $h\colon \Pi \to P_F$ such that $f = \bar{\pi}h$.

Define $h_\vartheta: \Pi \to P_\vartheta$ in the following way:

$$h_\vartheta = \begin{cases} 0 & \text{if } \vartheta \notin F \\ \pi_\vartheta h_\vartheta & \text{if } \vartheta \in F \end{cases}$$

where $\pi_\vartheta: P_F \to P_\vartheta$ is the canonical projection.

It can be easily verified that $\overline{h} = (h_\vartheta)_{\vartheta \in \Theta}$ is a continuous endomorphism of Π_R and that $\pi\overline{h} = f$.

(5) *Π_R is a selfgenerator.* In fact, let $W \leqslant V$ be open submodules of Π_R, with $W \neq V$. If $x \in V \setminus W$, put $V' = W + xR$. Then V'/W is finitely generated and so there exists a maximal submodule Z of V' containing W. Assume that $V'/Z \cong S_\vartheta$; since $t_\tau(V')$ is dense in V' by 3.6, we have $t_\tau(V') + Z = V'$ and

$$S_\vartheta \cong \frac{V'}{Z} = \frac{t_\tau(V') + Z}{Z} \cong \frac{t_\tau(V')}{t_\tau(V') \cap Z} .$$

Consider $X = t_\tau(V')$ but endowed with the discrete topology: then $X \in \text{Mod-}R_\tau$ and there is a morphism $f: P_\vartheta \to X$ such that $\operatorname{Im} f \nleqslant Z \cap t_\tau(V')$, so that $\operatorname{Im} f \nleqslant Z$ and therefore $\operatorname{Im} f \nleqslant W$, regarding now X as a submodule of Π_R. The composition h of the morphisms

$$\Pi \xrightarrow{\pi_\vartheta} P_\vartheta \xrightarrow{f} X = t_\tau(V') \hookrightarrow V' \hookrightarrow V$$

is clearly continuous and $\operatorname{Im} h = \operatorname{Im} f \nleqslant W$.

Lemma 3.5 now gives us that $\Sigma_\Pi(V)$ is dense in V. ∎

3.8 Definition. The module Π_R defined in Theorem 3.7 is called the *canonical τ-progenerator* of R_τ. The ring $B_\beta = \operatorname{Chom}_R^u(\Pi, \Pi)$ is called the *basic ring* of R_τ.

It must be noted that, since Π is complete, the basic ring B_β is also complete.

The property of being d-semiperfect is invariant under similarity of linearly topologized rings [4]; an invariant of the class of similarity is the basic ring.

3.9 Theorem. *Let R_τ and A_σ be right linearly topologized rings.*

(a) If R_τ and A_σ are similar and R_τ is d-semiperfect, then A_σ is d-semiperfect and both rings have topologically isomorphic basic rings.

(b) If R_τ and A_σ are d-semiperfect with topologically isomorphic basic rings, then they are similar.

PROOF. (*a*) Let $T_R \in$ LT-R_τ be a τ-progenerator with

$$\mathrm{Chom}_R^u(T, T) \cong A_\sigma \quad [4]:$$

a generic finitely generated module in Mod-A_σ is then one of the form $N = \mathrm{Chom}_R(T, M)$, where M is a finitely generated module in Mod-R_τ; if now (Q, q) is a τ-projective cover of M, with Q discrete, consider the pair $(Q_* = \mathrm{Chom}_R(T, Q),\ q_* = \mathrm{Chom}_R(T, q))$. It is clear that $Q_* \in$ Mod-A_σ is σ-projective and that q_* is surjective. Moreover

$$\ker q_* = \{\xi \in N | q_*(\xi) = q\xi = 0\} = \{\xi \in N | \mathrm{Im}\, \xi \leqslant \ker q\}.$$

Any submodule of Q_* has the form $\mathfrak{I}(X) = \{\eta \in Q_* | \mathrm{Im}\, \eta \leqslant X\}$, where X is a submodule of Q_R. Assume now that $\mathfrak{I}(X) + \ker q_* = Q_*$. If $x \in Q$, there exist $f_i \in \mathrm{Chom}_R(T, Q)$ and $y_i \in T$ $(i = 1, \dots, n)$ with

$$x = \sum_{i=1}^{n} f_i(y_i).$$

By hypothesis there are $g_i \in \mathfrak{I}(X)$, $h_i \in \ker q_*$ such that $f_i = g_i + h_i$, for $i = 1, \dots, n$. Hence

$$x = \sum_{i=1}^{n} f_i(y_i) = \sum_{1}^{n} g_i(y_i) + \sum_{i=1}^{n} h_i(y_i) \in X + \ker q$$

and thus $X + \ker q = Q$, whence $X = Q$ and $\mathfrak{I}(X) = Q_*$.

It is now obvious that the canonical τ-progenerator and the canonical σ-progenerator correspond to each other: if Π_R is the former, the latter is just

$$\Pi_* = \mathrm{Chom}_R^u(T, \Pi) = \prod_{\vartheta \in \Theta} \mathrm{Chom}_R(T, P_\vartheta)$$

and that $\mathrm{Chom}_A^u(\Pi_*, \Pi_*)$ is topologically isomorphic to $\mathrm{Chom}_R^u(\Pi, \Pi)$.

Part (*b*) is now clear since, by 3.7, a *d*-semiperfect ring is similar to its basic ring. ■

A *d*-semiperfect ring will be called *basic* if it is (topologically isomorphic to) the basic ring of a *d*-semiperfect ring.

3.10 COROLLARY. *Let B_β e a d-semiperfect ring and let Σ_B be its canonical β-progenerator. Then B_β is basic if and only if $B_\beta \cong \Sigma_B$ as topological right B-modules.*

PROOF. Assume that B_β is the basic ring of R_τ and Π_R is the canonical τ-progenerator. Then, as in the proof of 3.9, it is $\Sigma = \mathrm{Chom}_R^u(\Pi, \Pi) = B_\beta$.

Conversely, if $B_\beta \cong \Sigma_B$, then the basic ring of B_β is $\mathrm{Chom}_B^{\mathcal{U}}(B, B) = B_\beta$. ■

We want to see now when the basic ring of a d-semiperfect ring is discrete: in this case it is obvious that the basic ring will be semiperfect. These rings can then be easily characterized by using the theory developed so far.

3.11 THEOREM. *Let R_τ be a right linearly topologized d-semiperfect ring. The following conditions are equivalent:*

(a) the basic ring of R_τ is discrete;

(b) R_τ is similar to a discrete ring, i.e. Mod-R_τ *is equivalent to* Mod-A, *where A is a discrete ring;*

(c) there is in Mod-R_τ *only a finite number of non isomorphic simple modules.*

If these conditions are satisfied, then the basic ring of R_τ is semiperfect.

PROOF. $(a) \Rightarrow (b)$ is obvious by Theorem 2.7.

$(b) \Rightarrow (c)$ A is semiperfect, so that in Mod-A and hence also in Mod-R_τ there is only a finite number of nonisomorphic simple modules.

$(c) \Rightarrow (a)$ If (c) holds, then the canonical τ-progenerator is discrete. ■

We shall now see that the canonical τ-progenerator of the d-semiperfect ring R_τ satisfies the hypothesis of 3.3. We shall denote by $(S_\vartheta)_{\vartheta \in \Theta}$ a system of representative of all non isomorphic simple modules in Mod-R_τ and by $(P_\vartheta, p_\vartheta)$ a (discrete) τ-projective cover of S_ϑ, for $\vartheta \in \Theta$.

3.12 LEMMA. *If $\Pi = \prod_{\vartheta \in \Theta} P_\vartheta$ is the canonical τ-progenerator of R_τ, then*

$$\mathrm{rad}_t(\Pi_R) = \prod_{\vartheta \in \Theta} \mathrm{rad}_t(P_\vartheta) = \prod_{\vartheta \in \Theta} \ker p_\vartheta .$$

PROOF. Set $X = \prod_{\vartheta \in \Theta} \mathrm{rad}_t(P_\vartheta)$. It is clear that $\mathrm{rad}_t(\Pi) \leqslant X$.

If we put $X_\vartheta = \mathrm{rad}_t(P_\vartheta) = \ker p_\vartheta$, we have to prove that given an open maximal submodule V of Π, then V contains X. Let F be a finite subset of Θ such that $V \geqslant \mathcal{W}(F)$: passing to the quotients modulo $\mathcal{W}(F)$

yields an isomorphism

$$\frac{\Pi}{\mathfrak{W}(F)} \cong \bigoplus_{\vartheta \in F} P_\vartheta$$

and $V/\mathfrak{W}(F)$ corresponds to a maximal submodule $\overline{V}$ of $\bigoplus_{\vartheta \in F} P_\vartheta$. Then, by 1.13,

$$\overline{V} \supseteq \operatorname{rad}_t\left(\bigoplus_{\vartheta \in F} P_\vartheta\right) = \bigoplus_{\vartheta \in F} \operatorname{rad}_t(P_\vartheta)$$

and so

$$\frac{V}{\mathfrak{W}(F)} \supseteq \frac{X + \mathfrak{W}(F)}{\mathfrak{W}(F)},$$

whence $V \supseteq X$. ∎

3.13 Lemma. *Let Π_R be the canonical τ-progenerator of R_τ and let $B_\beta = \operatorname{Chom}_R^{\mathfrak{u}}(\Pi, \Pi)$ be the basic ring. If V is an open submodule of Π_R such that $\mathfrak{I}(V) \supseteq J(B_\beta)$, then $V \supseteq \operatorname{rad}_t(\Pi_R)$.*

Proof. Assume that V does not contain $\operatorname{rad}_t(\Pi_R)$ and that F is a finite subset of Θ such that $\mathfrak{W}(F) \subseteq V$. Then $V \not\supseteq \mathfrak{W}(F) + \operatorname{rad}_t(\Pi)$ and, factoring modulo $\mathfrak{W}(F)$ we obtain a submodule $\overline{V}$ of

$$P = \bigoplus_{\vartheta \in F} P_\vartheta$$

which does not contain $\operatorname{rad}_t(P)$. Since $(P_\vartheta)_{\vartheta \in \Theta}$ is a family of τ-generators (2.8), there exist a suitable $\vartheta \in \Theta$ and a morphism $f\colon P_\vartheta \to P$ such that $\operatorname{Im} f \leqslant \operatorname{rad}_t(P)$ but $\operatorname{Im} f \not\leqslant \overline{V} \cap \operatorname{rad}_t(P)$. If we now consider the projection $\pi_\vartheta\colon \Pi \to P_\vartheta$ and the embedding $j_F\colon P \to \Pi$, we get a continuous morphism $g = j_F f \pi_\vartheta\colon \Pi \to \Pi$ with $\operatorname{Im} g \leqslant \operatorname{rad}_t(\Pi)$ and $\operatorname{Im} g \not\leqslant V$, so that $g \in J(B_\beta)$ but $g \notin \mathfrak{I}(V)$. ∎

3.14 Theorem. *Let B_β be a right linearly topologized basic d-semiperfect ring. Then $B_\beta/J(B_\beta)$, with the quotient topology, is topologically isomorphic to a product of division rings (each one endowed, of course, with the discrete topology).*

Proof. By 3.2, 3.3 and 3.13 we have that $B_\beta/J(B_\beta)$ is topologically embedded as a dense subring in $\operatorname{Chom}_R^{\mathfrak{u}}(\overline{\Sigma}, \overline{\Sigma})$, where Σ_B is the canonical β-progenerator and $\overline{\Sigma} = \Sigma/\operatorname{rad}_t(\Sigma)$. It now follows from 3.12 that $B_\beta/J(B_\beta) \cong \overline{\Sigma}$ is complete, being $\overline{\Sigma} = \prod_{\vartheta \in \Theta} S_\vartheta$ (with obvious meaning of the symbols). Hence

$$B_\beta/J(B_\beta) \cong \prod_{\vartheta \in \Theta} \mathrm{End}_B(S_\vartheta)$$

is topologically isomorphic to a product of division rings. ■

4. Linearly compact rings.

4.1. In this section we shall prove the following result: *Every linearly compact ring is t-semiperfect* and some of its consequences.

This should not be a surprising result, at least if one recalls that every linearly compact discrete ring is semiperfect. Moreover, every linearly compact ring admits a duality theory, which generalizes Morita duality (see [2] and [3]). We will use, in this particular case, the duality defined by Menini and Orsatti in [7]. In what follows we denote by R_τ a fixed right linearly compact ring.

4.2. It is well known that the category Mod-R_τ has injective hulls: indeed, if $M \in$ Mod-R_τ, its injective hull is

$$E_\tau(M) = t_\tau(E(M)),$$

where $E(M)$ is the injective hull of M as an abstract R-module.

If, as usual, $(S_\vartheta)_{\vartheta \in \Theta}$ is a system to representative of the simple modules in Mod-R_τ then

$$W_R = E_\tau\left(\bigoplus_{\vartheta \in \Theta} S_\vartheta\right)$$

is the minimal injective cogenerator of Mod-R_τ.

Set $C = \mathrm{End}(W_R)$ and endow it with the (left linear) W-topology γ, with a local basis consisting of the annihilators in C of the finite subsets of W; C_γ is called the *cobasic ring* of R_τ [3, Section 7].

4.3 Lemma. *The canonical ring morphism* $R \to \mathrm{End}({}_CW)$ *is an isomorphism,* τ *is finer than the* W*-topology on* R *and* ${}_CW$ *is an injective cogenerator, with essential socle, of* C_γ-Mod.

Proof. See [3, Corollary 2.12]. ■

4.4 Remark. It is not difficult to show that the W-topology on R is *equivalent* to τ, in the sense that the two topologies have the same closed right ideals (see [3, 1.5 and 1.6]).

4.5. Consider the category $\mathcal{B}(W_R)$ consisting of all modules in LT-R_τ which are topologically isomorphic to a submodule of a product W_R^X of copies of the discrete module W_R. An equivalent characterization is the following: a module $M \in$ LT-R_τ belongs to $\mathcal{B}(W_R)$ if and only if the topology on M coincides with the weak topology of a family of continuous morphisms $M_R \to W_R$.

Consider now the analogous category $\mathcal{B}(_CW)$: the left C-module $\mathrm{Chom}_R(M, W)$ with the topology of pointwise convergence, belongs to $\mathcal{B}(_CW)$; moreover, if $f: M \to N$ is a morphism in $\mathcal{B}(W_R)$, the transposed morphism

$$f^*: \mathrm{Chom}_R(N, W) \to \mathrm{Chom}_R(M, W)$$

$$\xi \mapsto \xi f$$

is a morphism in $\mathcal{B}(_CW)$.

Thus, making analogous definitions on the category $\mathcal{B}(_CW)$ we have a pair of functors

$$D_1: \mathcal{B}(_CW) \to \mathcal{B}(W_R),$$

$$D_2: \mathcal{B}(W_R) \to \mathcal{B}(_CW).$$

If $M \in \mathcal{B}(W_R)$, then we can consider the evaluation morphism $\omega_M: M \to D_1D_2(M)$ defined by $x \mapsto \omega_M(x) = \tilde{x}$, where $\tilde{x}(\xi)$, for $x \in M$ and $\xi \in \mathrm{Chom}_R(M, W)$; of course, we can do similarly for $N \in \mathcal{B}(_CW)$.

This is a particular case of a setting investigated by Menini and Orsatti. From their Theorem 5.3 in [7] we get the following

4.6 THEOREM. *Let R_τ be a right linearly compact ring, W_R be the minimal injective cogenerator of* Mod-R_τ *and $C_\gamma = \mathrm{End}(W_R)$ be the cobasic ring of R_τ. Then the pair of functors*

$$D_1: \mathcal{B}(_CW) \to \mathcal{B}(W_R), \qquad D_2: \mathcal{B}(W_R) \to \mathcal{B}(_CW)$$

is a duality. For $M \in \mathcal{B}(W_R)$ and $N \in \mathcal{B}(_CW)$, the evaluation morphisms $\omega_M: M \to D_1D_2(M)$ and $\omega_N: N \to D_2D_1(N)$ are topological isomorphisms.

We have also an important property of the modules W_R and $_CW$, which is stated in Lemma 6.4 of [7], descending from the fact that they are cogenerators of Mod-R_τ and C_γ-Mod respectively.

4.7 PROPOSITION. *If the discrete module K_R is a cogenerator of* Mod-R_τ *then, for any $M \in$* LT-R_τ *and $x \in M$, $x \neq 0$, there exists a continuous morphism $\xi: M_R \to K_R$ such that $\xi(x) \neq 0$.*

4.8. Let $S \in \text{Mod-}R_\tau$ be a simple module. Then $T = D_2(S)$ is a simple C-module in C_γ-Mod, by Proposition 4.7; by applying Lemma 4.3, we get that its injective hull $E = E_\gamma(T)$ is a direct summand of ${}_CW$ and there exists an idempotent $e \in R = \text{End}({}_CW)$ such that $E = We$. Set $P = eR$ with the relative topology of τ and $P^w = D_2(E) = \text{Hom}_C(E, W)$: then P and P^w are isomorphic as abstract modules and P^w has a topology coarser than and equivalent to the topology of P (see Remark 4.4), so that P^w and P have the same closed submodules. From the inclusion $T \to E$, we get a continuous morphism $p\colon P^w \to S$ which is non-zero, hence surjective.

4.9 Proposition. *(P, p) is a τ-projective cover of S.*

Proof. First, p is continuous, since it is continuous as a morphism $P^w \to S$. Then it is almost obvious that P_R is τ-projective, as every morphism of R into any $M \in \text{Mod-}R_\tau$ is continuous with respect to τ.

We have to prove that $\ker p$ is inessential. So, let V be a closed submodule of P such that $V + \ker p = P$ and assume, by contradiction, that $V \neq P$.

By 4.7, V is closed also in $P^w \in \mathcal{B}(W_R)$; then, by giving P^w/V the quotient topology, there exists a non-zero continuous morphism of $P^w/V \to W$. Composing this to the canonical projection yields a non-zero continuous morphism $P^w_R \to W_R$ which is zero on V. Now we recall that $P^w = D_1(E)$ and that ω_E is a topological isomorphism (Theorem 4.6), so that any continuous morphism $P^w_R \to W_R$ has the form $(x)\,\omega_E = \tilde{x}$, for some $x \in E$.

Consider $V^\perp = \{x \in E \mid \tilde{x}(V) = 0\}$: this is a non-zero submodule of ${}_CE$ and so it contains some non-zero element $t \in T$, which is the (essential) socle of E; denote by $i\colon T \to E$ the inclusion.

Now, for any $\xi \in P^w = \text{Chom}_C(E, W)$ there are $\alpha \in V$ and $\beta \in \ker p$ with $\xi = \alpha + \beta$. Thus

$$\tilde{t}(\xi) = (t)\xi = (t)\,\alpha + (t)\,\beta = (t)\,\alpha + (t)\,i\beta = 0$$

since $(t)\,\alpha = \tilde{t}(\alpha) = 0$, as $t \in V^\perp$, and $(t)\beta = 0$ as $(t)\beta = (t)\,i\beta$ and $i\beta = D_1(i)(\beta) = p(\beta) = 0$.

This is a contradiction, because it implies that $t = 0$. Hence $V = P$. ∎

4.10 Theorem. *Every linearly compact ring is t-semiperfect.*

Proof. Apply Proposition 4.9 and Theorem 2.9. ∎

The discussion above has another consequence; we want to state it, for convenience, as a separate result.

4.11 PROPOSITION. *With the notations as in* 4.2 *and* 4.8, *let* $S \in$ Mod-R_τ *be a simple module. Then there exists a* τ*-projective cover of* S_R *of the form* eR, *where* e *is an idempotent of* R *such that* We, *as left* C*-module, is the injective hull of* $D_2(S) \in C_\gamma$*-Mod.*

4.12. The last part of this section is dedicated to the study of d-semiperfect linearly compact rings. Before starting, we need a result due to Warner [8]: if R_τ is a right linearly compact ring, then, among the linearly compact topologies equivalent to τ (see Remark 4.4), there is a finest one, which we denote by τ^*; τ^* is a linearly compact ring topology.

If W_R is, as usual, the minimal injective cogenerator of Mod-R_τ and $C_\gamma = \mathrm{End}({}_C W)$ is the cobasic ring, then a local basis for τ^* consists of the annihilators of the linearly compact submodules of the discrete module ${}_C W$ [6, Theorem 1.6].

4.13 THEOREM. *If* R_τ *is a d-semiperfect linearly compact ring, then* $\tau = \tau^*$.

PROOF. If S is a simple module in R_τ then it has, up to isomorphisms, a unique τ-projective cover of the form eR, for some idempotent $e \in R$ (see Proposition 2.4 and Proposition 4.11); the relative topology τ on this eR is discrete by hypothesis.

Consider a linearly compact submodule X of ${}_C W$ (the notations are as in 4.12).

Then X has finitely generated and essential socle, say

$$\mathrm{Soc}(X) = \bigoplus_{i=1}^{n} T_i,$$

where the T_i are simple submodules of ${}_C W$. Thus

$$E_\gamma(X) = \bigoplus_{i=1}^{n} E_\gamma(T_i)$$

is a direct summand of ${}_C W$ and so there exist idempotents $e_1, \dots, e_n \in R$ with

$$E_\gamma(X) = \bigoplus_{i=1}^{n} We_i;$$

like in 4.11, $e_i R$ is a τ-projective cover of $\mathrm{Hom}_C(T_i, W)$ and so it is discrete. Now

$$\mathrm{Ann}_R(X) \supseteq \mathrm{Ann}_R(E_\gamma(X)) = \bigcap_{i=1}^{n} \mathrm{Ann}_R(We_i) = \bigcap_{i=1}^{n} \mathrm{Ann}_R(e_i)$$

is open in R_τ. Thus $\tau^* \supseteq \tau$ and we are done. ∎

4.14 REMARK. The condition $\tau = \tau^*$ in the previous theorem is not sufficient for the d-semiperfectness of R_τ. In fact, consider a commutative local ring A which admits a linearly compact topology σ, but is not linearly compact in the discrete topology d. We can assume that $\sigma = \sigma^*$.

Then A_σ is the σ-projective cover of the unique simple module and it is not discrete.

A ring as above can be constructed as follows (the example is due to A. Orsatti): let F be a field and consider the product vector F-space $M = F^X$ where X is an infinite set. Set $A = F \oplus M$, where the addition is defined component-wise and the product is $(\alpha, x)(\beta, y) = (\alpha\beta, x\beta + y\alpha)$. Then A is a commutative local ring and $M = 0 \oplus M$ is the maximal ideal of A; it is not linearly compact in the discrete topology, since its socle is not finitely generated, but the topology having as a local basis the submodules of M which are open in the product topology of the discrete topology on F is obviously linearly compact.

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