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Sylow theory of *CC*-groups

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Sylow Theory of CC-groups.

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1. Introduction.

Groups with Černikov conjugacy classes or CC-*groups* were first considered by Polovickiĭ [9] as an extension of the concept of FC-groups, that is, groups in which every element has only a finite number of conjugates. A group G is said to be a CC-*group* if $G/C_G(x^G)$ is a Černikov group for each $x \in G$. Polovickiĭ's characterization of CC-groups assures that if G is a CC-group then x^G is Černikov-by-cyclic and $[G, x]$ is Černikov for every x in G (see [10; 4.36]).

In [1] and [6] a Sylow theory for CC-groups was initiated from a classical point of view and for a single prime. Since then, the authors of the present paper have been working on extensions of these results to arbitrary sets of primes π.

Here we present an account of the «conjugacy theory» inherent in the theory of Sylow π-subgroups (π an arbitrary set of primes). Results of this type are highlighted in Theorem 3.5, where we state the characterization of the Sylow π-subgroups, and Theorem 4.12, where we give equivalent conditions for the conjugacy of Sylow bases; amongst these is the property that the group be locally nilpotent-by-finite. The situation is very similar to the corresponding theory in periodic FC-groups [13, 12]. Using Theorem 3.5 as a starting point, we develop a theory of Sylow bases, Carter subgroups, etc. in locally soluble CC-groups.

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Throughout our group-theoretic notation is standard and is taken from [10] and [13], to which we refer for the basic definitions and the setting of the problems we are considering.

2. Locally inner automorphisms.

This is an auxiliary section which is devoted to properties of locally inner automorphisms of CC-groups needed in the sequel, most of which were shown in [8] and which we include here for the reader's convenience. We recall the definitions. An automorphism φ of a group G is said to be *locally inner* if, for each finite set of elements $x_1, \ldots, x_n \in G$, there is an element $g \in G$ such that $x_i^{\varphi} = x_i^{g}$, for every i. The locally inner automorphisms of G clearly form a subgroup of Aut G which we denote by Linn(G). Two subgroups H and K of G are said to be *locally conjugate* in G if there is an automorphism $\varphi \in$Linn G such that $H^{\varphi} = K$.

Our first result is a useful property of subgroups of CC-groups.

LEMMA 2.1. *A subgroup H of a* CC*-group G cannot be locally conjugate to a subgroup of itself.*

PROOF. Let $\varphi \in$Linn(G) and assume that $H^{\varphi} \leqslant H$. If $h \in H$, we put $N = h^G$ so that $h \in H \cap N$. By [8; Corollary 2.2], there is an element $g \in G$ such that $N^{\varphi} = N^g$ and then $(H \cap N)^g = (H \cap N)^{\varphi} \leqslant H \cap N$. By [1; Lemma 2], we have $(H \cap N)^g = H \cap N$ so that $(H \cap N)^{\varphi} = H \cap N$ and $h \in H^{\varphi}$.

Locally inner automorphisms of subgroups and quotients of CC-groups are induced by locally inner automorphisms of the whole group (see [8; Theorem 2.5]).

THEOREM 2.2. *Let G be a* CC*-group.*

1) *If $H \leqslant G$ and $\varphi \in$* Linn, *H, then there is a $\theta \in$* Linn *G such that $\theta|_H = \varphi$.*

2) *If $N \trianglelefteq G$ and $\varphi \in$* Linn *G/N, then there is a $\theta \in$* Linn *G such that $\theta|_{(G/N)} = \varphi$.*

The set of all subgroups of G which are locally conjugate to H is $Lcl_G(H)$, *the local conjugacy class* containing H. We shall denote by $Cl_G(H)$ *the conjugacy class* containing H. The characterization of the coincidence of $Lcl_G(H)$ and $Cl_G(H)$ for a subgroup H was done in [8; Theorem 6.7]. We summarize it in the next result.

THEOREM 2.3. *Let H be a subgroup of the residually Černikov* CC-*group G. Then the following are equivalent*:

1) $Lcl_G(H) = Cl_G(H)$.

2) $Lcl_G(H)$ *is countable.*

3) H^G/H_G *is Černikov.*

3. Sylow subgroups.

If π is a set of primes, we say that a CC-group G is *a C_π-group* if, in each Černikov subgroup H of G, the Sylow π-subgroups of H are conjugate in H. Then a CC-group G is a C_p-group, for every prime p, and a locally soluble CC-group G is a C_π-group, for every π. The proof of the local conjugacy of Sylow π-subgroups of a C_π-group can be done as in the case where π consists of a single prime ([1; Theorem 1], so we omit it.

THEOREM 3.1. *The Sylow π-subgroups of a C_π-group are locally conjugate.*

It is now an easy consequence of this that the Sylow π-subgroups are well behaved with respect to normal sugroups, factor groups and intersections. These facts will be implicitly used in what follows.

COROLLARY 3.2. *Let G be a C_π-group.*

1) *If N is an arbitrary normal subgroup of G and $P \in \mathrm{Syl}_\pi(G)$, then $P \cap N \in \mathrm{Syl}_\pi(N)$ and all Sylow π-subgroups of N have this form.*

2) *If N is a torsion normal subgroup of G and $P \in \mathrm{Syl}_\pi(G)$, then $PN/N \in \mathrm{Syl}_\pi(G/N)$ and all Sylow π-subgroups of G/N have this form.*

3) *If $\{N_i/i \in I\}$ is a family of torsion normal subgroups of G and $P \in \mathrm{Syl}_\pi(G)$, then we hawe $\cap\{PN_i/i \in I\} = P(\cap\{N_i/i \in I\})$.*

PROOF. 1) Given P, let $Q \in \mathrm{Syl}_\pi(N)$ such that $P \cap N \leqslant Q$. By Theorem 3.1, a Sylow π-subgroup of G has the form P^α, where $\alpha \in \mathrm{Linn}(G)$. Thus $Q \leqslant P^\alpha$, for some α, and so $Q = P^\alpha \cap N$. By Lemma 2.1, $P \cap N =$ $= P^\alpha \cap N = Q \in \mathrm{Syl}_\pi(N)$. Clearly a Sylow π-subgroup of N has the above form.

2) Let Σ be a local system of G consisting of normal closures of finite subsets of G. Given $K \in \Sigma$, by (1), $P \cap K \in \mathrm{Syl}_\pi(K)$. If S is the torsion subgroup of K, S is Černikov and it is clear that $P \cap K =$

$= P \cap S \in \mathrm{Syl}_\pi(S)$ and $N \cap K \leqslant S$. Since S is Černikov we have that $(P \cap S)(N \cap S)/(N \cap S) \in \mathrm{Syl}_\pi(S/N \cap S)$ and therefore $(P \cap S)N/N \in$ $\in \mathrm{Syl}_\pi(SN/N)$. Since N is periodic, it is clear that SN/N is the torsion subgroup of KN/N so that $(P \cap K)N/N = (PN/N) \cap (KN/N) \in$ $\in \mathrm{Syl}_\pi(KN/N)$. From this it is immediate that the union of all these subgroups, which is exactly PN/N, is a Sylow π-subgroup of G/N.

Conversely, let $Q/N \in \mathrm{Syl}_\pi(G/N)$. Given P, we have just seen that $PN/N \in \mathrm{Syl}_\pi(G/N)$. By Theorem 3.1 Q/N and PN/N are locally conjugate in G/N and so, by Theorem 2.2 there is some $\alpha \in \mathrm{Linn}(G)$ such that $Q/N = P^\alpha N/N$.

3) By (2), we may assume that $\cap\{N_i \mid i \in I\} = 1$. Set $Q =$ $= \cap\{PN_i \mid i \in I\}$. Then $P \leqslant Q$ and $PN_i = QN_i$, for each $i \in I$. Hence $Q/(Q \cap N_i)$ is a π-group so Q is residually a π-group. We note that Q is periodic so Q is in fact π-group. Hence $P = Q$.

To arrive at our main «conjugacy result» we require several lemmas.

LEMMA 3.3. *Let G be a* CC-*group.*

1) *$O_\pi(G)$ contains any radicable π-subgroup of G.*

2) *If $P \in \mathrm{Syl}_\pi(G)$, then $P/O_\pi(G)$ is an* FC-*group. Moreover, if $P/O_\pi(G)$ is Černikov, then it is finite.*

3) *If $P \in \mathrm{Syl}_\pi(G)$ and G is a C_π-group, then the normal closure P^G of P in G is Černikov if and only if G satisfies the minimal condition for π-subgroups. Moreover, in such a case, the Sylow π-subgroups of G are conjugate and G has only countably many of them.*

PROOF. (1) and (2) can be proved exactly as in [6;2.2].

3) We first note that, by Theorem 3.1, P^G contains any Sylow π-subgroup of G so that P^G and G have the same set of Sylow π-subgroups. Thus, if P^G is Černikov, then $G \in$Min-π and the last assertions follow because they are true for a Černikov group. Suppose that $G \in$Min-π. Thus P is Černikov and so, by (2), $P/O_\pi(G)$ is finite. Therefore $(P/O_\pi(G))^{G/O_\pi(G)}$ is Černikov. Since $O_\pi(G)$ is Černikov by hypothesis, it follows that P^G is again Černikov.

LEMMA 3.4. *Let G be a periodic C_π-group and put $Z = Z(G)$, the centre of G, and $L = G/Z$. If π is a set of primes, then $O_\pi(L) = O_\pi(G)Z/Z$, the Sylow π-subgroups of $G/O_\pi(G)$ are isomorphic to these of $L/O_\pi(L)$ and there is a bijection between these families of Sylow π-subgroups.*

PROOF. Let Q/Z be a normal π-subgroup of L. If $P \in \mathrm{Syl}_\pi(G)$, then we have $Q \leqslant PZ$, $Q = (P \cap Q)Z$ and it follows that $P \cap Q$ is normal in Q. By Corollary 3.2, $P \cap Q \in \mathrm{Syl}_\pi(Q)$. Then $P \cap Q$ is a characteristic subgroup of Q and normal in G. Therefore $P \cap Q \leqslant O_\pi(G)$ and so $Q/Z \leqslant O_\pi(G)Z/Z$. Then $O_\pi(L) = O_\pi(G)Z/Z$.

The other assertion are clear and can be shown as [6; 2.3].

We now come to the main result regarding conjugacy.

THEOREM 3.5. *Let π be a set of primes. Then for a C_π-group the following are equivalent.*

1) *The Sylow π-subgroups of G are conjugate in G.*

2) *G has a countable number of Sylow π-subgroups.*

3) *$G/O_\pi(G)$ satisfies the minimal condition for π-subgroups.*

4) *The Sylow π-subgroups of $G/O_\pi(G)$ are Černikov groups.*

5) *The Sylow π-subgroups of $G/O_\pi(G)$ are finite groups.*

6) *The torsion subgroup of G is a finite extension of a π'-extension of $O_\pi(G)$.*

7) *G is a finite extension of a π^*-extension of $O_\pi(G)$.*

PROOF. Making use of Lemma 3.3, we see that (3), (4) and (5) are equivalent. The equivalence among (5), (6) and (7) can be shown as in [6].

Let $P \in \mathrm{Syl}_\pi(G)$. It T is the torsion subgroup of G, then it is clear that $P \leqslant T$ and $P_G = O_\pi(T)$. By Theorem 2.2, $Lcl_G(P) = Lcl_T(P)$ and, as in the proof of [1; Theorem 2], it can be shown that the Sylow π-subgroups of G are conjugate in G if and only if they are conjugate in T. Thus, to show the equivalence among (1), (2) and (4), we may assume that G is periodic. Further, by Lemma 3.4 and Theorem 2.2, we may replace G by G/Z to assume that G is residually Černikov. Then the required equivalence follows from Theorem 2.3.

4. Sylow bases, complement systems, basis normalizers and Carter subgroups.

We now introduce the elements of the theory we are studying in this section. *A Sylow basis of a group G* is a set $S = \{S_p\}$ of Sylow p-subgroups of G, one for each prime p, such that the subgroup $\langle S_p \mid p \in \pi\rangle$ is a π-group, for each set π of primes. If $S = \{S_p\}$ is a set of Sylow p-subgroups of the CC-group G, then it is easy to show that S is a Sylow basis of G if and only if $S_p S_q = S_q S_p$, for all primes p, q.

Standard results on Sylow bases follow and they are proved by using the behaviour of Sylow subgroups with respect to normal subgroups and images.

LEMMA 4.1. *Let* $\boldsymbol{S} = \{S_p\}$ *be a Sylow basis of a* CC-*group* G.

1) *If* N *is a normal subgroup of* G, *then* $\boldsymbol{S} \cap N = \{S_p \cap N\}$ *is a Sylow basis of* N *and, if* N *is furthermore periodic,* $\boldsymbol{S}/N = \{S_p N/N\}$ *is a Sylow basis of* G/N.

2) *If* π *is a set of primes, then* $S_\pi = \langle S_p / p \in \pi \rangle \in \mathrm{Syl}_\pi(G)$. *(This* S_π *is called the Sylow* π*-subgroup of* G *associated to* $\boldsymbol{S}$*).*

In order to state the existence and the local conjugacy of Sylow basis, we follow the original approach due to P. Hall. As usual, if p is a prime number, we denote by p' the set of all primes different from p. We also recall that a *Sylow complement system of a group* G is a set $\boldsymbol{K} = \{S_{p'}\}$ of Sylow p'-subgroups of G, one for each prime p. Thus, it is clear that any group G has a complement system. As a consequence of Lemma 4.1, a Sylow basis $\boldsymbol{S} = \{S_p\}$ of a CC-group G determines a Sylow complement system $\boldsymbol{K} = \{S_{p'}\}$ of G given by $S_{p'} = \langle S_q \,|\, q \in p' \rangle$ for each prime p. Proceeding as in the finite case, it is possible to show that the correspondence between Sylow bases and complement systems is one to one in the locally soluble case.

LEMMA 4.2. *Let* $\boldsymbol{K} = \{S_{p'}\}$ *be a complement system of a locally soluble* CC-*group* G. *If we define* $S_p = \cap\{S_{q'} \,|\, q \neq p\}$, *then* S_p *is a Sylow* p*-subgroup of* G, $\boldsymbol{S} = \{S_p\}$ *is a Sylow basis of* G *and* $S_{p'}$ *is the Sylow* p'*-subgroup associated* $\boldsymbol{S}$.

Moreover, the above correspondence between Sylow bases and complement systems is one to one.

We are now in a position of extending results of Gol'berg and Stonehewer to CC-groups (see [13; 5.22]).

THEOREM 4.3. *A* CC-*group* G *has a Sylow basis if and only if* G *is locally soluble. In this case, any two Sylow bases of* G *are locally conjugate in* G.

PROOF. If G is locally soluble and $\boldsymbol{K}$ is a complement system of G, by Lemma 4.2, $\boldsymbol{K}$ gives rise to a Sylow basis $\boldsymbol{S}$ of G. Conversely suppose that G has a Sylow basis. Let Σ be a local system of G consisting of normal closures of finite subsets of G. Given $H \in \Sigma$, the torsion subgroup T

of H is a Černikov normal subgroup of G and H/T is abelian. By Lemma 4.1, T has a Sylow basis and, making use of the corresponding result in the finite case, it is not hard to see that T is soluble. Hence H is soluble and G is locally soluble.

The local conjugacy of the Sylow bases can be obtained in a straightforward way.

It follows from Theorem 4.3 that the complement systems of G are locally conjugate. We also remark that the local conjugacy of the Sylow bases of a locally soluble CC-group G and Lemma 2.2 allow us to extend the statement of Lemma 4.1, proceeding as we did in Corollary 3.2: *The Sylow bases of a normal subgroup N of G and of the quotient G/N, provided N is periodic, have the form prescribed in Lemma 4.1.*

Another consequence of the relationship between Sylow bases and complement systems is that we have $\cap\{N_G(S_p)|p \text{ prime}\} = \cap\{N_G(S_{p'})|p \text{ prime}\}$, whenever $\boldsymbol{S} = \{S_p\}$ is a Sylow basis and $\boldsymbol{K} = \{S_{p'}\}$ is the corresponding complement system of the locally soluble CC-group G. This subgroup is called *the basis normalizer associated to* $\boldsymbol{S}$. It should be remarked that these normalizers are also locally conjugate as well as they will play an important role in what follows. We may extend in a natural way classical results such as those given in ([13] 5.9, 5.14, 6.9 and 6.12) to obtain the following properties of a basis normalizer.

LEMMA 4.4. *Let D be the basis normalizer associated to a Sylow basis $\boldsymbol{S}$ of a periodic locally soluble* CC*-group G and let N be a normal subgroup of G.*

1) *D is locally nilpotent.*

2) *DN/N is the basis normalizer associated to the Sylow basis $\boldsymbol{S}N/N$ of G/N and all these normalizers have this form.*

3) *If G/N is locally nilpotent, then $G = DN$.*

In order to introduce Carter subgroups, we start recalling some definitions and well-known facts. A subgroup K of a group G is said to be *abnormal in G* if $g \in \langle K, K^g \rangle$ for every $g \in G$. It is clear that if K is abnormal in G, then $K = N_G(K)$ and every subgroup of G containing K is also abnormal in K. K is said to be *quasiabnormal in G* if every subgroup of G that contains K is self-normalizing. We add to the above concepts those of *a locally nilpotent projector of G*, that is, a locally nilpotent subgroup K of G such that $L = MK$ whenever $K \leqslant L \leqslant G$ and L/M is locally nilpotent, and of *Carter subgroup of G*, that is, a self-normalizing locally nilpotent subgroup of G. Since a locally nilpotent CC-group is hypercentral ([5; 1.1]) and, in particular, an N-group (N-group = normalizer condition), we note that a Carter subgroup of a CC-group G is a maximal locally nilpotent subgroup of G.

To extend to CC-groups the equivalence among these concepts we need an auxiliary fact.

LEMMA 4.5. *Let G be a periodic* CC-*group. Suppose that H is a subgroup of G and that K is the normal closure of a finite subset of G. Then $C_H(K)$ is normal in HK and $HK/C_H(K)$ is Černikov.*

PROOF. Clearly $C_H(K)$ is a normal subgroup of HK. We may write $HK/C_H(K) = (H/C_H(K))(KC_H(K)/C_H(K))$. Since G is a periodic CC-group, $H/C_H(K)$ and K are Černikov groups. Therefore $HK/C_H(K)$ is Černikov.

THEOREM 4.6. *Let L be a locally nilpotent subgroup of a periodic locally soluble* CC-*group G. Then the following are equivalent.*

1) *L is a locally nilpotent projector of G.*

2) *L is abnormal in G.*

3) *L is quasiabnormal in G.*

4) *L is a Carter subgroup of G.*

PROOF. (1) $\Rightarrow$ (2). Let $x \in G$. Clearly $\langle L, L^x \rangle \leqslant Lx^G$. By Lemma 4.5, $Lx^G/C_L(x^G)$ is Černikov and in particular soluble. The standard properties of projectors allow us to conclude that $L/C_L(x^G)$ is a locally nilpotent projector of $Lx^G/C_L(x^G)$. It is clear that a soluble Černikov group is a U-group in the sense of [3] and we note that in [3] Carter subgroup means locally nilpotent projector ([3; p. 202]). By [3; 5.6], $L/C_L(x^G)$ is abnormal in $Lx^G/C_L(x^G)$. Therefore, it is clear that $xC_L(X^G) \in \langle L, L^x \rangle / C_L(x^G)$ and so $x \in \langle L, L^x \rangle$.

(2) $\Rightarrow$ (3). This is a trivial consequence of the definitions.

(3) $\Rightarrow$ (4). This is clear.

(4) $\Rightarrow$ (1). We suppose that there is a subgroup H of G containing L with a factor H/K which is locally nilpotent but such that $V = KL$ is properly contained in H. Thus V/K is a proper subgroup of H/K and, since H/K is an N-group, V/K is properly contained in its normalizer N/K in H/K. Thus we may take a finite non-trivial nilpotent subgroup U/V of N/V. Write $U = VX^U$, for some finite set X of U. It is clear that $L/C_L(X^U)$ is a Carter subgroup of $LX^U/C_L(X^U)$. By Lemma 4.5, $LX^U/C_L(X^U)$ is a soluble Černikov group so, by [11; 2.2] and [3;5.6], $L/C_L(X^U)$ is a locally nilpotent projector of $LX^U/C_L(X^U)$. Clearly

$C_L(X^U) \leqslant V \cap LX^U$. Moreover, the factor $LX^U/V \cap LX^U \cong U/V$ is nilpotent, so we have $LX^U = L(V \cap LX^U)$ and then $V = U$, a contradiction.

The key point to establish the local conjugacy of Carter subgroups and the characterization of the conjugacy of all these concepts is to deal with the poly-locally nilpotent case. We first state an auxiliary result.

LEMMA 4.7. *Let G be a periodic group and suppose that $G = MN$, where M is normal in G and M and N are locally nilpotent. If π is a set of primes, then $O_\pi(M)O_\pi(N) \in \mathrm{Syl}_\pi(G)$.*

PROOF. Since M and N are locally nilpotent and M is normal in G, we have $M = O_\pi(M)O_{\pi'}(M)$ and $N = O_\pi(N)O_{\pi'}(N)$. Then $G = MN = (O_\pi(M)O_\pi(N))(O_{\pi'}(M)O_{\pi'}(N))$. Since $O_\pi(M)O_\pi(N)$ is a π'-group, by [13; 5.9], we find that $O_\pi(M)O_\pi(N) \in \mathrm{Syl}_\pi(G)$, as required.

THEOREM 4.8. *Let G be a periodic locally nilpotent-by-locally* CC-*group. Then*

1) *The Carter subgroups of G are exactly the basis normalizers of G.*

2) *There is a bijection between the Sylow bases of G and the basis normalizers of G which is stable under conjugation.*

PROOF. Let H be the Hirsch-Plotkin radical of G so that G/H is locally nilpotent.

1) Let D be a basis normalizer of G. By Lemma 4.4, D is locally nilpotent and $G = DH$. Suppose we have enumerated all the primes numbers $p_1, p_2, \ldots$. Denote by D_i and H_i the respective Sylow p_i'-subgroups of D and H. By Lemma 4.7, $Q_i = D_iH_i$ is a Sylow p_i'-subgroup of G so that $\boldsymbol{K} = \{Q_i\}$ is a complement system of G. By the local conjugacy of basis normalizers, there is some $\varphi \in \mathrm{Linn}(G)$ such that $\cap\{N_G(Q_i)|i \geqslant 1\} = D^\varphi$. It is clear that we have $D \leqslant N_G(Q_i)$, $i \in I$, so that $D \leqslant D^\varphi$. By Lemma 2.1, $D = D^\varphi$. Then it follows for each $i \in I$ that $N_G(D) \leqslant N_G(D_i) \leqslant N_G(Q_i)$ so that $N_G(D) = D$. Therefore D is a Carter subgroup of G.

Conversely, let C be a Carter subgroup of G. By Theorem 4.6, C is a locally nilpotent projector of G so that $G = CH$. Reasoning as above we find that C is contained in a basis normalizer D of G. Then $C = D$ because C is a maximal locally nilpotent subgroup of G.

2) Let $\boldsymbol{S} = \{S_p\}$ and $\boldsymbol{T} = \{T_p\}$ be two Sylow bases of G having the same basis normalizer D. Then $S_p \cap D = T_p \cap D \in \mathrm{Syl}_p(D)$, for every p. Since $G = DH$, by Lemma 4.7, we have that $(S_p \cap D)(S_p \cap H)$ and $(T_p \cap D)(T_p \cap H)$ are Sylow p-subgroups of G and $S_p \cap H = T_p \cap H$, for every p. It follows that $\{(S_p \cap D)(S_p \cap H)\}$ is a Sylow basis of G so that $S_p = (S_p \cap D)(S_p \cap H)$, for every p. Similarly, $T_p = (T_p \cap D)(T_p \cap H)$, for every p, and hence $\boldsymbol{S} = \boldsymbol{T}$. The above construction shows that this bijection is stable under conjugation.

THEOREM 4.9. *A periodic locally soluble* CC*-group G has Carter subgroups and any two Carter subgroups of G are locally conjugate in G.*

PROOF. By Theorem 4.8, we may think of Carter subgroups as locally nilpotent projectors so that we shall use the properties of projectors in the next argument. Let R be the radicable part of G. By [6; 2.1], R is abelian and G/R is an FC- group. By [13; 6.19] we may choose a locally nilpotent projector L/R of G/R. Then L is locally nilpotent-by-locally nilpotent so that Theorems 4.3 and 4.8 assure that L has a locally nilpotent projector, say C. It is clear that C is then a Carter subgroup of G (see [13; 6.17]).

Now let C_1 and C_2 be two Carter subgroups of G. Then C_1R/R and C_2R/R are locally nilpotent projectors of G/R so that, by [13; 6.19], there is some $\varphi \in \mathrm{Linn}(G/R)$ such that $(C_1R/R)^\varphi = C_2R/R$. By Theorem 2.2, we may extend φ to an element $\theta \in \mathrm{Linn}(G)$. Thus $(C_1)^\theta$ and C_2 are Carter subgroups of C_2R and, since C_2R is locally nilpotent-by-locally nilpotent, $(C_1)^\theta$ and C_2 are locally conjugate in C_2R. Extending the corresponding locally inner automorphism of C_2R to G we conclude that C_1 and C_2 are locally conjugate in G.

By investigating a CC-group modulo its radicable part of its Hirsch-Plotkin radical we shall obtain structural consequences on the corresponding quotients. To translate this information back to G we shall need the following characterization.

LEMMA 4.10. *For a periodic* CC*-group G the following conditions are equivalent.*

1) *G is locally nilpotent-by-finite.*

2) *G is locally nilpotent-by-Černikov.*

3) *G is Černikov-by-locally nilpotent*

4) *There exists a finite subset X of G such that G/X^G is locally nilpotent.*

PROOF. Let H be the Hirsch-Plotkin radical of G and denote by Z the centre of G. By [7; Theorem B], G/H is residually finite.

(1) $\Rightarrow$ (4). Since G/H is finite, we may write $G = HX^G$, for some finite subset X of G. Thus G/X^G is locally nilpotent.

(4) $\Rightarrow$ (3). This is clear: X^G is Černikov.

(3) $\Rightarrow$ (2). Let C be a Černikov normal subgroup of G such that G/C is locally nilpotent. Since G/Z is residually Černikov and CZ/Z satisfies the minimal condition, there is a normal subgroup N of G containing Z such that $N \cap C \leqslant Z$ and G/N is Černikov. Thus $N/N \cap C$ is locally nilpotent and so is N. Therefore (2) follows.

(2) $\Rightarrow$ (1). Here G/H is Černikov and residually finite, so finite.

We remark that the condition «finite-by-locally nilpotent» is not equivalent to any of those of Lemma 4.10. For, let G be one of the groups described in the section 7 of [2] (Propositions 3 and 4). Here G is a Černikov soluble group which is not finite-by-abelian but in which every proper subgroup is abelian or finite. Suppose that G has a finite normal subgroup F such that G/F is locally nilpotent. If C is a Carter subgroup of G, then $G = CF$ and, since G cannot be finite, it follows that C has to be abelian, a contradiction.

By definition, *the locally nilpotent residual of a group* G is the intersection of all normal subgroups N of G such that G/N is locally nilpotent. The next result is known but we give a proof for the reader's convenience.

LEMMA 4.11. *If L is the locally nilpotent residual of the locally finite group G, then G/L is locally nilpotent.*

PROOF. Let S/L be a finite subgroup of G/L. There exists a finite subgroup F of G such that $S = LF$. Since F is finite there exists a normal subgroup N of G such that G/N is locally nilpotent and $N \cap FL = L$. Then S/L is isomorphic to a subgroup of G/N and hence S/L is nilpotent. Therefore, G/L is locally nilpotent.

We finally give the main result relating to conjugacy.

THEOREM 4.12. *For a periodic locally soluble* CC*-group G the following conditions are equivalent.*

1) *The Sylow bases of G are conjugate.*

2) *G has a countable number of Sylow bases.*

3) *The Sylow complement systems of G are conjugate.*

4) *G has a countable number of Sylow complement systems.*

5) *For each set of primes π, the Sylow π-subgroups of G are conjugate.*

6) *For each set of primes π, G has a countable number of Sylow π-subgroups.*

7) *The Carter subgroups of G are conjugate.*

8) *G has a countable number of Carter subgroups.*

9) *The basis normalizers of G are conjugate.*

10) *G has a countable number of basis normalizers.*

11) *G is locally nilpotent-by-finite.*

12) *G is a U-group.*

13) *Any one of the conditions (1)-(12) hold in G/L', where L is the locally nilpotent residual of G.*

PROOF. In what follows we shall denote by H the Hirsch-Plotkin radical of G and by R the radicable part of G. We recall that R is abelian, $R \leqslant H$, G/R and G/H are FC-groups and G/H is residually finite ([6; 2.1] and [7; Theorem B]).

The equivalences (1)$\Leftrightarrow$(3) and (2)$\Leftrightarrow$(4) are a clear consequence of Lemma 4.2 and the equivalence (5)$\Leftrightarrow$(6) follows from Theorem 3.5. The assertions (1)$\Rightarrow$(9) and (2)$\Rightarrow$(10) are trivial.

(3)$\Rightarrow$(5) and (4)$\Rightarrow$(6). Given a set of primes π we fix $P \in \mathrm{Syl}_\pi(G)$. For each $p \notin \pi$ there is $S_{p'} \in \mathrm{Syl}_\pi(G)$ such that $P \leqslant S_{p'}$. Therefore we have that $P = \cap\{S_{p'} \,|\, p \notin \pi\}$, since the latter is a π-group. Thus a Sylow π-subgroup of G can always be obtained as the intersection of some members $\{S_{p'} \,|\, p \notin \pi\}$ of a certain complement system of G so that the two implications follow.

(11)$\Rightarrow$(2). By hypothesis G/H is finite. Denote by σ the set of all primes occurring as the prime divisors of the orders of the elements of G/H. If $p \notin \sigma$, then $O_p(G)$ is the unique Sylow p-subgroup of G. If $p \in \sigma$ and $P \in \mathrm{Syl}_p(G)$ then $P/O_p(G)$, being isomorphic to a subgroup of G/H, is finite and it is clear that then G only has countably many Sylow p-subgroups. Since σ is a finite set, we conclude that G only can have a countable number of Sylow bases.

(5) or (6)$\Rightarrow$(1) and (7). It is clear that each subgroup of G also satisfies (6) (and so (5)). Then we may apply [4; Theorem E] to deduce that G is a U-group. Therefore (1) and (7) follow

from [3; 2.10 and 5.4]. Moreover, it is clear that (5) is equivalent to (12).

(1) $\Rightarrow$ (11). Clearly every subgroup and every factor of G satisfy (1). Since G/H is an FC-group, by [12; Theorem D], G/H is locally nilpotent-by-finite. Thus, in order to show that G/H is finite, we may assume that G/H is locally nilpotent. It is clear that H/Z is the Hirsch-Plotkin radical of G/Z, so we may replace G by G/Z to assume that G is residually Černikov. Since the condition (9) holds, if D is a basis normalizer of G, then D/D_G is Černikov by Theorem 2.3. By Lemma 4.4, $G = DH$ and, since $D_G \leqslant H$, it follows that G/H is Černikov. By Lemma 4.10, G/H is then finite.

(7), (8), (9) or (10) $\Rightarrow$ (11). First of all we recall that G/R satisfies (i) provided G satisfies (i), where $i = 7$, 8, 9 or 10. Since G/R is an FC-group, in the cases (8) or (10) we may assure that G/R has a finite class of conjugacy of Carter subgroups or basis normalizers (see [13; chapter 4]). In any case we apply [12; Theorem D] and [13; 6.31] to deduce that G/R is finite-by-locally nilpotent. Let F be a normal subgroup of G containing R such that F/R is finite and G/F is locally nilpotent. F is then an abelian-by finite CC-group and then, by [5; 1.1] and [10; 4.23], F' is Černikov. Now G/F' is abelian-by-locally nilpotent so that, by Theorem 4.8, we have that G/F' satisfies (1) or (2). We have already shown that G/F' is then locally nilpotent-by-finite. Then G has a normal subgroup L of finite index such that L is Černikov-by-locally nilpotent. By Lemma 4.10, L is locally nilpotent-by-finite and so is G.

(2) $\Rightarrow$ (8). Due to the equivalence between (2) and (6), G is again a $\boldsymbol{U}$-group. We note that G is also locally nilpotent-by-finite. Clearly G has a countable number of basis normalizers. By [11; 3.6] and Theorem 4.6, every Carter subgroup of G contains at least one basis normalizer and every basis normalizer is contained in at least one Carter subgroup of G, which is unique by [3; 5.10]. Thus there is an onto map between the basis normalizers of G and the Carter subgroups of G so that the number of Carter subgroups of G is countable.

Thus we have shown that the conditions (1)-(12) are equivalent for a periodic locally soluble CC-group. Obviously (11) $\Rightarrow$ (13). Conversely, suppose that (13) holds. Then G/L' is locally nilpotent-by-finite. By Lemma 4.10, there is a finite subset X of G such that $G/X^G L'$ is locally nilpotent. Then $L \leqslant X^G L'$ and so $L = (L \cap X^G)L'$. Thus the quotient $L/L \cap X^G$ is a perfect group and it has to be trivial. Then $L \leqslant X^G$ and, by Lemma 4.11, G/X^G is locally nilpotent. By Lemma 4.10, G is locally nilpotental-by-finite and the proof is now complete.

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