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On Multipliers of Heineken-Mohamed Type Groups.

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1. Introduction.

In this paper, a Heineken-Mohamed type group is a non-nilpotent p-group G (p a prime) in which every proper subgroup is both subnormal and nilpotent. The first example of such a group was given in [6]. Subsequently, and using a variety of techniques, several authors constructed Heineken-Mohamed type groups (see [4], [5], [7] and [9]). In particular, in [9] Meldrum constructs an uncountable number of isomorphism types of Heineken-Mohamed type groups (see also [7]). Meldrum's groups are presented as direct limits of finite p-groups, a fact which enables that author to determine the isomorphism types of the groups constructed. All of the groups G constructed in the above references are metabelian with $|\zeta(G)| \leqslant p$. The question of the existence of solvable Heineken-Mohamed groups of derived length greater than two arises in the study of certain minimality conditions (see [2; p. 50]).

In this paper, we will show that if G is any of the p-groups (p an odd prime) constructed by Meldrum, then the Schur multiplier $M(G)$ is infinite. Further, it will be shown that if H is any stem extension of $M(G)$ by G, then H is a three step solvable (and non-metabelian) group of Heineken-Mohamed type. Evidently, the

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group H has infinite center, and it is easily seen that the multiplier $M(H) = 0$.

The existence of solvable Heineken-Mohamed groups of derived length greater than three remains an open question. Also, it seems likely that any metabelian group of Heineken-Mohamed type has infinite multiplier, but we have not been able to verify that this is so. As will be evident in the sequel, our methods are amenable only to those groups which are presented as certain direct limits of finite groups; it is for this reason that we primarily discuss the groups of Meldrum.

2. Description of the groups and statement of the theorems.

We begin with a brief description of Meldrum's construction. As much as is possible, we will use the same terminology as that of [9].

Let x_3 be a positive integer satisfying

$$p^2 + p < x_3 \leqslant p^3 - 1, \tag{2.1}$$

and $\{r_k \mid k \geqslant 4\}$ a sequence of positive integers with

$$1 \leqslant r_k \leqslant p. \tag{2.2}$$

The sequence $\{x_k \mid k \geqslant 3\}$ is defined recursively by (2.1) and by

$$x_{k+1} = px_k - p + r_{k+1}. \tag{2.3}$$

For the positive integer $k \geqslant 3$, let A_k be the elementary Abelian p-group of rank x_k, and fix a basis $\{a(i,k) \mid 1 \leqslant i \leqslant x_k\}$ of A_k.

The group

$$B_k = \langle b_k \rangle, \qquad (b_k)^{p^k} = 1, \tag{2.4}$$

is a cyclic group of order p^k which acts on A_k via

$$a(i,k)^{b_k} = a(i,k) + a(i-1,k) \quad \text{if} \quad 1 < i \leqslant x_k \tag{2.5}$$

and

$$a(1,k)^{b_k} = a(1,k).$$

Equivalently, $[a(i,k), b_k] = a(i-1,k)$ if $1 < i \leqslant x_k$, and $[a(1,k), b_k] = 1$

It is shown in [9; p. 438] that B_k acts faithfully on A_k.

Now, let

(2.6) $D_k = \langle a(x_k - 1, k), b_k \rangle = \langle \{a(i, k), 1 \leqslant i \leqslant x_k - 1\}, b_k \rangle$;

the function $\theta_k : D_k \to D_{k+1}$ defined by

(2.7) $\theta_k(a(i,k)) = a(pi - p + r_{k+1}, k+1)$, and

$$\theta_k(b_k) = (b_{k+1})^p a(x_{k+1} - p + 1, k + 1)$$

is an embedding of D_k into D_{k+1}. The direct limit

$$G = G(\{r_k\}) = \varinjlim \{D_k, \theta_k\}$$

is a group of Heineken-Mohamed type, and $G(\{r_k\})$ is isomorphic to $G(\{(r^*)_k\})$ if and only if the sequences $\{r_k\}$ and $\{(r^*)_k\}$ are equal [9; pp. 440-444]. Thus, the direct limits $G(\{r_k\})$ provide an uncountable number of isomorphism types of groups of Heineken-Mohamed type. These groups are clearly metabelian. Further, Meldrum shows that $G = G(\{r_k\})$ has a non-trivial center if and only if there is a k_0 such that for all $k \geqslant k_0$, $r_k = 1$. Finally, it is not difficult to show that for odd primes, the sequence provided by

$$x_3 = p^3 - p^2 \quad \text{and} \quad r_k = p \quad \text{for} \quad k \geqslant 4,$$

yields a $G(\{r_k\})$ which is isomorphic to the group constructed in [6].

We now state our two principal results:

THEOREM 1. *For any odd prime p and any sequence* $\{r_k\}$ *as described above, the group* $G = G(\{r_k\})$ *has infinite multiplier* $M(G)$*: further,* $M(G)$ *is an elementary Abelian p-group.*

THEOREM 2. *For any of the groups G in Theorem 1, let H be a stem extension of* $M(G)$ *by G. Then H is a three-step solvable and non-metabelian group of Heineken-Mohamed type.*

COROLLARY. *For each odd prime p, there are an uncountable number of isomorphism types of p-groups G which have each of the following properties.*

(i) *G is three-step solvable;*

(ii) *G is non-metabelian;*

(iii) *G is of Heineken-Mohamed type;*

(iv) *G has infinite center.*

Theorems 1 and 2 will be proved in the following Section 3; the Corollary is an obvious consequence of Theorems 1 and 2. The results embodied in Theorems 1 and 2 remain open for the prime $p = 2$.

3. Additional notation and proofs.

The essential ingredient of the proof is to apply the direct limit property of multipliers; *i.e.*, the multiplier of a direct limit is a direct limit of the multipliers [1: p. 57]. In Section 3.1, we develop the notation necessary to apply this tecnique: in subsequent subsections, we use the methods of § 3.1 in the proofs of the theorems.

3.1. *Notation.* – The spirit of what follows is as in [8; pp. 296-300]. We assume the notation of Section 2: in particular, we assume that the sequences $\{x_k\}$ and $\{r_k\}$ satisfy the conditions (2.1), (2.2), and (2.3) and that

$$D_k = \langle a(x_k - 1, k), b_k \rangle = \langle \{a(i,k), 1 \leqslant i \leqslant x_k - 1\}, b_k \rangle.$$

Throughout, we will use the generators

$$\{t(b_k), s(a(1,k)), \ldots, s(a(x_k - 1, k))\}$$

and relations

$$(3.1.1)\quad \begin{cases} u(b_k) = (t(b_k))^{p^k}; & \\ v(a(i,k)) = s(a(i,k))^p, & 1 \leqslant i \leqslant x_k - 1; \\ w(a(i,k), a(j,k)) = [s(a(i,k)), s(a(j,k))], & 1 \leqslant i < j \leqslant x_k - 1; \\ y(b_k, a(i,k)) = & \\ \quad = s((i,k))^{t(b_k)} s(a(i,k))^{-1} s(a(i-1,k))^{-1} \text{ if } 1 < i \leqslant x_k - 1; & \\ y(b_k, a(1,k)) = s(a(1,k))^{b_k} s(a(1,k))^{-1} & \end{cases}$$

as a presentation of D_k.

Thus, if F_k is free on the generators

$\{t(b_k), s(a(1,k)), \ldots, s(a(x_k - 1, k))\}$ and $\alpha_k : F_k \to D_k$ is the homomorphism defined by

$$t(b_k)\alpha_k = b_k \quad \text{and} \quad s(a(i,k))\alpha_k = a(i,k),$$

then $\ker(\alpha_k) = R_k = \langle u(b_k),\ \{v(a(i,k)), y(b_k, a(i,k)) | 1 \leqslant i \leqslant x_k - 1\},$

$$\{w(a(i,k), a(j,k)) | 1 \leqslant i < j \leqslant x_k - 1\} \rangle^{F_k}.$$

Further, there is an embedding $\eta_k : F_k \to F_{k+1}$ defined by

(3.1.2) $\quad (t(b_k))\eta_k = (t(b_{k+1}))^p s(a(x_{k+1} - p + 1, k+1)),$ and

$$s(a(i,k))\eta_k = s(a(pi - p + r_{k+1}, k+1)),$$

and it is easily checked that the following diagram is commutative

$$\begin{array}{ccccc} R_k & \xrightarrow{\mathrm{id}} & F_k & \xrightarrow{\alpha_k} & D_k \\ \downarrow \eta_k & & \downarrow \eta_k & & \downarrow \theta_k \\ R_{k+1} & \xrightarrow[\mathrm{id}]{} & F_{k+1} & \xrightarrow[\alpha_{k+1}]{} & D_{k+1}. \end{array}$$

Denote the Schur multiplier of D_k by $M(D_k)$; the above diagram implies (see [1; Section 1.3]) that the homomorphism

$$\pi_k : M(D_k) = R_k \cap F'_k / [R_k, F_k] \to M(D_{k+1}) = R_{k+1} \cap F'_{k+1} / [R_{k+1}, F_{k+1}]$$

induced by θ_k (see 2.7) is given by

(3.1.3) $\quad (g[R_k, F_k])\pi_k = (g\eta_k)[R_{k+1}, F_{k+1}]$ for all $g \in R_k \cap F'_k$.

Finally, $M(G) = M(G(\{r_k\}))$ = the multiplier of G, is given by

$$M(G) = \varinjlim (M(D_k), \pi_k)$$

(see [1; pp. 56-57]).

3.2. *The multipliers of the finite groups D_k.* – In the free group F_k, let

$$U_k = \langle s(a(1,k)), \ldots, s(a(x_k - 1, k)) \rangle,$$

and

$$S_k = \langle \{w(a(i,k), a(j,k)), 1 \leqslant i < j \leqslant x_k - 1\}, \{v(a(i,k)), 1 \leqslant i \leqslant x_k - 1\} \rangle^{U_k}$$

(see (3.1.1)).

Then U_k/S_k is isomorphic to the elementary Abelian group

$$E_k = \langle a(i,k) | 1 \leqslant i \leqslant x_k - 1 \rangle \subseteq D_k = E_k \rtimes j \langle b_k \rangle .$$

Further,

$$W_k = U_k' / [S_k , U_k] \tag{3.2.1}$$

is isomorphic to the Schur multiplier of E_k: also, with

$$y_{i,j}(k) = [s(a(i,k), s(a(j,k))][S_k , U_k], \tag{3.2.2}$$

the elements $\{y_{i,j}(k) | 1 \leqslant i < j \leqslant x_k - 1\}$ form a basis for the group W_k.

The automorphism b_k of E_k induces an endomorphism β_k of U_k: here

$$s(a(i,k))^{\beta_k} = s(a(i,k)) s(a(i-1,k)) \quad \text{if } 1 < i \leqslant x_k - 1,$$

and

$$s(a(1,k))^{\beta_k} = s(a(1,k)).$$

The subgroups U_k' and $[S_k , U_k]$ are both β_k-invariant, and β_k induces an automorphism (also called β_k) on the quotient W_k. In particular, we have (in additive notation)

$$\begin{cases} y_{i,j}(k)^{\beta_k} = [s(a(i,k))^{\beta_k}, s(a(j,k))^{\beta_k}][S_k , U_k] = \\ = y_{i,j}(k) + y_{i,j-1}(k) + \\ + y_{i-1,j}(k) + y_{i-1,j-1}(k) \;\; \text{if } 1 < i \text{ and } i+1 < j; \\ y_{i,j}(k)^{\beta_k} = y_{i,j}(k) + y_{i-1,j}(k) + y_{i-1,j-1}(k) \;\; \text{if } 1 < i \text{ and } i+1 = j; \\ y_{i,j}(k)^{\beta_k} = y_{1,j}(k) + y_{1,j-1}(k) \;\; \text{if } i = 1 \text{ and } 2 < j; \\ y_{i,j}(k)^{\beta_k} = y_{1,2}(k) \;\; \text{if } i = 1 \text{ and } j = 2. \end{cases} \tag{3.2.3}$$

The equations (3.2.3) can be stated in the following, sometimes more convenient, form:

$$(3.2.4)\qquad \begin{cases} [y_{i,j}(k), \beta_k] = y_{i,j}(k)^{(\beta_k - 1)} = \\ (a) = \tau_{i,j} = y_{i,j-1}(k) + \\ \qquad + y_{i-1,j-1}(k) + y_{i-1,j}(k) \ \text{ if } 1 < i \text{ and } i+1 < j \leqslant x_k - 1; \\ (b) = \sigma_{i,j} = y_{i-1,j}(k) + \\ \qquad + y_{i-1,j-1}(k) \ \text{ if } 1 < i \text{ and } i+1 < j \leqslant x_k - 1; \\ (c) = \rho_j = y_{1,j-1} \ \text{ if } i = 1 \text{ and } 2 < j; \\ (d) = 0 \ \text{ if } i = 1 \text{ and } j = 2. \end{cases}$$

It has been shown by Evens [3; pp. 170-171] that, for odd primes p, the group $W_k/[W_k, \beta_k]$ is isomorphic to a subgroup of $M(D_k) = R_k \cap F_k'/[R_k, F_k]$; specifically if we put

$$(3.2.5)\qquad v_{i,j}(k) = [s(a(i,k)), s(a(j,k))][R_k, F_k],$$

the mapping

$$(3.2.6)\qquad \upsilon_k : W_k/[W_k, \beta_k] \to R_k \cap F_k'/[R_k, F_k] = M(D_k)$$

$$\text{defined by } (y_{i,j}(k)[W_k, \beta_k])\upsilon_k = v_{i,j}(k)$$

is an embedding (this is the only point in all of our proofs where we use the fact that p is odd). We are able to determine a basis and to derive other specific information regarding the vector space $W_k/[W_k, \beta_k]$. This, in conjunction with (3.2.6), will give considerable information about the multiplier $M(D_k)$ and the homomorphism π_k of (3.1.3). We collect various properties of the vector space $W_k/[W_k, \beta_k]$ in the following

LEMMA 1. *Put* $n = x_k - 1$ *and* $w_{i,j} = y_{i,j}(k)[W_k, \beta_k]$:

(i) *If* $n = 2m$, *then* $w_{i,j} = 0$ *if* $i \leqslant m - 1$ *and* $j \leqslant n - i$. *If* $n = 2m + 1$, *then* $w_{i,j} = 0$ *if* $i \leqslant m$ *and* $j \leqslant n - i$.

(ii) *If* $n = 2m$, *then* $\{w_{i,i+1} \mid m \leqslant i \leqslant n - 1\}$ *is a basis for* $W_k/[W_k, \beta_k]$; *thus,* $\dim(W_k/[W_k, \beta_k]) = m$. *If* $n = 2m + 1$, *then* $\{w_{i,i+1} \mid m + 1 \leqslant i \leqslant n - 1\}$ *is basis for* $W_k/[W_k, \beta_k]$ *thus,* $\dim(W_k/[W_k, \beta_k]) = m$.

(iii) *Consider the element* $w_{r,r+k}$ *where* $1 \leqslant k \leqslant n - r$, $r \geqslant 1$. *Then* *there are elements* $\{e_i(r,k)\}$ *in* $GF(p)$, *the field of* p *elements, such that:*

$$(3.2.7) \qquad w_{r,r+k} = \pm w_{r,r+1} + \sum\{e_i(r,k)w_{i,i+1} \,|\, r+1 \leqslant i \leqslant r+s_k\}$$

$$\text{where } s_k = \frac{k}{2} \text{ if } k \text{ is even, and } s_k = \frac{k+1}{2} \text{ if } k \text{ is odd.}$$

(iv) *Let* $\{\gamma_1,\ldots,\gamma_t\}$ *and* $\{\tau_1,\ldots,\tau_t\}$ *be sets of positive integers satisfying the following three conditions:*

(a) $\gamma_1 \geqslant m$ *if* $n = 2m$ *and* $\gamma_1 \geqslant m+1$ *if* $n = 2m+1$; $\gamma_i < \tau_i$ *for* $i = 1,\ldots,t$.

(b) $\gamma_{i+1} > \gamma_i + \mu_i$ *where* $\mu_i = \dfrac{\tau_i - \gamma_i}{2}$ *if* $\tau_i - \gamma_i$ *is even and* $\mu_i = \dfrac{\tau_i - \gamma_i + 1}{2}$ *if* $\tau_i - \gamma_i$ *is odd.*

(c) $\tau_t \leqslant n$.

Then $\{w_{\gamma_i,\tau_i} \,|\, 1 \leqslant i \leqslant t\}$ *is an independent set.*

Proof. Throughout this proof we replace $y_{i,j}(k)$ by $y_{i,j}$; recall also that $y_{i,j}$ is defined for $1 \leqslant i < j \leqslant n$. The equations (3.2.4) can be translated to read

$$(\alpha)\ w_{i,j-1} + w_{i-1,j-1} + w_{i-1,j} = 0 \qquad \text{if } 1<i \text{ and } i+1<j\leqslant n;$$

$$(\beta)\ w_{i-1,i+1} + w_{i-1,i} = 0 \qquad \text{if } 1<i\leqslant n-1;$$

$$(\delta)\ w_{1,j} = 0 \qquad \text{if } 2\leqslant j\leqslant n-1.$$

Part (i) of the lemma follows easily from (α), (β) and (δ). We now move to the proof of part (iii). Notice first that for all relevant r, $w_{r,r+2} = -w_{r,r+1}$; thus the equations (3.2.7) hold for $k=1$ and $k=2$ (for all r). Suppose (3.2.7) true for $2 \leqslant h < k$ (with h in place of k in (3.2.7)); we complete the inductive proof by showing that (3.2.7) holds for k. By (α) above, with $i-1=r$ and $j=r+k$, the induction hypothesis yields

$$w_{r,r+k} = -w_{r,r+k-1} - w_{r+1,r+1+k-2} =$$

$$-(\pm w_{r,r+1} + \sum(e_i(r,k-1)w_{i,i+1} \,|\, r+1 \leqslant i \leqslant r+s_{k-1}\} -$$

$$-(\pm w_{r+1,r+2} + \sum(e_i'(r+1,k-2)w_{i,i+1} \,|\, r+2 \leqslant i \leqslant r+1+s_{k-2}\}.$$

Now, if k is even, we have $r+s_{k-1} = r+1+s_{k-2} = r+s_k$, while, if k is odd, $r+s_{k-1} < r+1+s_{k-2} = r+s_k$. It is then clear that the above sum is of the desired type and this completes the proof of part (iii).

Observe that part (i), together with the equations (3.2.7), show that the set of vectors in (ii) (of cardinality m) forms a spanning set of $W_k/[W_k, \beta_k]$. If we can show that $W_k/[W_k, \beta_k]$ contains a set of m independent vectors ($n = 2m$ or $n = 2m + 1$), the proof of part (ii) would be complete. Producing such a set of vectors in $W_k/[W_k, \beta_k]$ appears difficult, and we instead show that the null space of $\beta_k - 1 (= NS(\beta_k - 1))$ has dimension at least m. Since $W_k/[W_k, \beta_k]$ and $NS(\beta_k - 1)$ have the same dimension we conclude from this that $\dim(W_k/[W_k, \beta_k]) = m$, and part (ii) will now follow.

To show that $NS((\beta_k - 1))$ has dimension at least m, note that the elements y_r of W_k, $1 \leqslant r \leqslant m$, defined by

$$y_r = \sum\left\{\binom{r-1}{j} y_{1, r+1+j} \,|\, 0 \leqslant j \leqslant r-1\right\} -$$

$$- \sum\left\{\binom{r-2}{j} y_{2, r+1+j} \,|\, 0 \leqslant j \leqslant r-2\right\} +$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+(-1)^{t+1} \sum\left\{\binom{r-t}{j} y_{t, r+1+j} \,|\, 0 \leqslant j \leqslant r-t\right\} +$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+(-1)^r (y_{r-1, r+1} + y_{r, r+2}) + (-1)^{r+1} y_{r, r+1}$$

are in $NS(\beta_k - 1)$. The proof of this fact is routine, and we will not present it here. The essential observation now is that y_r contains the term $\pm y_{r, r+1}$ and all other terms in y_r are of the form $y_{i,j}$, where $i < r$ and $j \geqslant r + 1$. From this observation, it is routine to show that the set $\{y_1, \ldots, y_m\}$ is an independent subset of $NS(\beta_k - 1)$. Further, the vectors y_r are defined in both of the cases $n = 2m$ and $n = 2m + 1$. We conclude that $NS(\beta_k - 1)$ has dimension at least m, and in view of the remarks above, this concludes the proof of part (ii).

For the proof of part (iv), observe that the formulas (3.2.7) together with the conditions on γ_i and τ_i imply that the equation (3.2.7) for the vector w_{γ_i, τ_i} involves the term $\pm w_{\gamma_i, \gamma_{i+1}}$ while this vector does not occur in the expression (3.2.7) of any other w_{γ_j, τ_j}, $j \neq i$. The independence of the set $\{w_{\gamma_i, \tau_i} \,|\, 1 \leqslant i \leqslant t\}$ follows easily, and this completes the proof of the lemma.

LEMMA 2. *Let p be an odd prime and use the notation of* 3.2.5 *and* 3.2.6.
Then

(i) υ_k *is an embedding and*

$$\dot{M}(D_k) = \mathrm{Im}(\upsilon_k) \oplus \langle [s(a(1,k)), t(b_k)][R_k, F_k] \rangle .$$

(ii) *If* $x_k - 1 = n = 2m$, $\{v_{i,i+1}(k) | m \leqslant i \leqslant n-1\}$ *is a basis of* $\mathrm{Im}(\upsilon_k)$, *while if* $x_k - 1 = n = 2m+1$, $\{v_{i,i+1}(k) | m+1 \leqslant i \leqslant n-1\}$ *is a basis of* $\mathrm{Im}(\upsilon_k)$.

It follows that $\mathrm{Im}(\upsilon_k)$ *is an elementary Abelian p-group of order* p^m; *thus, from* (i), $M(D_k)$ *is an elementary Abelian p-group of order* p^{m+1}.

(iii) *If the sets of positive integers* $\{\gamma_i\}$ *and* $\{\tau_i\}$ *satisfy the hypotheses of* (iv) *of Lemma 1, then* $\{v_{\gamma_i, \tau_i}(k) | 1 \leqslant i \leqslant t\}$ *is an independent subset of* $M(D_k)$.

Proof. The first statement of (i) is a consequence of a theorem of Evens [3: p. 167], as it has been noticed before. Moreover it is not hard to prove that $\langle [s(a(1,k)), t(b_k)][R_k, F_k] \rangle$ is a subgroup of $M(D_k)$ of order p and is isomorphic to $H_1(\langle b_k \rangle, E_k)$; further, $\langle [s(a(1,k)), t(b_k)][R_k, F_k] \rangle$ is not contained in $\mathrm{Im}(\upsilon_k)$. From these considerations and again from the work of Evens [3; Theorem 2.1], (i) follows. Part (ii) is a consequence of (i) and of part (ii) of Lemma 1, while part (iii) follows from (i) and from (iv) of Lemma 1; this completes the proof of Lemma 2.

3.3. *The proof of Theorem 1.* – Recall that G is the direct limit of the finite groups D_k with the embeddings θ_k (see (2.6) and (2.7)) and that $M(G)$ is the direct limit of the groups $M(D_k)$ with homomorphisms π_k (as in (3.1.3)). Keep in mind also that D_k is defined only for $k \geqslant 3$.

The fact that $M(G)$ is an elementary p-group follows directly from part (ii) of Lemma 2. We proceed to show that $M(G)$ is infinite.

Let $k \geqslant 3$, $n_k = x_k - 1$, and $\pi_{k,0}$ be the identity map on $M(D_k)$; for the positive integer $s \geqslant 1$ define $\pi_{k,s}$ by

(3.3.1) $\pi_{k,s} = \pi_k \pi_{k+1} \dots \pi_{k+s-1} = \pi_{k,s-1} \pi_{k+s-1}$, (hence $\pi_{k,1} = \pi_k$).

To avoid excessive subscripting in this proof, we write

$$v_{i,j}(k) = v(i,j,k) \text{ (see (3.2.5))}.$$

For any i with $1 \leqslant i \leqslant n_k - 1$, and $s \geqslant 1$, write

(3.3.2) $$v(i, i+1, k)\pi_{k,s} = v(\gamma_s(i), \tau_s(i), k+s).$$

It follows easily from (2.7), (3.1.2), (3.1.3) and (3.2.5) that

(3.3.3) $$\begin{cases} \gamma_1(i) = p(i-1) + r_{k+1}; \\ \text{for } s > 1, \ \gamma_s(i) = p(\gamma_{s-1}(i) - 1) + r_{k+s}; \\ \tau_1(i) = pi + r_{k+1}; \\ \text{for } s > 1, \ \tau_s(i) = p(\tau_{s-1}(i) - 1) + r_{k+s}. \end{cases}$$

Further, it follows from (2.3) that the dimensions $n_k = x_k - 1$ satisfy

(3.3.4) $$n_{k+s} = pn_{k+s-1} + r_{k+s} - 1.$$

The essential features of the proof of Theorem 1 are embodied in

(3.3.5) $$\begin{cases} (a) \ \gamma_s(i+1) - \gamma_s(i) = p^s \quad \text{for} \quad 1 \leqslant i \leqslant n_k - 1. \\ (b) \ \tau_s(i) = \gamma_s(i) + p^s. \\ (c) \ \text{If } i \geqslant \dfrac{n_k}{2} + 1, \quad \text{then } \gamma_s(i) \geqslant \dfrac{n_{k+s}}{2} + 1. \end{cases}$$

For the proof of parts (a) and (b) of (3.3.5), use (3.3.3) and an obvious induction on s. We prove part (c) also by induction on s. If $s = 1$, we have $\gamma_1(i) = p(i-1) + r_{k+1}$ and $n_{k+1} = pn_k + r_{k+1} - 1$ (see (2.3)). Then

$$\gamma_1(i) \geqslant p(n_k/2) + r_{k+1} = \frac{pn_k}{2} + \frac{r_{k+1}}{2} + \frac{r_{k+1}}{2} \geqslant \frac{n_{k+1}}{2} + 1$$

(see (3.3.4)), and this verifies the $s = 1$ case.

Suppose now that $\gamma_t(i) \geqslant \dfrac{(n_{k+t})}{2} + 1$. Then

$$\gamma_{t+1}(i) = p(\gamma_t(i) - 1) + r_{k+t+1} \geqslant p\frac{n_{k+t}}{2} +$$

$$+ r_{k+t+1} \geqslant \frac{pn_{k+t}}{2} + \frac{r_{k+t+1}}{2} + \frac{1}{2} = \frac{n_{k+t+1}}{2} + 1$$

(by (3.3.4)) and this completes the proof of part (c) of (3.3.5).

Now that (3.3.5) is established, let $k \geqslant 3$ and consider the set of vectors

$$L = \left\{ v(i, i+1, k) \,\middle|\, \frac{n_k}{2} + 1 \leqslant i \leqslant n_k - 1 \right\}.$$

It is routine to verify that

$$|L| = m_k - 1 \geqslant \frac{n_k}{2} - 2$$

in all cases. We will now prove

(3.3.6): For $s \geqslant 1$, $L_s = \left\{ v(i,i+1,k)\pi_{k,s} \,\middle|\, \frac{n_k}{2} + 1 \leqslant i \leqslant n_k - 1 \right\}$

is an independent subset of $M(D_{k+s})$ and $|L_s| \geqslant \dfrac{n_k}{2} - 2$.

For the proof of (3.3.6) let i_1, $i_2 = i_1 + 1$, $i_3 = i_2 + 1, \ldots, i_t$, be all the integers between $\dfrac{n_k}{2} + 1$ and $n_k - 1$ (including endpoints where appropriate). Consider now the sets of integers $\{\gamma_s(i_j), j = 1, 2, \ldots, t\}$, and $\{\tau_s(i_j), j = 1, 2, \ldots, t\}$: we will prove that L_s is linearly independent by using (iii) of Lemma 2; in particular we will prove that the two sets above satisfy, for all s, conditions (a) (b) and (c) of (iv) of Lemma 1. The independence of L_s easily implies that cardinality assertion.

For the verification of the hypotheses of Part (iv) of Lemma 1 note that (c) of (3.3.5) gives $\gamma_s(i_1) \geqslant \dfrac{n_{k+s}}{2} + 1$ and from this it is easy to deduce that (a) of Lemma 1 (iv) holds. Notice now that, from our definition, $\gamma_s(i_{j+1}) = \gamma_s(i_j + 1)$; thus by (a) of (3.3.5) we have $\gamma_s(i_{j+1}) - \gamma_s(i_j) = p^s = \tau_s(i_j) - \gamma_s(i_j)$, $j = 1, \ldots, t$, and so (b) of Lemma 1 (iv) also holds; since (c) of Lemma 1 (iv) is trivially verified, the proof of (3.3.6) is complete.

It now follows that in the direct limit $M(G) = \varinjlim\{M(D_k), \pi_k\}$, the equivalence classes

$$\left\{ [\{v(i, i+1, k)\pi_{k,s} \,|\, s \geqslant 1\}] \,\middle|\, \frac{n_k}{2} < i < n_k \right\}$$

form an independent set. Thus, the rank of $M(G)$ must be greater than or equal to $n_k/2 - 2$ for all $k \geqslant 3$. Since the sequence $\{n_k\}$ tends to infinity, $M(G)$ is an elementary Abelian group (as a direct limit of elementary Abelian groups) of infinite rank, and this concludes the proof of Theorem 1.

3.4. *The proof of Theorem 2*. – Let G be any of the groups in Theorem 1 and $M = M(G)$ be the multiplier of G. Also let H be any central «stem» extension of M by G (for existence see [1; p. 92]). We view M as a subgroup of H and so $M \subseteq H' \cap \zeta(H)$. The group H is obviously three step solvable; we show that $H'' \neq 1$. As we have seen in Theorem 1, M is an infinite elementary abelian p-group. Let S be a finite subgroup of M of rank $\geqslant 2$. There is then a finite subgroup K of H such that

(1) for some $k \geqslant 3$, $KM/M \cong K/K \cap M \cong D_k$ and

(2) $S \leqslant K' \cap M$.

Now, since $K \cap M$ is an elementary Abelian p-group, $K' \cap M$ is a direct factor of $K \cap M$; thus there exists $X \subseteq K \cap M$ such that $K \cap M = (K' \cap M) \oplus X$. Now $W = K/X$ is a central «stem» extension of $K \cap M/X$ by $(K/X)/(K \cap M/X) \cong K/K \cap M \cong D_k$.

Moreover $K \cap M/X \cong K' \cap M$ and $S \subseteq K' \cap M$. Thus $K \cap M/X$ has rank $\geqslant 2$. We will show that $W'' \neq 1$, which in turn implies that $H'' \neq 1$.

Using the notation of [1; Prop. 3.4 pp. 92-93], there is a normal subgroup T_k of F_k satisfying:

(i) $[F_k, R_k] \subseteq T_k \subseteq R_k$;

(ii) $T_k(F_k' \cap R_k) = R_k$;

(iii) $F_k/T_k \cong W$;

(iv) $R_k/T_k \cong K \cap M/X$ has rank at least 2 and is a homomorphic image of $M(D_k)$.

From Lemma 2 (i) we see that there must be an i such that $[s(a(i,k)), s(a(i+1,k))] \notin T_k$; since

$$[s(a(i,k)), s(a(i+1,k))] \equiv$$

$$\equiv [[s(a(i+1,k)), t(b_k)], [s(a(i+2,k)), t(b_k)]] (\mathrm{mod}\, T_k)$$

we see that $[s(a(i,k)), s(a(i+1,k))]T_k \in (F_k/T_k)''$. Thus $W \cong F_k/T_k$ is not metabelian.

To complete the proof of Theorem 2 we must prove that H is a group of Heineken-Mohamed type. To this end, let N be a proper subgroup of H; then NM is a proper subgroup of H, for otherwise $H' = N' \subseteq N$, and since $M \subseteq H' \subseteq N$, we have $N = H$, a contradiction. Thus NM/M is a proper subgroup of $H/M \cong G$ and so is subnormal in H/M and nilpotent. This implies that NM is subnormal in H and, since M is central in H, that N is nilpotent and subnormal. This completes the proof of Theorem 2.

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