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Mittag-Leffler modules, reduced products, and direct products

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Mittag-Leffler Modules, Reduced Products, and Direct Products.

ALBERTO FACCHINI (*)

SUMMARY - In this paper we prove that every Mittag-Leffler module is a pure submodule of a reduced product of pure-projective modules. This leads us to look for characterizations of the rings over which all reduced products of Mittag-Leffler modules are Mittag-Leffler modules. Here we focus our attention on the rings over which every direct product of Mittag-Leffler modules is a Mittag-Leffler module.

1. Introduction.

All rings have an identity and all modules are unitary. Recall that a left module M over a ring R is said to be a *Mittag-Leffler module* if the canonical homomorphism

$$(\prod A_\lambda) \otimes_R M \to \prod (A_\lambda \otimes_R M)$$

is monic for every family $\{A_\lambda\}$ of right R-modules [9]. For instance, every finitely presented module is a Mittag-Leffler module. The class of Mittag-Leffler modules is closed for (arbitrary) direct sums, pure submodules and pure extensions. In particular pure-projective modules, which are exactly the direct summands of direct sums of finitely presented modules, are Mittag-Leffler modules. Raynaud and Gruson, who studied and defined Mittag-Leffler modules first, proved that

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every countable subset of a Mittag-Leffler module M is contained in a countably generated pure-projective pure submodule of M ([9, Théorème II.2.2.1]).

In this paper we prove that every Mittag-Leffler module is isomorphic to a pure submodule of a *reduced product* $\prod_{i\in I} N_i/\mathcal{F}$, where $\mathcal{F}$ is a countably complete filter on a set I and $\{N_i \mid i \in I\}$ is a family of pure-projective modules (Proposition 2.1). In order to try to invert this result, that is, in order to try to characterize the rings for which Mittag-Leffler modules are exactly the pure submodules of reduced products of pure-projective modules, we are naturally led to the study of the rings with the property that every direct product of pure-projective modules is a Mittag-Leffler module (or, equivalently, with the property that every direct product of Mittag-Leffler modules is a Mittag-Leffler module). We show that this property is «two-sided», in the sense that it holds for the right modules over a ring if and only if it holds for the left modules over the same ring (Theorem 3.2). Then we prove that a left coherent ring such that every direct product of Mittag-Leffler modules is a Mittag-Leffler module has the property that any intersection of finitely generated left ideals is a finitely generated left ideal and all its left annihilator ideals are finitely generated (Theorem 3.3).

Theorem 3.2 has a very interesting consequence concerning the conjecture according to which a ring with right pure global dimension zero has left pure global dimension zero also. Namely suppose that R is a ring with right pure global dimension zero. Then every right R-module is a Mittag-Leffler module, so that all direct products of pure-projective left R-modules are Mittag-Leffler modules by Theorem 3.2, i.e., every countable subset of any direct product $\prod_{\lambda} M_\lambda$ of pure-projective left R-modules M_λ is contained in a pure-projective pure submodule of $\prod_{\lambda} M_\lambda$. This is slightly less than saying that any direct product of pure-projective left R-modules is pure-projective, which would be equivalent to saying that R has left pure global dimension zero.

In the last section of the paper we show that the property that every direct product of Mittag-Leffler modules is a Mittag-Leffler module doesn't hold for the ring $\boldsymbol{Z}$ of integers, that it holds for a valuation domain if and only if the valuation domain is a field, and that it holds for a commutative von Neumann regular ring if and only if the ring is self-injective.

There is a close connection between this paper and the papers [6] and [15], where similar closure properties of various classes of modules with respect to direct product are studied.

I wish to take this opportunity to thank M. Dugas for some explanations about the techniques employed in his paper [3].

2. Mittag-Leffler modules and reduced products.

Let R be an associative ring with identity and let M be a left R-module. The module M is said to be a *Mittag-Leffler module* if the canonical homomorphism

$$(\prod A_\lambda) \otimes_R M \to \prod (A_\lambda \otimes_R M)$$

is monic for every family $\{A_\lambda\}$ of right R-modules [9]. For instance, every pure-projective module is a Mittag-Leffler module. Raynaud and Gruson proved that a module M is a Mittag-Leffler module if and only if every countable subset of M is contained in a countably generated pure-projective pure submodule of M ([9, Théorème II.2.2.1] or [1, Theorem 6]).

Let I be any index set and $\mathcal{F}$ a *countably complete* filter on I, that is, a filter $\mathcal{F}$ such that intersections of $\aleph_0$ elements of $\mathcal{F}$ belong to $\mathcal{F}$. If $\{N_i \mid i \in I\}$ is a family of left R-modules, the *reduced product* $\prod_{i\in I} N_i/\mathcal{F}$ is $\prod_{i\in I} N_i/U(\mathcal{F})$, where $\prod_{i\in I} N_i$ is the direct product of the modules N_i and $U(\mathcal{F})$ is the subgroup

$$U(\mathcal{F}) = \left\{(x_i)_{i\in I} \in \prod_{i\in I} N_i \mid \{i \in I \mid x_i = 0\} \in \mathcal{F}\right\}$$

of $\prod_{i\in I} N_i$.

The proof of our first result is a modification of the proof of [3, Lemma 1.1].

PROPOSITION 2.1. *Let M be a Mittag-Leffler left R-module. Then M is isomorphic to a pure submodule of a reduced product $\prod_{i\in I} N_i/\mathcal{F}$ of pure-projective left modules N_i modulo a countably complete filter $\mathcal{F}$.*

PROOF. Let $I = \{i \mid i \subseteq M, |i| \leq \aleph_0\}$ be the set of all countable subsets of M. For every $i \in I$ let N_i denote a fixed countably generated pure-projective pure submodule of M containing i. Let $\mathcal{F}$ be the set of all $S \subseteq I$ with the following property:

$$\exists i \in I \, \forall j \in I \ \ (i \subseteq j \Rightarrow j \in S).$$

Then $\mathcal{F}$ is a countably complete filter on I, because if $S_n \in \mathcal{F}$ for every

$n \in N$ and $i_n \in I$ is such that $\forall j \in I (i_n \subseteq j \Rightarrow j \in S_n)$, then $i = \bigcup_{n \in N} i_n$ belongs to I and for every $j \in I$ $i \subseteq j$ implies $j \in S_n$ for each $n \in N$, so that $\bigcap_{n \in N} S_n \in \mathcal{F}$.

Set $\overline{M} = \prod_{i \in I} N_i / \mathcal{F}$, and for each $x \in M$ set $\tilde{x} = (x_i)_{i \in I} \in \prod_{i \in I} N_i$, where $x_i = x$ if $x \in N_i$ and $x_i = 0$ if $x \notin N_i$. Let $\varphi : M \to \overline{M}$ be defined by $\varphi(x) = \tilde{x} + U(\mathcal{F})$ for every $x \in M$. We shall prove that φ is a pure monomorphism.

Let $x, y \in M$. In order to show that $\varphi(x+y) = \varphi(x) + \varphi(y)$ we must show that $\widetilde{(x+y)} - \tilde{x} - \tilde{y} \in U(\mathcal{F})$, that is, $\{i \in I \mid (x+y)_i = x_i + y_i\} \in \mathcal{F}$. This holds because $\{x, y, x+y\} \in I$, and for every $j \in I$ if $\{x, y, x+y\} \subseteq j$ then $x, y, x+y \in N_j$, so that $(x+y)_j = x + y = x_j + y_j$. Similarly $\varphi(rx) = r\varphi(x)$ for every $r \in R$ and $x \in M$, i.e., φ is a homomorphism.

Now let $x \in M, x \neq 0$, be such that $\varphi(x) = 0$. Then $\tilde{x} \in U(\mathcal{F})$, i.e., $\{j \in I \mid x_j = 0\} = \{j \in I \mid x_j \notin N_j\} \in \mathcal{F}$. Therefore there exists $i \in I$ with the property that for every $j \in I$ such that $i \subseteq j$ one has $x_j \notin N_j$. In particular if $j = i \cup \{x\}$, then $x_j = x \in N_j$ and $x_j \notin N_j$. This contradiction shows that φ is monic.

Finally, in order to prove that the monomorphism φ is pure suppose that

$$\text{(S)} \qquad \sum_{l=1}^{m} r_{kl} x_l = \varphi(b_k), \qquad k = 1, 2, \ldots, n,$$

is a system of equations with $r_{kl} \in R$ and $b_k \in M$, and suppose that (S) has a solution $(a_{1i})_{i \in I} + U(\mathcal{F}), \ldots, (a_{ni})_{i \in I} + U(\mathcal{F})$ in $\overline{M}$. To show that φ is a pure monomorphism it is sufficient to prove that (S) has a solution in $\varphi(M)$ [13, Proposition 3]. Set $\varphi(b_k) = (b_{ki})_{i \in I} + U(\mathcal{F}), k = 1, \ldots, n$. Since the $(a_{li})_{i \in I} + U(\mathcal{F})$'s, $l = 1, \ldots, m$, are a solution of the system, it follows that the sets

$$\left\{ i \in I \mid \sum_{l=1}^{m} r_{kl} a_{li} = b_{ki} \right\}, \qquad k = 1, \ldots, n,$$

are elements of $\mathcal{F}$. But $\mathcal{F}$, which is a filter, is closed for finite intersection, so that

$$\left\{ i \in I \mid \sum_{l=1}^{m} r_{kl} a_{li} = b_{ki} \text{ for every } k = 1, \ldots, n \right\}$$

belongs to $\mathcal{F}$. Therefore there exists $i \in I$ such that for every $j \in I$ with $i \subseteq j$ one has $\sum_{l=1}^{m} r_{kl} a_{lj} = b_{kj}$ for every $k = 1, \ldots, n$. In particular for $j =$

$= i \cup \{b_1, \dots, b_n\}$ we get $\sum_{l=1}^{m} r_{kl} a_{lj} = b_{kj} = b_k$ for every $k = 1, \dots, n$, i.e., $\varphi(a_{1j}), \dots, \varphi(a_{mj}) \in \varphi(M)$ is a solution of the system (S). Hence φ is pure by [13, Proposition 3]. □

Proposition 2.1 cannot be inverted: it is not true that every pure submodule of a reduced product of pure-projective left modules modulo a countably complete filter $\mathcal{F}$ is a Mittag-Leffler module. For instance, every direct product $\prod_{i \in I} N_i$ is a reduced product of this type (take $F = \{I\}$), and we shall see in Section 4 that there exist direct products of pure-projective left modules over suitable rings which are not Mittag-Leffler modules. Thus we are naturally led to study when a direct product of pure-projective modules is a Mittag-Leffler module and in the rest of the paper we shall try to describe the rings for which this happens.

3. Mittag-Leffler modules and direct products.

We begin this section with an easy but interesting lemma.

LEMMA 3.1. *Let M be a left R-module. Then M is a Mittag-Leffler module if and only if the canonical homomorphism $\Psi: (\prod F_\lambda) \otimes_R M \to \prod (F_\lambda \otimes_R M)$ is monic for every family $\{F_\lambda\}$ of finitely presented right R-modules.*

PROOF. We need the following

CLAIM. *If A_R, ${}_RM$ are R-modules, $a_1, \dots, a_n \in A$, $x_1, \dots, x_n \in M$ and $\sum_{i=1}^{n} a_i \otimes x_i$ is the zero element of $A \otimes_R M$, then there exist a finitely presented module F_R, a homomorphism $\varphi: F \to A$, and $b_1, \dots, b_n \in F$ such that $\sum_{i=1}^{n} b_i \otimes x_i$ is the zero element of $F \otimes_R M$ and $\varphi(b_i) = a_i$ for every $i = 1, \dots, n$.*

In order to prove the claim fix a direct system of finitely presented right R-modules A_α, $\alpha \in \Delta$, with $A \cong \varinjlim A_\alpha$. Then there exists $\beta \in \Delta$ such that all the elements $a_1, \dots, a_n$ are in the image of A_β. Since $A \otimes M \cong \varinjlim (A_\alpha \otimes M)$ and $\sum_i a_i \otimes x_i = 0$ in $A \otimes_R M$, there exist $\gamma \geq \beta$ in Δ and $b_1, \dots, b_n$ in A_γ such that $\sum_i b_i \otimes x_i = 0$ in $A_\gamma \otimes_R M$ and a_i is the image of b_i in the canonical homomorphism $A_\gamma \to A$. Therefore A_γ is the required finitely presented module, and the canonical homomorphism $A_\gamma \to A$ is the required homomorphism φ. This proves the Claim. □

In order to prove the Lemma it is sufficient to prove that if ${}_RM$ is a left module that is not a Mittag-Leffler module, then there exists a family of finitely presented right modules $\{F_\lambda\}_\lambda$ for which the canonical homomorphism Ψ is not monic.

But if ${}_RM$ is not a Mittag-Leffler module, there exists a family of right modules $\{A_\lambda\}_\lambda$ for which the canonical homomorphism $(\prod A_\lambda) \otimes_R M \to \prod(A_\lambda \otimes_R M)$ is not monic. Let $\sum_i (a_{\lambda,i})_\lambda \otimes x_i$ be a non-zero element in the kernel of this homomorphism. Then the elements $\sum_i a_{\lambda,i} \otimes x_i$ are the zero elements in $A_\lambda \otimes M$ for every λ. By the Claim just proved, for each λ there exist a finitely presented module F_λ, a homomorphism $\varphi_\lambda \colon F_\lambda \to A_\lambda$ and $b_{\lambda,i} \in F_\lambda$ such that $\varphi_\lambda(b_{\lambda,i}) = a_{\lambda,i}$ and $\sum_i b_{\lambda,i} \otimes x_i = 0$ in $F_\lambda \otimes M$. Now construct the commutative diagram

$$\begin{array}{ccc} (\prod F_\lambda) \otimes M & \xrightarrow{\Psi} & \prod(F_\lambda \otimes M) \\ \Phi \downarrow & & \downarrow \Phi' \\ (\prod A_\lambda) \otimes M & \longrightarrow & \prod(A_\lambda \otimes M) \end{array},$$

where the vertical arrows Φ and Φ' are induced by the φ_λ's.

The element $e = \sum_i (b_{\lambda,i})_\lambda \otimes x_i \in (\prod F_\lambda) \otimes M$ is non-zero because its image via Φ is $\sum_i (a_{\lambda,i})_\lambda \otimes x_i \neq 0$. Since $\Psi(e) = (\sum_i b_{\lambda,i} \otimes x_i)_\lambda = 0 \in \in \prod_\lambda (F_\lambda \otimes M)$, the mapping Ψ cannot be monic. This concludes the proof of the Lemma. □

THEOREM 3.2. *Let R be a ring. The following statements are equivalent*:

- (a) *Every direct product of Mittag-Leffler left R-modules is a Mittag-Leffer module;*
- (a′) *Every direct product of Mittag-Leffler right R-modules is a Mittag-Leffler module;*
- (b) *Every direct product of pure-projective left R-modules is a Mittag-Leffler module;*
- (b′) *Every direct product of pure-projective right R-modules is a Mittag-Leffler module;*
- (c) *Every direct product of finitely presented left R-modules is a Mittag-Leffler module;*

(c′) *Every direct product of finitely presented right R-modules is a Mittag-Leffler module;*

(d) *Every direct power M^X of a Mittag-Leffler left R-module M is a Mittag-Leffler module;*

(d′) *Every direct power M^X of a Mittag-Leffler right R-module M is a Mittag-Leffler module;*

(e) *Every direct power M^X of a pure-projective left R-module M is a Mittag-Leffler module;*

(e′) *Every direct power M^X of a pure-projective right R-module M is a Mittag-Leffler module.*

PROOF. The implications (a′) ⇒ (b′) ⇒ (c′) are trivial.

(c′) ⇒ (a). If (c′) holds and $\{M_\alpha\}_\alpha$ is a family of Mittag-Leffler left R-modules, we must prove that $\prod_\alpha M_\alpha$ is a Mittag-Leffler module, i.e., that if $\{F_\lambda\}_\lambda$ is a family of finitely presented right R-modules the homomorphism $(\prod_\lambda F_\lambda) \otimes (\prod_\alpha M_\alpha) \to \prod_\lambda (F_\lambda \otimes \prod_\alpha M_\alpha)$ is monic (Lemma 3.1). Consider the commutative diagram

$$\begin{array}{ccc} (\prod_\lambda F_\lambda) \otimes (\prod_\alpha M_\alpha) & \longrightarrow & \prod_\lambda (F_\lambda \otimes \prod_\alpha M_\alpha) \\ \downarrow & & \downarrow \\ \prod_\alpha ((\prod_\lambda F_\lambda) \otimes M_\alpha) & \longrightarrow & \prod_{\lambda,\alpha} (F_\lambda \otimes M_\alpha) \end{array}.$$

The arrow on the left is monic because $\prod_\lambda F_\lambda$ is a Mittag-Leffler module by (c′), and the lower horizontal arrow is monic because the M_α's are Mittag-Leffler modules. Therefore the upper horizontal arrow is monic. This proves (a).

Thus we have shown that (a′) ⇒ (b′) ⇒ (c′) ⇒ (a). By symmetry, that is, by passing to the opposite ring, we obtain that (a) ⇒ (b) ⇒ (c) ⇒ (a′). This proves that the first six conditions (a)-(c′) are equivalent.

(e) ⇒ (b). If $\{M_\alpha\}_{\alpha \in X}$ is a family of pure-projective left R-modules, its direct sum $M = \bigoplus_{\alpha \in X} M_\alpha$ is trivially a pure-projective left R-module. If (e) holds, then M^X is a Mittag-Leffler module with a direct summand isomorphic to $\prod_{\alpha \in X} M_\alpha$. This proves (b).

Since (a) ⇒ (d) ⇒ (e) are trivial implications, we get that (d) and (e) also are equivalent to the first six conditions. The conclusion follows immediately. □

A ring R is said to be *left coherent* if every finitely generated left ideal is finitely presented, or, equivalently, if every direct product of flat right R-modules is flat [12, § I.13]. It is possible to prove that R is left coherent if and only if the left annihilator $l(x) = \{r \in R \mid rx = 0\}$ of every element $x \in R$ is a finitely generated left ideal and the intersection of two finitely generated left ideals is finitely generated. Also recall that a left annihilator ideal of R is a left ideal I in R for which there exists a subset $S \subseteq R$ such that $I = l(S) = \{r \in R \mid rx = 0 \text{ for every } x \in S\}$.

THEOREM 3.3. *Let R be a left coherent ring such that every direct product of Mittag-Leffler left R-modules is a Mittag-Leffler module. Then all left annihilator ideals of R are finitely generated, and so are arbitrary intersections of finitely generated left ideals.*

PROOF: Let $\{I_\lambda\}_\lambda$ be a family of finitely generated left ideals of R. Then the left modules R/I_λ are finitely presented, so that $\prod_\lambda R/I_\lambda$ is a Mittag-Leffler module. Let $\varphi\colon R/\bigcap_\lambda I_\lambda \to \prod_\lambda R/I_\lambda$ be the canonical monomorphism. By [7, Theorem 1] φ factors through a monomorphism $\psi\colon R/\bigcap_\lambda I_\lambda \to N$, where N is finitely presented. Since R is left coherent, the module $R/\bigcap_\lambda I_\lambda$ also is finitely presented, i.e., the left ideal $\bigcap_\lambda I_\lambda$ is finitely generated [12, Proposition I.3.2]. Therefore arbitrary intersections of finitely generated left ideals are finitely generated.

Now let $l(S)$ be a left annihilator ideal of R. Then $l(S) = \bigcap_{x \in S} l(x)$ is an intersection of left ideals that are finitely generated because R is left coherent. Therefore $l(S)$ is finitely generated. □

REMARK 3.4. The class $\mathcal{C}$ of all the rings R such that every direct product of Mittag-Leffler R-modules is a Mittag-Leffler module could have a close connection with the class $\mathcal{C}'$ of all rings R such that every direct product of locally projective left R-modules is locally projective [15]. We shall see in Example 4.1 that the ring $\boldsymbol{Z}$ of integers is not in $\mathcal{C}$. Since every right noetherian ring is in $\mathcal{C}'$ [15, Corollary 4.3] it follows that $\mathcal{C}'$ is not contained in $\mathcal{C}$. We do not know if $\mathcal{C}$ is a proper subclass of $\mathcal{C}'$. If this were true every ring in $\mathcal{C}$ would be right and left coherent and our Theorem 3.3 would immediately follow from [15, Corollary 4.3].

4. Examples.

EXAMPLE 4.1. THE RING $\boldsymbol{Z}$. The ring $\boldsymbol{Z}$ of integers does not have the equivalent properties stated in Theorem 3.2. To see this let $\boldsymbol{Z}(p)$ be the cyclic group with p elements for each prime p. Let us show that condition (c) of Theorem 3.2 does not hold by proving that the direct product $\prod_p \boldsymbol{Z}(p)$, where p ranges in the set of all primes, is not a Mittag-Leffler $\boldsymbol{Z}$-module. By making use of [1, Proposition 7] it is sufficient to prove that $\prod_p \boldsymbol{Z}(p)/t(\prod_p \boldsymbol{Z}(p))$ is not an $\aleph_1$-*free* group, where t denotes the torsion subgroup, and $\aleph_1$-free means that all its countable subgroups are free [4, § 19]. It is very easy to show that $t(\prod_p \boldsymbol{Z}(p)) = \bigoplus_p \boldsymbol{Z}(p)$ and that $\prod_p \boldsymbol{Z}(p)/\bigoplus_p \boldsymbol{Z}(p)$ is a divisible abelian group. Therefore $\prod_p \boldsymbol{Z}(p)/t(\prod_p \boldsymbol{Z}(p))$ is a nonzero torsion-free divisible abelian group; in particular it cannot be $\aleph_1$-free.

Note that every direct power M^X of a finitely presented $\boldsymbol{Z}$-module M is a Mittag-Leffler $\boldsymbol{Z}$-module. In fact, if M is any finitely presented $\boldsymbol{Z}$-module, the exact sequence $0 \to t(M) \to M \to M/t(M) \to 0$ induces a short exact sequence $0 \to (t(M))^X \to M^X \to (M/t(M))^X \to 0$. Now $M/t(M)$ is a finite direct product of copies of $\boldsymbol{Z}$, so that $(M/t(M))^X$ is a direct product of copies of $\boldsymbol{Z}$, and therefore it is $\aleph_1$-free [4, Theorem 19.2]. Moreover, $t(M)$ is torsion of bounded order, so that $t(M)^X$ is torsion of bounded order. Therefore M^X, extension of an $\aleph_1$-free abelian group by a torsion group of bounded order, is a Mittag-Leffler $\boldsymbol{Z}$-module [1, Proposition 7]. This proves that the condition *Every direct power M^X of a finitely presented R-module M is a Mittag-Leffler module* is strictly weaker than the ten equivalent conditions of the statement of Theorem 3.2.

EXAMPLE 4.2. VALUATION RINGS. If R is a valuation domain that is not a field, R does not have the equivalent conditons stated in Theorem 3.2.

To prove this, suppose R is a valuation domain that is not a field and with the property that every direct product of Mittag-Leffler R-modules is a Mittag-Leffler R-module. Under these hypotheses there exists a non-zero element $r \in R$ that is not invertible. Then $P = \bigcap_{n \geq 1} Rr^n$ is a prime ideal [5, Proposition I.1.6(d)]. By Theorem 3.3 P is a finitely generated ideal because it is an intersection of finitely generated ideals and any valuation domain is coherent. Since R is a valuation domain, P is a principal prime ideal, so that either $P = 0$ or P is maximal [5, Proposition I.1.6(b)]. Since r is not invertible, one has $Rr \supset Rr^2 \supseteq P$, and therefore P cannot be the maximal ideal. This proves that $P = 0$. Now consider the module $M = \prod_{n \geq 1} R/Rr^{2n}$. Since M is a direct product of finitely presented modules, M is a Mittag-Leffler module. For each $t \geq 1$ let $x_t = (x_{n,t})_{n \geq 1}$ be the element of M defined by $x_{n,t} = r^{n-t} + Rr^{2n}$ if $n \geq t$, and $x_{n,t} = Rr^{2n}$ if $n < t$. Let N be the submodule of M generated by the set $\{x_t \mid t \geqslant 1\}$. Since M is a Mittag-Leffler module, there exists a pure-projective submodule Q of M with $N \leqslant Q$ [9, Théorème II.2.2.1]. Then Q is cyclically presented [5, Theorem II.3.4 and Proposition II.4.3], so that $Q = F \oplus T$, where F is free and T is torsion. Then $N \leqslant Q \leqslant M$ implies that $N + t(M)/t(M) \leqslant Q + t(M)/t(M) \leqslant M/t(M)$, where $t(M)$ is the torsion submodule of M and $Q + t(M)/\ t(M) \cong Q/Q \cap t(M) = Q/T \cong F$ is free.

Consider the torsion-free module $\overline{N} = N/t(N) = N/N \cap t(M) \cong N + t(M)/t(M)$. It is generated by the set $\{\overline{x}_t \mid t \geqslant 1\}$, where the bar denotes reduction modulo $t(N)$, and it is easy to verify that $r\overline{x}_{t+1} = \overline{x}_t$ for every $t \geqslant 1$. It follows that $r\overline{N} = \overline{N}$. But if x is a nonzero element of R then $x \notin Rr^n$ for some n (because $\bigcap_n Rr^n = 0$). Therefore $Rx \supsetneq Rr^n$, so that $x\overline{N} \supseteq r^n\overline{N} = \overline{N}$, i.e., $\overline{N}$ is a divisible R-module. Hence $\overline{N} \cong N + t(M)/t(M)$ is a torsion-free divisible R-module contained in the free R-module $Q + t(M)/t(M)$. This implies that $\overline{N} = 0$, i.e., $N \subseteq t(M)$ is torsion. This contradiction proves that a valuation domain that is not a field does not have the equivalent conditions stated in Theorem 3.2.

Most of what we proved above for valuation domains can be extended to coherent valuation rings that are not integral domains. Namely, if R is a coherent valuation ring (i.e., a coherent commutative ring whose ideals are linearly ordered by set inclusion), which has the property that every direct product of Mittag-Leffler modules is a Mittag-Leffler module, then any intersection of principal ideals is principal by Theorem 3.3. It follows (as in the case of valuation domains), that R has exactly one prime ideal [5, Proposition I.1.6]. Moreover its value semigroup G, which is a *0-segmental semigroup*, i.e., it is the extended positive cone of a totally ordered abelian group modulo an ideal [10], is complete as a lattice (compare [6, Example 5.3]).

Example 4.3. Von Neumann regular rings.

Proposition 4.4. *Let R be a von Neumann regular ring.*

(1) *Every direct product of left Mittag-Leffler R-modules is a Mittag-Leffler R-module if and only if the direct power R^R is a Mittag-Leffler right R-module.*

(2) *If R is a left self-injective ring, then every direct product of left Mittag-Leffler R-modules is a Mittag-Leffler R-module.*

(3) *If R is commutative, then every direct product of Mittag-Leffler R-modules is a Mittag-Leffler R-module if and only if R is self-injective.*

PROOF. (1) The «only if» part follows immediately from Theorem 3.2 (a) ⇒ (e′). For the converse suppose that $\{M_\lambda\}$ is a family of Mittag-Leffler left R-modules and that R^R is a Mittag-Leffler right R-module. By [7, Corollary], in order to prove that $\prod M_\lambda$ is a Mittag-Leffler left R-module it is sufficient to prove that the natural map $R^R \otimes (\prod M_\lambda) \to (\prod M_\lambda)^R$ is injective. For this it is enough to observe that the natural map $R^R \otimes (\prod M_\lambda) \to (\prod M_\lambda)^R$ can be written as the composition of the three canonical maps $R^R \otimes (\prod M_\lambda) \to \prod (R^R \otimes M_\lambda) \to \prod (M_\lambda^R) \to (\prod M_\lambda)^R$, where the first map is injective because R^R is a Mittag-Leffler right module, the second map is injective because the M_λ's are Mittag-Leffler modules, and the third map is the obvious isomorphism.

(2) If R is a left self-injective regular ring, then R^R is a *regular left module* in the sense of Zelmanowitz [14], that is, given any $m \in R^R$ there exists $f \in \mathrm{Hom}_R(R^R, R)$ such that $f(m) \cdot m = m$ [14, Theorem 2.10]. By [14, (1.4)] all finitely generated submodules of R^R are regular, hence projective [14, Corollary 1.7]. Therefore R^R is a Mittag-Leffler left R-module [7, Corollary]. By Part (1) every direct product of right Mittag-Leffler R-modules is a Mittag-Leffler R-module. The conclusion follows immediately by Theorem 3.2.

(3) The «if» part has just been proved. The converse follows from [7, Theorem 2, (b) ⇒ (c)]. □

5. Remarks.

(1) We saw in Proposition 4.4 that over a von Neumann regular ring R the class of Mittag-Leffler modules is closed for products if and only if R^R is a Mittag-Leffler right module (by Theorem 3.2 this happens if and only if the left module R^R is a Mittag-Leffler module). The behaviour of noetherian rings is completely different because, as we show in the next paragraph, over a right noetherian ring R every direct power R^X is a Mittag-Leffler left R-module (this is also proved in [15, Corollary 4.3] and in [9, Example II.2.4.1], but here we give a very easy proof of this fact). Note that over the ring $\boldsymbol{Z}$ there are products of finitely presented modules that are not Mittag-Leffler modules (Example 4.1).

Let us prove that if R is a right noetherian ring every direct power R^X is a Mittag-Leffler left R-module. If R is a right noetherian ring and A_R is a right R-module, then the natural map $A \otimes R^X \to A^X$ is monic [7, Theorem 1]. Therefore for every family of right R-modules $\{A_\lambda\}$ the vertical arrows in the diagram

$$\begin{array}{ccc} (\prod_\lambda A_\lambda)\otimes R^X & \longrightarrow & \prod_\lambda(A_\lambda \otimes R^X) \\ \downarrow & & \downarrow \\ (\prod_\lambda A_\lambda)^X & \longrightarrow & \prod_\lambda(A_\lambda^X) \end{array}$$

are monomorphisms, and the lower horizontal map is an isomorphism. Therefore the upper horizontal arrow is a monomorphism, i.e., R^X is a Mittag-Leffler left R-module.

(2) The class $\mathcal{C}$ of the rings over which every direct product of Mittag-Leffler modules is a Mittag-Leffler module contains

(a) the rings with left pure global dimension zero or with right pure global dimension zero, because all left (right) modules over these rings are Mittag-Leffler modules ([11, Theorem 6.3] or [1, Theorem 8]);

(b) the self-injective von Neumann regular rings (Proposition 4.4);

(c) the endomorphism ring $R = \mathrm{End}_K(V)$ of a vector space V over a field K. This ring R is a von Neumann regular ring, which is not self-injective if V is of infinite dimension. B. Zimmermann-Huisgen proved in [15, Example 4.7] that the left R-modules R^X are regular, so that R^R is a Mittag-Leffler R-module (repeat the proof of Proposition 4.4(2)). From Proposition 4.4(1) we conclude that every direct product of Mittag-Leffler modules over the ring $R = \mathrm{End}_K(V)$ is a Mittag-Leffler module.

(3) The class $\mathcal{C}$ of the rings we have studied in this paper, that is, the rings over which every direct product of Mittag-Leffler modules is a Mittag-Leffler module, is exactly the class of the rings over which every direct product of pure-projective modules is a Mittag-Leffler module (Theorem 3.2). Suppose we want to study the subclass $\mathcal{P}$ of $\mathcal{C}$ consisting of the rings over which every direct product of pure-projective left R-modules is pure-projective. Then the rings in this subclass $\mathcal{P}$ are exactly the rings with left pure global dimension zero. I would like to thank professor Daniel Simson for this interesting remark.

(4) In Section 2 we proved that every Mittag-Leffler module is a pure submodule of a reduced product of pure-projective modules. In order to reverse this result we should have studied when all pure sub-

modules of reduced products of pure-projective modules are Mittag-Leffler modules, or, equivalently, when all *reduced products* of pure-projective modules are Mittag-Leffler modules. Since this problem turned out to be almost intractable, we decided to study when *direct products* of pure-projective modules are Mittag-Leffler modules and construct some examples and counterexamples. This is what we did in Sections 3 and 4. We conclude this paper with a remark about a particular case of reduced products of pure-projective modules.

Suppose that R is an artin algebra (that is, a ring which is finitely generated as a module over a commutative artinian ring) with the property that every direct product of Mittag-Leffler modules is a Mittag-Leffler module. If M is a finitely generated R-module, then given any index set I and any filter $\mathcal{F}$ on I, the reduced product $M^I/\mathcal{F}$ of copies of M modulo $\mathcal{F}$ is a Mittag-Leffler R-module.

To prove this, note that under these hypotheses the submodule $U(\mathcal{F})$ of M^I (notations as in Section 2) is a direct summand of M^I (proof of Theorem F in [8]). Therefore $M^I/\mathcal{F}$ is isomorphic to a direct summand of M^I, which is a Mittag-Leffler module. The conclusion follows immediately.

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