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Some Properties of Butler Modules over Valuation Domains.

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In this paper all modules are over a valuation domain R, *i.e.*, a commutative domain with 1 in which the ideals form a chain under inclusion. Q will be the field of quotients of R with $Q \neq R$.

The study of Butler modules over valuation domains was initiated by L. Fuchs and the author in [4], proving that Butler modules of rank $\leqslant \aleph_1$ as well as those whose projective dimension is at most 1 are completely decomposable. In this paper we continue this study and find some new properties of Butler modules of any rank, but a satisfactory classification is far from being complete.

In the first section it is proved that reduced Butler modules B have the following property p:

for any pure submodule N of rank k (k a cardinal) of B and any rank one k'-generated module J with $k' > k$, each epimorphism $f : N \to J$ is splitting.

From this property we can infer, as a corollary, that finite rank pure submodules of Butler modules are completely decomposable as is proved in [4] in another way. The idea of the proof comes from the techniques used by Griffith for the solution of Baer's problem, in the case of finite rank abelian groups [6], and later by Eklof and Fuchs in [2], for Baer modules over valuation domains.

In the second section we introduce *striped modules* as torsion free modules whose reduced summands satisfy property p. We study some properties of this class of modules and, using an example due to L. Fuchs, we prove that the class of striped modules properly contains the class of Butler modules.

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In the third section we find a property of TEP-submodules and a property of completely decomposable torsion free modules.

Now we recall some definitions which will be useful later. A short exact sequence of R-modules

$$0 \to A \to B \xrightarrow{f} C \to 0,$$

is *balanced* if, for every R-modules J/I with $0 \leqslant I < J \leqslant Q$, the induced map $\operatorname{Hom}(J/I, B) \to \operatorname{Hom}(J/I, C)$ is surjective. N is a *balanced submodule* of M if the sequence $0 \to N \hookrightarrow M \to M/N \to 0$ is balanced exact. The elements of $\operatorname{Ext}^1(C, A)$ corresponding to balanced short exact sequences form a submodule of $\operatorname{Ext}^1(C, A)$ called $\operatorname{Bext}^1(C, A)$. As Warfield observed in [7], balanced short exact sequences form a proper class and we can define the functors $\operatorname{Bext}^n(C, A)$, for all $n \in \mathbf{N}$, in such a way that one gets the usual long exact sequences.

A torsion free R-module B is *Butler* if $\operatorname{Bext}^1(B, T) = 0$ for every torsion R-module T. A torsion free R-module is *completely decomposable* if it is a direct sum of rank one R-modules. A torsion free R-module is *separable* if every finite set of its elements is contained in a completely decomposable summand (of finite rank).

1. – The main property.

We are going to define a useful non-splitting sequence. For every non-free R-submodule J of Q, we can construct an R-module $\mathcal{J}_J$ in the following way:

If J is k-generated (without loss of generality we may assume that k is a regular cardinal), let $\{g_\sigma\}_{\sigma < k}$ be a well ordered set such that we have the chain $0 \neq Rg_0 < Rg_1 < \ldots < Rg_\sigma < Rg_{\sigma+1} < \ldots \bigcup_{\sigma < k} Rg_\sigma = J$. Let $r_\sigma g_\sigma = g_0$, for every $0 < \sigma < k$, with $r_\sigma \in R$ and $r_0 = 1$. Let $\{b_\sigma\}_{\sigma < k}$ be a set of symbols which satisfy just the relations $r_\sigma b_\sigma = b_0$, for every $0 < \sigma < k$, and let $\mathcal{J}_J$ be the R-module generated by $\{b_\sigma\}_{\sigma < k}$. We call $\mathcal{J}_J$ a *fan module*. Let $\varphi : \mathcal{J}_J \to J$ be the homomorphism defined by $\varphi(b_\sigma) = g_\sigma$, for every $\sigma < k$.

Then, setting $T = \ker \varphi$, we will prove the following, in Remarks 1-4, for the exact sequence

$$0 \to T \hookrightarrow \mathcal{J}_J \xrightarrow{\varphi} J \to 0:$$

it is non-splitting and, defining $a_{\sigma\tau} = r_\tau r_\sigma^{-1} b_\tau - b_\sigma$, for every $0 \leqslant \sigma < \tau < k$, we have that T is torsion, $T = t\mathcal{J}_J$, $T = \sum_{0 \leqslant \sigma < \tau < k} Ra_{\sigma\tau}$ with
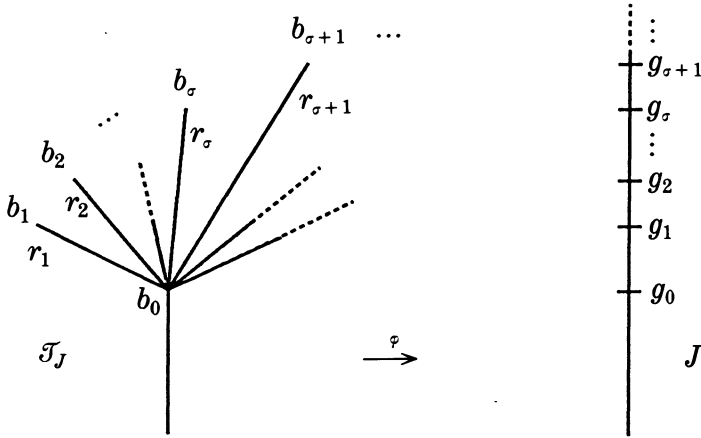
Ann $(a_{\sigma\tau}) = Rr_\sigma$, $T \cap Rb_0 = 0$. If $k = \aleph_0$, then $T = \underset{n \in \mathbb{N}}{\oplus} Ra_{n,n+1}$ (this was studied by Fuchs and Eklof in [2]).

REMARK 1. Consider the submodule $t\left(\sum_{\tau \leqslant \bar{\sigma}} Rb_\tau\right)$ of $\mathcal{T}_J$ where $\bar{\sigma} < k$ is fixed. Then

$$\mathcal{T}_J = t\left(\sum_{\tau \leqslant \bar{\sigma}} Rb_\tau\right) \oplus \sum_{\bar{\sigma} \leqslant \tau < k} Rb_\tau.$$

In fact, for every $\tau < \bar{\sigma}, b_\tau = r_{\bar{\sigma}} r_\tau^{-1} b_{\bar{\sigma}} + (b_\tau - r_{\bar{\sigma}} r_\tau^{-1} b_{\bar{\sigma}})$ and $b_\tau - r_{\bar{\sigma}} r_\tau^{-1} b_{\bar{\sigma}} \in t\left(\sum_{\tau \leqslant \bar{\sigma}} Rb_\tau\right)$. Hence $\mathcal{T}_J = t\left(\sum_{\tau \leqslant \bar{\sigma}} Rb_\tau\right) + \sum_{\bar{\sigma} \leqslant \tau < k} Rb_\tau$. To prove that the sum is direct, we observe that, by the definition of $\mathcal{T}_J, \left(\sum_{\tau \leqslant \bar{\sigma}} Rb_\tau\right) \cap \left(\sum_{\bar{\sigma} \leqslant \tau < k} Rb_\tau\right) = Rb_{\bar{\sigma}}$ and $t\left(\sum_{\tau \leqslant \bar{\sigma}} Rb_\tau\right) \cap Rb_{\bar{\sigma}} = 0$.

REMARK 2. T is torsion. In fact, if T were not torsion, there would be an element $t \in T$ such that, for some non zero $s, r \in R$, $st = rb_0$. Then $\varphi(st) = \varphi(rb_0) = rg_0 \neq 0$ which is a contradiction, since $T = \ker\varphi$.

REMARK 3. $T = \sum_{0 \leqslant \sigma < \tau < k} Ra_{\sigma\tau}$. In fact, it is obvious that $T \geqslant \sum_{0 \leqslant \sigma < \tau < k} Ra_{\sigma\tau}$. We have to prove the inverse inclusion. Let $t = s_0 b_{\sigma_0} + s_1 b_{\sigma_1} + \ldots + s_n b_{\sigma_n} \in T$ where $\sigma_0 < \sigma_1 < \ldots < \sigma_n$. Then

$b_{\sigma_i} = r_{\sigma_n} r_{\sigma_i}^{-1} b_{\sigma_n} - a_{\sigma_i \sigma_n}$ for $0 \leqslant i < n$ and then

$$t = \sum_{1 \leqslant i \leqslant n-1} s_i (r_{\sigma_n} r_{\sigma_i}^{-1} b_{\sigma_n} - a_{\sigma_i \sigma_n}) + s_n b_{\sigma_n} =$$

$$= \Big(\sum_{1 \leqslant i \leqslant n-1} s_i r_{\sigma_n} r_{\sigma_i}^{-1} + s_n \Big) b_{\sigma_n} - \sum_{1 \leqslant i \leqslant n-1} s_i a_{\sigma_i \sigma_n}.$$

As T is torsion, $\displaystyle\sum_{1 \leqslant i \leqslant n-1} s_i r_{\sigma_n} r_{\sigma_i}^{-1} + s_n = 0$ and $t = - \displaystyle\sum_{1 \leqslant i \leqslant n-1} s_i a_{\sigma_i \sigma_n}$

REMARK 4. The exact sequence $0 \to T \hookrightarrow \mathcal{T}_J \overset{\varphi}{\to} J \to 0$ doesn't split. In fact, if we consider the free R-module $F = \underset{0 \leqslant \sigma < k}{\oplus} R c_\sigma$, then $\mathcal{T}_J = F/F'$, where $F' = \underset{0 < \sigma < k}{\oplus} R(r_\sigma c_\sigma - c_0)$; hence p.d. $\mathcal{T}_J = 1$. If $k > \aleph_0$, then p.d. $J > 1$ and so the sequence cannot split as $\mathcal{T}_J$ cannot have a summand of projective dimension > 1. If $k = \aleph_0$, it is proved in [2] that the sequence doesn't split.

Now we give a very technical, but crucial, result.

LEMMA 1.1. *Let J be an R-module of rank 1 which is k-generated (k is a regular cardinal) and let $0 \to T \to \mathcal{T}_J \overset{\varphi}{\to} J \to 0$ be the non-splitting sequence defined above. Then, if M is a torsion free R-module of rank less than k, there are no homomorphisms $\psi : M \to \mathcal{T}_J$ such that $\mathcal{T}_J = T + \psi M$.*

PROOF. Suppose that there is $\psi : M \to \mathcal{T}_J$ such that $\mathcal{T}_J = T + \psi M$. As $J = \varphi(\mathcal{T}_J) = \varphi(T + \psi M) = \varphi \psi M$, there is $t_1 \in t \Big(\displaystyle\sum_{1 \leqslant \tau \leqslant \mu_1} R b_\tau \Big)$, for some ordinal $\mu_1 (1 \leqslant \mu_1 < k)$, such that $b_1 + t_1 \in \psi M$. By remark 1, $K_1 = t \Big(\displaystyle\sum_{1 \leqslant \tau \leqslant \mu_1} R b_\tau \Big)$ is a summand of $\mathcal{T}_J$ and, if we set $\mathcal{T}_J^1 = \displaystyle\sum_{\mu_1 \leqslant \tau < k} R b_\tau$ we have $\mathcal{T}_J = K_1 \oplus \mathcal{T}_J^1$. Let $\eta_1 : \mathcal{T}_J \to \mathcal{T}_J^1$ be the canonical projection with $\ker \eta_1 = K_1$. Then the exact sequence $0 \to t \mathcal{T}_J^1 \to \mathcal{T}_J^1 \overset{\varphi}{\to} J \to 0$ doesn't split and $\eta_1(b_1 + t_1) = \eta_1(b_1) = \eta_1((b_1 - r_{\mu_1} r_1^{-1} b_{\mu_1}) + r_{\mu_1} r_1^{-1} b_{\mu_1}) = r_{\mu_1} r_1^{-1} b_{\mu_1} \in \eta_1 \psi M$. Hence $b_0 \in \eta_1 \psi M$ and, if we set $r_{\mu_1} r_1^{-1} b_{\mu_1} = x_1$, $x_1 \in \eta_1 \psi M$.

By transfinite induction, construct the canonical projections

$$\eta_2, \eta_3, \ldots, \eta_\alpha, \eta_{\alpha+1}, \ldots \qquad (\alpha < k)$$

with $\eta_{\alpha+1} : \mathcal{T}_J^\alpha \to \mathcal{T}_J^{\alpha+1}$ for every $1 \leqslant \alpha < k$, where $\mathcal{T}_J^\alpha$ is a fan module which is a submodule of $\mathcal{T}_J$ generated by $\{b_\sigma\}_{\sigma \in I_\alpha} \cup \{b_\sigma\}_{\mu_\alpha \leqslant \sigma < k}$ where μ_α is fixed and I_α is an ordered subset of cardinals less than or equal to μ_α, which contains μ_α. Moreover, for every $\alpha < k$, set $\mathcal{T}_J^\alpha = A_\alpha \oplus B_\alpha$ with

$A_\alpha = t\left(\sum_{\sigma \in I_\alpha} Rb_\sigma\right)$ and $B_\alpha = \sum_{\mu_\alpha \leq \sigma < k} Rb_\sigma$. This is possible by Remark 1. Now

let the $\nu_{\alpha+1}$ be ordinals such that $0 \leq \mu_\alpha < \nu_{\alpha+1} \leq \mu_{\alpha+1} < k$. Then

$$B_\alpha = t\left(\sum_{\mu_\alpha \leq \sigma \leq \nu_{\alpha+1}} Rb_\sigma\right) \oplus t\left(\sum_{\nu_{\alpha+1} \leq \sigma \leq \mu_{\alpha+1}} Rb_\sigma\right) \oplus \sum_{\mu_{\alpha+1} \leq \sigma < k} Rb_\sigma$$

by remark 1. If we call $K_{\alpha+1} = t\left(\sum_{\nu_{\alpha+1} \leq \sigma \leq \mu_{\alpha+1}} Rb_\sigma\right)$, we have

$$\mathcal{T}_J^\alpha = A_\alpha \oplus t\left(\sum_{\mu_\alpha \leq \sigma \leq \nu_{\alpha+1}} Rb_\sigma\right) \oplus K_{\alpha+1} \oplus B_{\alpha+1}$$

Let $\eta_{\alpha+1}$ be the canonical projection

$$\eta_{\alpha+1} : \mathcal{T}_J^\alpha \to A_\alpha \oplus t\left(\sum_{\mu_\alpha \leq \sigma \leq \nu_{\alpha+1}} Rb_\sigma\right) \oplus B_{\alpha+1} = A_{\alpha+1} \oplus B_{\alpha+1} = \mathcal{T}_J^{\alpha+1},$$

defining $A_{\alpha+1} = A_\alpha \oplus t\left(\sum_{\mu_\alpha \leq \sigma < \nu_{\alpha+1}} Rb_\sigma + Rb_{\mu_{\alpha+1}}\right)$, as $\ker \eta_{\alpha+1} = K_{\alpha_1}$ and, if

b_τ is a generator of $\mathcal{T}_J^\alpha$, $\eta_{\alpha+1} b_\tau = r_{\mu_{\alpha+1}} r_\tau^{-1} b_{\mu_{\alpha+1}}$, for every $\nu_{\alpha+1} \leq \tau < \mu_{\alpha+1}$,
but $\eta_{\alpha+1} b_\tau = b_\tau$ otherwise. Hence $I_{\alpha+1} = I_\alpha \cup \{\sigma | \mu_\alpha \leq \sigma < \nu_{\alpha+1}\} \cup \{\mu_{\alpha+1}\}$
and $A_{\alpha+1} = t\left(\sum_{\sigma \in I_{\alpha+1}} Rb_\sigma\right)$. Observe that $\eta_{\alpha+1}$ is the identity map on $B_{\alpha+1}$.
We are led by the intuitive idea of closing the ribs between $b_{\nu_{\alpha+1}}$ and
$b_{\mu_{\alpha+1}}$ of the fan $\mathcal{T}_J^\alpha$.

If α is a limit ordinal less than k, we choose $\mu_\alpha = \sup_{1 \leq \beta < \alpha} \mu_\beta$, which is

less than k, and we set $I_\alpha \left(\bigcup_{\beta < \alpha} I_\beta\right) \cup \{\mu_\alpha\}$. Then $\mathcal{T}_J^\alpha = A_\alpha \oplus B_\alpha$ and we de-

fine η_α as the identity map on $\mathcal{T}_J^\alpha$. In fact we choose $\nu_\alpha = \mu_\alpha$ and $\ker \eta_\alpha = K_\alpha = 0$. In any case the exact sequence

$$0 \to t\mathcal{T}_J^\alpha \to \mathcal{T}_J^\alpha \to J \to 0,$$

doesn't split, for any $1 \leq \alpha < k$.

Let $\xi_\alpha : \mathcal{T}_J \to \mathcal{T}_J^\alpha$ be the homomorphism defined as follows:

$\xi_\alpha b_\sigma = b_\sigma$ if $\mu_\beta \leq \sigma < \nu_{\beta+1}$ for   some $\beta < \alpha$,

$\xi_\alpha b_\sigma = b_\sigma$ if $\mu_\alpha \leq \sigma < k$,

$\xi_\alpha b_\sigma = r_{\mu_\beta} r_\sigma^{-1} b_{\mu_\beta}$ if $\nu_\beta \leq \sigma < \mu_\beta$ for some $\beta < \alpha$, (we suppose $\nu_1 = 1$).

We observe that $\ker \xi_\alpha = \bigoplus_{\beta \leq \alpha} K_\beta$. $\operatorname{Im} \xi_\alpha = \mathcal{T}_J^\alpha$, $\xi_{\alpha+1} = \eta_{\alpha+1} \xi_\alpha$, for every
$\alpha < k$ and that ξ_α is the identity map on B_α.

Now we have to choose the ordinals ν_α and μ_α inductively. By
induction, suppose that there are elements

$x_1, x_2, \ldots, x_\alpha \in \mathcal{T}_j^\alpha \cap (\xi_\alpha \psi M) \cap \sum_{1 \leqslant \tau < \mu_\alpha} Rb_\tau$ defined, for every $\beta \leqslant \alpha$, as follows:

$$x_\beta = s_\beta \xi_\alpha b_{\nu_\beta} = s_\beta r_{\mu_\beta} r_{\nu_\beta}^{-1} b_{\mu_\beta}$$

with $s_\beta \in R$ and, if β is not limit, with $s_\beta b_{\nu_\beta} \notin Rb_0$.

Then $Rx_\beta \cap \sum_{1 \leqslant \tau \leqslant \alpha} Rx_\tau = Rb_0$ for every $1 \leqslant \beta \leqslant \alpha$. Let $\nu_{\alpha+1}$ be an arbitrary ordinal such that $\mu_\alpha < \nu_{\alpha+1} < k$. Then there is $t_{\alpha+1} \in t\mathcal{T}_j^\alpha$ such that $b_{\nu_{\alpha+1}} + t_{\alpha+1} \in \xi_\alpha \psi M$, in fact $\mathcal{T}_j^\alpha = \xi_\alpha \mathcal{T}_j = \xi_\alpha (T + \psi M) = \xi_\alpha T + \xi_\alpha \psi M = t\mathcal{T}_j^\alpha + \xi_\alpha \psi M$ and $J = \varphi \mathcal{T}_j^\alpha = \varphi \xi_\alpha \psi M$. As $t\mathcal{T}_j^\alpha = A_\alpha \oplus t\left(\sum_{\mu_\alpha \leqslant \sigma \leqslant \nu_{\alpha+1}} Rb_\sigma\right) \oplus t\left(\sum_{\nu_{\alpha+1} \leqslant \sigma < k} Rb_\sigma\right)$, by remark 1, then $t_{\alpha+1} = t'_{\alpha+1} + t''_{\alpha+1}$ with $t'_{\alpha+1} \in A_\alpha \oplus t\left(\sum_{\mu_\alpha \leqslant \sigma \leqslant \nu_{\alpha+1}} Rb_\sigma\right)$ and $t''_{\alpha+1} \in t\left(\sum_{\nu_{\alpha+1} \leqslant \sigma < \mu_{\alpha+1}} Rb_\sigma\right)$ for some ordinal $\mu_{\alpha+1}$ such that $\nu_{\alpha+1} \leqslant \mu_{\alpha+1} < k$.

Let $Rs_{\alpha+1} = \operatorname{Ann}_R t'_{\alpha+1}$; then $s_{\alpha+1} b_{\nu_{\alpha+1}} \notin Rb_0$ as $t'_{\alpha+1} \in A_\alpha \oplus t\left(\sum_{\mu_\alpha \leqslant \sigma \leqslant \nu_{\alpha+1}} Rb_\sigma\right)$ and then $Rs_{\alpha+1} > Rr_{\nu_{\alpha+1}}$. We now obtain $s_{\alpha+1} \eta_{\alpha+1} (b_{\nu_{\alpha+1}} + t_{\alpha+1}) \in \xi_{\alpha+1} \psi M$ and, by the definition of $\eta_{\alpha+1}$ given above,

$$s_{\alpha+1} \eta_{\alpha+1} (b_{\nu_{\alpha+1}} + t_{\alpha+1}) = s_{\alpha+1} \eta_{\alpha+1} b_{\nu_{\alpha+1}} + s_{\alpha+1} t'_{\alpha+1} =$$

$$= s_{\alpha+1} \eta_{\alpha+1} b_{\nu_{\alpha+1}} = s_{\alpha+1} \xi_{\alpha+1} b_{\nu_{\alpha+1}},$$

as $\xi_\alpha b_{\nu_{\alpha+1}} = b_{\nu_{\alpha+1}}$, because $\xi_\alpha \upharpoonright B_\alpha$ is the identity map. Hence, if we set $y = s_{\alpha+1} b_{\nu_{\alpha+1}}$, then $\eta_{\alpha+1} y = s_{\alpha+1} \eta_{\alpha+1} b_{\nu_{\alpha+1}} = s_{\alpha+1} \xi_{\alpha+1} b_{\nu_{\alpha+1}} \in \xi_{\alpha+1} \psi M \setminus Rb_0$ as $\xi_{\alpha+1} b_0 = b_0$ and $\ker \xi_{\alpha+1} < T$. Now we can observe that $\sum_{1 \leqslant \beta \leqslant \alpha} Rx_\beta \leqslant \mathcal{T}_j^\alpha \cap \sum_{1 \leqslant \sigma \leqslant \mu_\alpha} Rb_\sigma$ implies $Ry \cap \sum_{1 \leqslant \beta \leqslant \alpha} Rx_\beta = Rb_0$. Moreover, as $\sum_{1 \leqslant \beta \leqslant \alpha} Rx_\beta + Ry \leqslant \sum_{1 \leqslant \sigma \leqslant \nu_{\alpha+1}} Rb_\sigma$, $\ker \eta_{\alpha+1} \cap \left(\sum_{1 \leqslant \beta \leqslant \alpha} Rx_\beta + Ry\right) \leqslant \ker \eta_{\alpha+1} \cap t\left(\sum_{1 \leqslant \sigma \leqslant \nu_{\alpha+1}} Rb_\sigma\right) = 0$. Hence the restriction of $\eta_{\alpha+1}$ to $\sum_{1 \leqslant \beta \leqslant \alpha} Rx_\beta + Ry$ is a monomorphism (with $\eta_{\alpha+1} b_0 = b_0$). Therefore, if we define $x_{\alpha+1} = \eta_{\alpha+1} y$, then $\sum_{1 \leqslant \beta \leqslant \alpha+1} Rx_\beta \leqslant \xi_{\alpha+1} \psi M \cap \sum_{1 \leqslant \sigma \leqslant \mu_{\alpha+1}} Rb_\sigma$, as $x_{\alpha+1} = \eta_{\alpha+1} y = s_{\alpha+1} r_{\mu_{\alpha+1}} r_{\nu_{\alpha+1}}^{-1} b_{\mu_{\alpha+1}}$ and $\eta_{\alpha+1} x_\beta = x_\beta$ for $1 \leqslant \beta \leqslant \alpha$. Hence

$$Rx_\beta \cap \sum_{\substack{1 \leqslant \tau \leqslant \alpha+1 \\ \tau \neq \beta}} Rx_\tau = Rb_0,$$

for $1 \leqslant \beta \leqslant \alpha+1$.

If α is a limit ordinal less than k, we choose $x_\alpha = b_0 = r_{\nu_\alpha} b_{\nu_\alpha}$,

where $\nu_\alpha = \mu_\alpha = \sup_{\beta < \alpha} \mu_\beta$, and we set $s_\alpha = r_{\nu_\alpha}$. Then $x_\alpha \in \xi_\alpha \psi M$ and the set $\{x_\beta\}_{\beta \leq \alpha}$ satisfies the conditions of the inductive hypothesis.

Let ξ be the endomorphism of $\mathcal{J}_J$ defined as follows:

$$\xi b_\sigma = \begin{cases} b_\sigma & \text{if } \mu_\beta \leq \sigma < \nu_{\beta+1} \text{ for some } \beta < k, \\ r_{\mu_\beta} r_\sigma^{-1} b_{\mu_\beta} & \text{if } \nu_\beta \leq \sigma < \mu_\beta \text{ for some } \beta < k. \end{cases}$$

Then, for every $\alpha < k$, $\xi = \chi_\alpha \xi_\alpha$, where χ_α is the homomorphism $\mathcal{J}_J^\alpha \to \mathcal{J}_J$ defined by: $\chi_\alpha b_\sigma = r_{\mu_\beta} r_\sigma^{-1} b_{\mu_\beta}$ if $\nu_\beta \leq \sigma < \mu_\beta$ for some $\alpha < \beta < k$ and $\chi_\alpha b_\sigma = b_\sigma$ otherwise. Let $\mathcal{J}_J^k = \xi \mathcal{J}_J$. Hence $\sum_{1 \leq \alpha < k} Rx_\alpha \leq \mathcal{J}_J^k \cap \xi \psi M$, as, for every $\alpha < k$, $x_\alpha \in \xi_\alpha \psi M$ and $x_\alpha = \chi_\alpha \xi_\alpha \psi M = \xi \psi M$. Let $\rho : \mathcal{J}_J^k \to \mathcal{J}_J^k / Rb_0$ be the natural epimorphism; then $\rho\left(\sum_{1 \leq \alpha < k} Rx_\alpha\right) \cong \bigoplus_{1 \leq \alpha < k} (Rx_\alpha / Rb_0)$ and the set of the non-zero summands has cardinality k.

Hence $\rho \xi \psi M$ has Goldie dimension k which is greater than the rank of M and we have a contradiction, as M is torsion free. ∎

As in [4], a homomorphism $\varphi : A \to B$ is called a *-*homomorphism* if every uniserial submodule of A is mapped into a cyclic submodule of B.

LEMM 1.2. *Let A be a pure submodule of the torsion free module B and let C be a pure-injective torsion free module. Then every *-homomorphism $\varphi \in \mathrm{Hom}(A, C)$ can be extended to a *-homomorphism $\psi \in \mathrm{Hom}(B, C)$.*

PROOF. Let $A = A_0 < A_1 < \dots < A_\alpha < \dots < A_k = B$ be a continuous ascending chain of pure submodules of B such that $rk(A_{\alpha+1}/A_\alpha) = 1$ for every $\alpha < k$. Then either A_α is a summand of $A_{\alpha+1}$ or A_α is pure essential in $A_{\alpha+1}$.

Suppose, by induction, that a *-homomorphism $\psi_\alpha : A_\alpha \to C$, which extends φ, is defined. Then, if $A_{\alpha+1} = A_\alpha \oplus J_{\alpha+1}$, for some rank 1 submodule $J_{\alpha+1}$ of $A_{\alpha+1}$, we define $\psi_{\alpha+1} \uparrow A_\alpha = \psi_\alpha$ and $\psi_{\alpha+1} \uparrow J_{\alpha+1} = 0$; moreover, if A_α is pure essential in $A_{\alpha+1}$, every homomorphism $\psi_{\alpha+1} \in \mathrm{Hom}(A_{\alpha+1}, C)$, which extends ψ_α, is a *-homomorphism by Lemma 4.2 of [4]. If λ is a limit ordinal, we define $\psi_\lambda = \bigcup_{\alpha < \lambda} \psi_\alpha$, which is a *-homomorphism. Then $\psi = \psi_k$ is a *-homomorphism. ∎

Now we need two easy preparatory lemmas. We recall a result by Facchini in [3] on the pure-injective hull of an ideal in R:

If J is a proper ideal of a valuation domain R, $J^* = \{r \in R | rJ \neq J\}$ is the prime ideal determined by J, R_{J^*} is the localization of R at J^* and $\tilde{R}_{J^*}$ is a maximal immediate extension of R_{J^*}, then $J\tilde{R}_{J^*}$ is a pure injective hull of the R-module J, i.e., $\hat{J} = J\tilde{R}_{J^*}$.

From now on, without loss of generality, we can suppose that $J = \sum\limits_{\sigma < k} Rg_\sigma$, where k is a regular cardinal and, for every $\sigma < \tau < k$, $g_\sigma g_\tau^{-1} \in J^*$. In fact, if for every σ and τ such that $\bar{\sigma} < \sigma < \tau < k$, for some fixed $\bar{\sigma}$, we have $g_\sigma g_\tau^{-1} \in R \setminus J^*$, then $J \sim J' \nleq J^*$, which is not true (see page 15 of [5]).

As $J < R$ we can write the non-splitting sequence

$$0 \to T \to \mathcal{J}_J \xrightarrow{\varphi'} J \to 0$$

similar to the one above, where $\varphi' = r\varphi$ for some $r \in R$. In any case, we will write φ instead of φ' to simplify the notation.

LEMMA 1.3. *If $E : 0 \to T \to \mathcal{J}_J \xrightarrow{\varphi} J \to 0$ is the non-splitting sequence defined above and $S = \tilde{R}_{J^*}$, then the pure exact sequence*

$$E' : 0 \to T \otimes_R S \to \mathcal{J}_J \otimes_R S \xrightarrow{\varphi \otimes \iota_S} J \otimes_R S \to 0,$$

is a non-splitting sequence of S-modules which is like E, but over the domain S instead of R (ι_S is the identity map on S).

PROOF. $J \otimes_R R_{J^*} \cong JR_{J^*}$ is a k-generated R_{J^*}-ideal and then $J \otimes_R S$ is a k-generated S-ideal (ex. 5, page 6 of [5]). $\mathcal{J}_J \otimes_R S$ is generated by $b_\sigma \otimes s$ for every $\sigma < k$ and $s \in S$ and it can be considered as an S-module by defining $s'(b_\sigma \otimes s) = b_\sigma \otimes s's$, for every $s' \in S$. As $b_\sigma \otimes s = s(b_\sigma \otimes 1_S)$, $\mathcal{J}_J \otimes_R S$ is a fan S-module generated by $\{Rb_\sigma\}_{\sigma < k}$ and, as $T \otimes_R S$ is a torsion R-module, $T \otimes_R S$ is a torsion S-module (as $R < S$) generated by $(r_\sigma r_\tau^{-1} b_\tau - b_\sigma) \otimes 1_S$ for every $0 \leqslant \sigma < \tau < k$. For every $t \in T$ with $\text{Ann}_R(t) = Rr$, $\text{Ann}_S(t) = Sr$, as S is a domain. As S is flat, the sequence E' is exact and it does not split, as a sequence of S-modules, because of the same arguments for which the sequence E does not split. ∎

Now we recall the following well-known lemma

LEMMA 1.4. *Let N be a torsion free R-module and S be a valuation domain such that $R < S$. Then $rk_R N = rk_S(N \otimes_R S)$, considering $N \otimes_R S$ as an S-module.*

THEOREM 1.5. *Let R be a valuation domain and Q be its field of quotients. Let B be a reduced Butler module, N be a rank k (cardinal) pure submodule of B, J be a k'-generated submodule of Q and $\pi: N \to J$ be an epimorphism. If $k' > k$, then $\ker \pi$ is a summand of N.*

PROOF. Let $k' > k$. If $J = Q$, by way of contradiction, suppose that $\ker \pi$ is not a summand of N. Then π is a $*$-homomorphism which can be extended to a $*$-epimorphism $\pi': B \to J$ by lemma 1.2. Write the non-splitting sequence defined above $0 \to T \to \mathcal{J}_J \xrightarrow{\varphi} J \to 0$ and form the pull-back diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & M & \longrightarrow & B & \longrightarrow & 0 \\
& & \| & & \downarrow{\scriptstyle\psi}\;\nearrow & & \downarrow{\scriptstyle\pi'} & & \\
0 & \longrightarrow & T & \longrightarrow & \mathcal{J}_J & \xrightarrow{\varphi} & J & \longrightarrow & 0.
\end{array}
$$

As π' is a $*$-homomorphism, by lemma 1.2 of [4], the top row is balanced, and so it splits, as B is Butler. Then we have a homomorphism $\psi: B \to \mathcal{J}_J$ such that $\pi' = \varphi\psi$ and moreover $\pi = \varphi(\psi \upharpoonright N)$. By lemma 1.1 we obtain the contradiction.

If $J \neq Q$, we can suppose that $J < R$ and we can write, as observed in lemma 1.3, the non-splitting sequences E and E', where J is k'-generated, $S = \tilde{R}_{J*}$ and $J \otimes_R S$ is the pure injective hull of J. If, by way of contradiction, $\ker \pi$ is not a summand of N, then π is a $*$-R-homomorphism and, if $v: J \to J \otimes_R S$ is the monomorphism defined by setting $v(g) = g \otimes_R 1_S$, for every $g \in J$, then $v\pi: N \to J \otimes_R S$ is a $*$-R-homomorphism, which, by lemma 1.2, can be extended to a $*$-R-homomorphism $\pi': B \to J \otimes_R S$. Now, considering the sequence E' as a sequence of R-modules, we can form the pull-back diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T \otimes_R S & \longrightarrow & H & \longrightarrow & B & \longrightarrow & 0 \\
& & \| & & \downarrow{\scriptstyle\psi}\;\nearrow & & \downarrow{\scriptstyle\pi'} & & \\
0 & \longrightarrow & T \otimes_R S & \longrightarrow & \mathcal{J}_J \otimes_R S & \xrightarrow{\varphi \otimes \iota_S} & J \otimes_R S & \longrightarrow & 0.
\end{array}
$$

The top row splits, as, by lemma 1.2 of [4], it is balanced and B is Butler. Hence there is an R-homomorphism $\psi: B \to \mathcal{J}_J \otimes_R S$ such that $\pi' = (\varphi \otimes_R \iota_S)\psi$. In particular $v\pi = (\varphi \otimes_R \iota_S)(\psi \upharpoonright N)$. Now we define $\bar\psi: N \otimes_R S \to \mathcal{J}_J \otimes_R S$ as $\bar\psi(n \otimes s) = s\psi(n)$, for every $n \in N$ and $s \in S$. We observe that $(\varphi \otimes_R \iota_S)\bar\psi = \pi \otimes_R \iota_S$ as S-homomorpohisms; in fact $(\varphi \otimes_R \iota_S)\bar\psi(n \otimes s) = (\varphi \otimes_R \iota_S)(s\psi(n)) = s(\varphi \otimes_R \iota_S)(\psi(n)) = sv\pi(n) = = s(\pi(n) \otimes 1_S) = \pi(n) \otimes s$, for every $n \in N$ and $s \in S$. This means that

$(\varphi \otimes_R \iota_S)\bar{\psi}$ is an S-epimorphism and we have the following commutative diagram of S-modules

$$0 \longrightarrow T \otimes_R S \longrightarrow \mathcal{T}_J \otimes_R S \underset{\varphi \otimes \iota_S}{\longrightarrow} J \otimes_R S \longrightarrow 0.$$

with $N \otimes_R S$ mapping via $\bar{\psi}$ and $\pi \otimes \iota_S$.

As, by lemma 1.4, $\mathrm{rk}_S (N \otimes_R S) = \mathrm{rk}_R N < k'$, we have a contradiction by lemmas 1.3 and 1.1. ∎

By theorem 1.5 we infer that all finite rank pure submodules of Butler modules are completely decomposable and then, like in theorem 2.2 of [4], we conclude:

COROLLARY 1.6. *Countable rank pure submodules of Butler modules are completely decomposable. In particular, countable rank Butler modules are completely decomposable.* ∎

2. – Striped modules.

A torsion free R-module M is *striped* if, for all of its reduced pure submodules N of rank k (cardinal), each epimorphism $f: N \to J$ is splitting, whenever J is a rank one k'-generated R-module with $k' > k$.

By theorem 1.5, Butler modules, hence completely decomposable modules, are striped.

It is obvious that *pure submodules of striped modules are striped* and, from the proof of corollary 1.6, we infer that *countable rank pure submodules of striped modules are completely decomposable.*

PROPOSITION 2.1. *Every finite rank pure submodule of a striped module is balanced.*

PROOF. Let F be a finite rank pure submodule of the striped module M. Let I be any rank one pure submodule of M/F and p be the canonical epimorphism $p: M \to M/F$.

Then $p^{-1}(I)$ is a finite rank pure submodule of M and is completely decomposable, as M is striped. Hence F is a summand of $p^{-1}(I)$, for every I, *i.e.*, F is balanced in M. ∎

PROPOSITION 2.2. *Let $0 \to F \hookrightarrow H \xrightarrow{f} M \to 0$ be an exact sequence, where H is striped, $\mathrm{rk} F$ is finite and M is torsion free. Then M is striped too.*

PROOF. For every epimorphism $g: N \to J$, where N is a reduced pure submodule of M and $\mathrm{rk}\, J = 1$, there is an epimorphism $g': f^{-1}(N) \to J$ such that $g' = gf$. If $\mathrm{rk}\, N$ is finite, then $\mathrm{rk}\, f^{-1}(N)$ is finite too and g' splits. In fact, if $f^{-1}(N) = N' \oplus D$, where N' is reduced and D is divisible, then $D \leqslant \ker f$, as N is reduced, and the epimorphism $g' \upharpoonright N$ splits, as H is striped. Hence g splits, as $\ker g' > \ker f$. For the same reason, if $\mathrm{rk}\, N = k$ (infinite cardinal), then $\mathrm{rk}\, f^{-1}(N) = k$ and, if g' splits, g splits as well. ∎

Now we observe that the class of striped modules *properly* contains the class of Butler modules; in fact the second one is not closed even for balanced submodules. To prove such a claim, we need two results due to L. Fuchs.

PROPOSITION 2.3. (L. Fuchs) *If M is Butler and has a free basic submodule B, then M is free.*

PROOF. In this case, as B is projective, $\mathrm{Bext}^1(M, T) = \mathrm{Ext}^1(M, T)$ for every torsion module T. Hence M is Butler if and only if M is Baer and therefore M is free (see [2]). ∎

At this point, we give an example of a balanced submodule of a free module which is not Butler, so that it is not completely decomposable.

EXAMPLE 2.4. (L. Fuchs) Let p.d. $Q = 3$. We have a pure exact sequence

$$0 \to F_3 \to F_2 \to F_1 \xrightarrow{\beta} F_0 \xrightarrow{\alpha} Q \to 0,$$

where F_i, for $0 \leqslant i \leqslant 3$, is free. Consider the exact sequence

$$0 \to \ker \beta \to F_1 \to \ker \alpha \to 0,$$

where p.d. $\ker \alpha = 2$ and p.d. $\ker \beta = 1$. It is balanced, in fact $\ker \alpha$ is separable, as a pure submodule of a free module, and then every rank one pure submodule is cyclic. Likewise $\ker \beta$ is a balanced submodule of a free module and then its basic submodules are free, but it is not Butler by proposition 2.3, as it is not free (p.d. $\ker \beta = 1$). ∎

PROBLEMS:

2.1. In chap. XIV of [5] the properties of separable torsion free modules are studied and we can observe that there are many analogies with the properties of striped modules. Are the classes of torsion free separable modules and of striped modules equal?

2.2. The class of completely decomposable torsion free modules is not closed even for balanced submodules (examples 2.4). Then, does the class of separable and striped modules coincide with the class of all pure submodules of completely decomposable torsion free modules?

2.3. Are Butler modules separable?

2.4. Are Butler modules completely decomposable?

3. – TEP-submodules in a Butler module.

A submodule A of an R-module B is said to have the *torsion extension property* (we say that A is a TEP-submodule of B) if, for each torsion R-module T, the map $\operatorname{Hom}(B, T) \to \operatorname{Hom}(A, T)$ induced by the inclusion $A \to B$ is surjective. This notion, introduced by Dugas-Rangaswamy in [1] investigating Butler groups, was very useful in [4] for the study of Butler modules over valuation domains of rank $> \aleph_0$.

LEMMA 3.1. *Let B be a Butler module. If N and N' are pure submodules of B such that $N < N' < B, N$ is TEP in N' and $\operatorname{rk}(N'/N) = 1$, then N is a summand of N'.*

PROOF. Suppose that N is not a summand of N'. Then, as $\operatorname{rk}(N'/N) = 1$, N is pure essential in N' and we can write the pure exact sequence $0 \to N \to N' \xrightarrow{\pi} J \to 0$ where $J = N'/N$. Consider the two non-splitting sequences E and E' which are defined in 1.3 and the monomorphism $v: J \to J \otimes_R S$ such that $v(g) = g \otimes_R 1_S$ for every $g \in J$.

By lemma 1.2, the *-homomorphism $v\pi: N' \to J \otimes_R S$ can be extended to a *-homomorphism $\pi': B \to J \otimes_R S$ and, by lemma 1.2 of [4], in the pull-back diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T \otimes_R S & \longrightarrow & H & \longrightarrow & B & \longrightarrow & 0 \\
& & \Big\| & & \Big\downarrow {\scriptstyle\psi} \!\!\!\swarrow & & \Big\downarrow {\scriptstyle\pi'} & & \\
0 & \longrightarrow & T \otimes_R S & \longrightarrow & \mathcal{T}_J \otimes_R S & \underset{\varphi \otimes \iota_S}{\longrightarrow} & J \otimes_R S & \longrightarrow & 0.
\end{array}
$$

the top row is balanced exact and hence is splitting, as B is Butler.

This means that there is a homomorphism $\psi: B \to \mathcal{T}_J \otimes_R S$ which

makes the diagram commute. Observe the following commutative diagram

$$
\begin{array}{c}
N' \\
\psi \upharpoonright N' \swarrow \quad \downarrow \nu\pi \\
0 \longrightarrow T \otimes_R S \longrightarrow \mathcal{J}_J \otimes_R S \underset{\varphi \otimes \iota_S}{\longrightarrow} J \otimes_R S \longrightarrow 0.
\end{array}
$$

As $\ker \nu\pi = N$, we have $\psi(N) \leqslant T \otimes_R S$, which is a torsion R-module and, as N is TEP in N', there is $\psi' \in \operatorname{Hom}(N', T \otimes_R S)$ such that the following diagram commutes

$$
\begin{array}{c}
N \longrightarrow N' \\
\psi \uparrow N \downarrow \quad \nearrow \psi' \\
T \otimes_R S
\end{array}
$$

Set $\chi = \psi \upharpoonright N - \psi'$. Then $\ker \chi = N$, as $\chi(N) = 0$ and $(\varphi \otimes \iota_S)\chi = (\varphi \otimes \iota_S)\psi \upharpoonright N' = \nu\pi$. Define the homomorphism $\chi' : N' \otimes_R S \to \mathcal{J}_J \otimes_R S$ such that $\chi'(x \otimes s) = s\chi(x)$, for every $x \in N'$ and $s \in S$. Then $\ker \chi' = N \otimes_R S$, $(\varphi \otimes \iota_S)\chi' = \pi \otimes \iota_S$ and $\bar{\chi} : (N' \otimes_R S)/(N \otimes_R S) \to \mathcal{J}_J \otimes_R S$ is a monomorphism defined by $\bar{\chi}(y + N \otimes_R S) = \chi'(y)$ for every $y \in N' \otimes_R S$.

As $0 \to N \otimes_R S \to N' \otimes_R S \overset{\pi \otimes \iota_S}{\longrightarrow} J \otimes_R S \to 0$ is an exact sequence, we have the isomorphism $\bar{\pi} : (N' \otimes_R S)/(N \otimes_R S) \to J \otimes_R S$ defined by $\bar{\pi}(y + N \otimes_R S) = (\pi \otimes \iota_S)(y)$ for every $y \in N' \otimes_R S$, and $\bar{\chi}\bar{\pi}^{-1}$ is a splitting map for the sequence E'. Hence we have the contradiction. ∎

THEOREM 3.2. *Let B be a Butler module and $N < M \leqslant B$ be pure submodules of B such that N is TEP in M; then N is balanced in M. Hence TEP-submodules of Butler modules are balanced.*

PROOF. N is TEP in every submodule N' such that $N \underset{\text{pure}}{<} N' \leqslant M$ and $\operatorname{rk}(N'/N) = 1$ and hence, by lemma 3.1, N is a summand of N'. Therefore the exact sequence $0 \to N \to M \overset{f}{\longrightarrow} M/N \to 0$ is balanced, as, for every rank one pure submodule J of $M/N, N$ is a summand of $f^{-1}(J)$. ∎

By lemma 3.2 and lemma 3.6 of [4] we conclude that:

THEOREM 3.3. *Let k be an uncountable regular cardinal and*

$$0 = M_0 < M_1 < \dots < M_\alpha < \dots < M \quad (\alpha < k)$$

be a well-ordered continuous ascending chain of submodules of M such that

   i) $M = \bigcup_{\alpha < k} M_\alpha$ *is torsion free,*

   ii) M_α *is pure in* $M_{\alpha+1}$, *for all* $\alpha < k$,

   iii) M_α *is a Butler module and* $\mathrm{rk}\, M_\alpha < k$, *for each* $\alpha < k$.

   If M is a Butler module, then there is a club C in k such that, for each $\alpha \in C$, M_α *is TEP and balanced in* M_β, *for every* $\beta > \alpha$. *Obviously* $M = \bigcup_{\alpha \in C} M_\alpha$ ■.

   Finally we establish a property of completely decomposable torsion free modules over valuation domains.

   PROPOSITION 3.4. *Let C be a completely decomposable torsion free R-module of infinite rank and let B be a proper basic submodule of C. Then*

$$\mathrm{p.\,d.}\,(C/B) = 1.$$

   PROOF. Suppose that p. d. $(C/B) \geq 2$. Then we have a pure exact sequence $0 \to S \hookrightarrow F \to C/B \to 0$ where F is free and p. d. $S \geq 1$. Form the pull-back diagram

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
B & = & B \\
\downarrow & & \downarrow \\
0 \to S \to & H & \to C \to 0 \\
\| & \downarrow & \downarrow \\
0 \to S \to & F & \to C/B \to 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

The top row is balanced, by lemma 1.2 of [4], and the middle column splits, as F is projective. Thus $H(\cong B \oplus F)$ is completely decomposable and then S is completely decomposable, as the top row splits (C is balanced projective). Hence S is free and we have contradiction to p.d. $S \geq 1$. ■

If we suppose that C is not completely decomposable, but just Butler, then S is TEP in H, which is completely decomposable, but we are not able to conclude that either S is Butler or S is a summand of H and to get the contradiction in both cases. (If S is Butler, S is free by proposition 2.3).

Hence we can propose the following problems:

PROBLEMS:

3.1 Are TEP-submodules of completely decomposable modules summands?

3.2 Are TEP-submodules of completely decomposable modules Butler?

REFERENCES

[1] M. DUGAS and K. M. RANGASWAMY, *Infinite rank Butler modules*, Trans. Amer. Math. Soc. **305** (1988), 129-142.
[2] P. C. EKLOF and L. Fuchs, *Baer modules over valuation domains*, Ann. Mat. Pura e Appl. **150** (1988), 363-373.
[3] A. FACCHINI, *Torsion-free covers and pure-injective envelopes over valuation domains*, Isr. J. Math. **52** (1985), 129-139.
[4] L. FUCHS and E. MONARI MARTINEZ, *Butler modules over valuation domains*, (to appear in Can J. Math. Vol. **43** (1), 1991).
[5] L. FUCHS and L. SALCE, *Modules over valuation domains*, Lecture Notes in Pure Appl. Math. **97**, Marcel Dekker, 1985.
[6] P. GRIFFITH, *A solution to the splitting mixed group problem of Baer*, Trans. Amer. Math. Soc. **139** (1969), 261-270.
[7] R. B. WARFIELD, *Purity and algebraic compactness for modules*, Pac. J. Math. **28** (1969), 699-719.