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**Boundary-value problems for a class of first order partial differential equations in Sobolev spaces and applications to the Euler flow**

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# Boundary-Value Problems for a Class of First Order Partial Differential Equations in Sobolev Spaces and Applications to the Euler Flow.

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## 1. Notations.

Let $\Omega$ be an open bounded subset of $\boldsymbol{R}^n$, $n \geq 2$, that lies (locally) on one side of its boundary $\Gamma$, a $C^4$ manifold. We denote by $\nu$ the unit outward normal to $\Gamma$.

For $h(x) = (h_{rs}(x))$, where $h_{rs}$ are real functions defined on $\Omega$, we define

$$(1.1) \qquad |D^l h(x)|^2 = \sum_{|\alpha|=l} \sum_{r=1}^{R} \sum_{s=1}^{S} |D^\alpha h_{rs}(x)|^2 ,$$

where $l$ is a nonegative integer, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, and $|\alpha| = \alpha_1 + \ldots + \alpha_n$. We set $|h| = |D^0 h|$, $|Dh| = |D^1 h|$. If for each couple of indices $r, s$ one has $h_{rs} \in X$, where $X$ is a function space, we simply write $h \in X$.

For $u = (u_1, \ldots, u_N)$, $w = (w_1, \ldots, w_N)$, $v = (v_1, \ldots, v_n)$, we define

$$(1.2) \qquad u \cdot w = \sum_{j=1}^{N} u_j w_j , \qquad |u|^2 = u \cdot u , \qquad (v \cdot \nabla) u = \sum_{i=1}^{n} v_i D_i u .$$

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We will use the abbreviate notations

$$D_i h = \frac{\partial h}{\partial x_i}, \qquad \int h = \int_\Omega h(x)\,dx, \qquad (u, w) = \int u\cdot w .$$

In general, if $X$ and $Y$ are Banach spaces, $\mathcal{L}(X, Y)$ denotes the Banach space of all bounded linear maps from $X$ into $Y$. We set $\mathcal{L}(X) = \mathcal{L}(X, X)$.

We denote by $C^k$, $k \geq 0$, the Banach space consisting of functions defined in $\bar{\Omega}$, and which are restrictions to $\bar{\Omega}$ of $C^k(\boldsymbol{R}^n)$ functions. The canonical norm in the above space is denoted by $[\ ]_k$. $C_0^k$ denotes the subspace of $C^k$ consisting of functions vanishing on $\Gamma$. We denote by $L^p$ the Banach space $L^p(\Omega)$, and by $|\ |_p$ its canonical norm (see below).

The real number $p \in ]1, +\infty[$, and the domain $\Omega$ are fixed once for all. For convenience these symbols will be dropped even from some standard notations. According to this convention, $W^k$ denotes the Sobolev space $W^{k,p}(\Omega)$ and $\|\ \|_k$ denotes the canonical norm $\|\ \|_{k,p}$ (see below). However, in sections 3 and 4 some functional spaces are defined with respect to an open set $B \neq \Omega$, and in section 5 some functional spaces and operators are defined with respect to a value $q \neq p$. In these cases, either the symbols $B$ or $q$ will be inserted in the notation, or a different notation will be used.

We define $\mathring{W}^k$, $k \geq 1$, as the closure of $C_0^\infty(\Omega)$ in $W^k$, and $W^{-k}$ as the dual space of $\mathring{W}^{k,q}$, where $q = p/(p-1)$. Finally

$$W_l^k = W^k \cap \mathring{W}^l \equiv W^{k,p}(\Omega) \cap \mathring{W}^{l,p}(\Omega)\,,$$

where $0 \leq l \leq k$. For convenience, we set $W_{k+1}^k = \mathring{W}^k$. Clearly, $W_0^k = W^k$, and $W_k^k = \mathring{W}^k$. Note that $W_l^k$, $l \geq 1$, is the subspace of $W^k$ consisting of functions vanishing on $\Gamma$ together with their derivatives of order less than or equal to $l-1$.

The above notation will also be used to denote function spaces whose elements are vector fields or matrices. For instance, both $L^p$ and $L^p \times \ldots \times L^p$ ($N$ times) will be denoted by the same symbol $L^p$, and the corresponding norms by the same symbol $|\ |_p$. Finally, for $h = (h_{rs}) \in W^k$, $k \geq 0$, we define

$$|D^l h|_p = \left(\int |D^l h(x)|^p\,dx\right)^{1/p}, \qquad \|h\|_k = \sum_{l=0}^k |D^l h|_p .$$

In the sequel, $T$ is a fixed positive real number, and $I = [-T, T]$. We use standard notations for functional spaces consisting of functions defined on $I$ with values in a Banach space. In particular, the canonical norm in the Banach space $L^\infty(I; C^k)$, $k \geq 0$, is denoted by $[\ ]_{I,k}$, that in $L^\infty(I; W^k)$ by $\|\ \|_{I,k}$, and that in $L^1(I; W^k)$ by $|||\ |||_{I,k}$.

In the sequel the symbol $c$ may denote different positive constants. The symbol $c(\Omega, n, N, p, k)$ means that $c$ depends *at most* on the variables inside brackets.

## 2. Results.

Let $a = (a_{jk})$ be a $N \times N$ matrix, $N \geq 1$, and $v = (v_1, \dots, v_n)$ be a vector field, both defined on $I \times \bar{\Omega}$. Mostly we will assume here that

$$v \cdot \nu = 0 \,, \qquad \forall (t, x) \in I \times \Gamma \,. \tag{2.1}$$

Let us consider the initial-boundary value problem

$$\begin{cases} D_t u + (v(t,x)\cdot\nabla)u + a(t,x)u = f\,, & \text{in } I \times \Omega\,, \\ u = \dots = D^{l-1} u = 0\,, & \text{on } I \times \Gamma\,, \\ u_{|t=0} = u_0 & \text{in } \Omega\,, \end{cases} \tag{2.2}$$

where $l$ is a fixed nonegative integer (if $l = 0$, the equation $(2.2)_2$ is dropped), $f = (f_1, \dots, f_N)$ is a given vector field in $I \times \Omega$, $u_0$ is a given vector field in $\Omega$ and $(v\cdot\nabla)u = (v\cdot\nabla u_1, \dots, v\cdot\nabla u_N)$.

In particular, we will show that the Cauchy-Dirichlet problem (2.2) admits a solution $u$ if and only if $f$ verifies the condition $f = \dots = D^{l-1} f = 0$. Problem (2.2) will be studied in Sobolev spaces $W^{k,p}(\Omega)$ for arbitrary $p \in ]1, +\infty[$. Moreover, in case that $l = 0$, the parameter $k$ is also allowed to be negative.

Our approach to the evolution problem looks interesting by itself: Following the author's paper [8], we will prove that the operator $\mathcal{A}(t)$, defined by equation (2.3), is the generator of a strongly continuous group of operators in suitable Sobolev spaces $W^k_l(\Omega)$. This enables us to apply the well known general theory developed by T. Kato [14], [15], [16]. Main points here are the study of the stationary problem (2.4), which has direct applications to interesting physical problems, and the proof that the abstract theory of Kato applies to the initial-boundary value problem (2.2). Other approaches to problems (2.2) and (2.4) are possible, and we believe that the results obtained here are partially known.

Let us consider the differential operator

$$\mathcal{A}(t)u \equiv (v\cdot\nabla)u + au\,, \qquad t\in I\,, \tag{2.3}$$

defined on vector fields $u = (u_1, \dots, u_N)$ in $\Omega$, and acting in the *distributional sense.* We set

$$D_l^k(t) \equiv \{u \in W_l^k\colon (v\cdot\nabla)u \in W_l^k\}\,,$$

for each fixed $t\in I$, and for each couple of integers $l, k$ such that $0 \le l \le k$, $1 \le k$. We denote by $A_l^k(t)$ the restriction of the operator $\mathcal{A}(t)$ to the domain $D_l^k(t)$, i.e., we define

$$A_l^k(t)u \equiv \mathcal{A}(t)u\,, \qquad \forall u \in D_l^k(t)\,.$$

The lower index $l$ means that there are $l-1$ boundary conditions. The definitions in case that $k \le 0$ will be postponed to section 5. On treating the time independent case we drop the symbol $t$ from the above symbols and definitions.

One has the following result.

THEOREM 2.1. *Let $k$ be a fixed integer, let $\Gamma \in C^{|k|+2}$, assume that*

$$(H_k) \qquad \begin{cases} v, a \in C^{|k|} & \text{if } k \ne 0\,, \\ v, a \in C^1 & \text{if } k = 0\,, \end{cases}$$

*and that* (2.1) *holds. If $0 \le l \le k$, the equation*

$$\lambda u + \big(v(x)\cdot\nabla\big)u + a(x)u = f(x) \quad \text{in } \Omega \tag{2.4}$$

*has a unique solution $u \in W_l^k$, for each $f \in W_l^k$, provided $|\lambda| > \theta_k$, where*

$$\theta_k \equiv c(\Omega, n, N, p, k)\big([v]_{|k|} + [a]_{|k|}\big)\,, \quad \text{if } k \ne 0\,, \tag{2.5}$$

*and $\theta_0 = \theta_1$ if $k = 0$. Here, $c$ denotes a suitable positive constant depending only on the variables inside brackets. Moreover, the solution $u$ verifies* *the estimate*

$$\|u\|_k \leqslant \frac{1}{|\lambda| - \theta_k} \|f\|_k .$$

*If $l = 0$ the above result holds for any integer $k$. Finally, if $0 = l \le k$, and without assuming that* (2.1) *holds, there exists a linear continuous map $G \in \mathcal{L}(W^k)$ such that $u = Gf$ is a solution of* (2.4), *for each $f \in W^k$. Moreover* (3.18) *holds. Here, we assume that $\Gamma \in C^k$ [resp. $C^1$, if $k = 0$].*

In particular theorem 2.1 shows that if $v$ and $a$ are time independent, then the operators $A^k$, for any integer $k$ [resp. $A^k_l$, for $0 \le l \le k$, $1 \le k$] generate strongly continuous groups of operators in the Banach spaces $W^k$ [resp. $W^k_l$].

It is worth noting that (if $k \ge 0$) the above estimates are trivially obtained (and well known), if the coefficients and the solution are sufficiently smooth. Here, we give a *simple and rigorous* prove of the existence result. Cf. the remark 2.4.

By combining the above result with Kato's results, we obtain the following theorem.

THEOREM 2.2. *Let $k$ be a fixed integer (not necessarily nonnegative). Assume that $\Gamma \in C^{|k|+2}$, that* (2.1) *holds, and that*

$$(H_k) \qquad v, a \in L^\infty(I; C^{|k|}) \cap C(I; C^{|k|-1}), \quad \text{if } k \neq 0 .$$

*If $k = 0$ assume that condition $(H_1)$ holds. Then, the family operators $\{A^k(t)\}_{t \in I}$, is $(1, \theta_k)$-stable in $W^k$, where by definition*

$$\theta_k = c(\Omega, n, N, p, k)\big([v]_{I,|k|} + [a]_{I,|k|}\big), \quad \text{if } k \neq 0 ,$$

*and $\theta_0 = \theta_1$ if $k = 0$. The map $t \to A(t)$ is norm continuous on $I$ with values in $\mathcal{L}(W^k, W^{k-1})$, for $k \geqslant 1$. In case that $k \ge 1$ all the above results hold for the family $\{A^k_l(t)\}$ in the space $W^k_l$, for each fixed $l = 0, \ldots, k$.*

*If $u_0 \in W^k$ and $f \in L^1(I; W^k)$, $k$ not necessarily nonegative, then the Cauchy problem $(2.2)_1$, $(2.2)_3$ has a unique strong solution $u \in C(I; W^k)$. Furthermore, if $0 \le l \le k$, $1 \le k$, and if $u_0 \in W^k_l$, $f \in L^1(I; W^k_l)$, the Cauchy-Dirichlet problem* (2.2) *has a unique strong solution $u \in C(I; W^k_l)$. Moreover,*

$$(2.6) \qquad \|u\|_{k,I} \le \big(\|u_0\|_k + |||f|||_{I,k}\big) \exp[\theta_k T] .$$

One also proves the following result.

COROLLARY 2.3. *Let* $0 = l \le k$, *let* $(H_k)$ *holds, and assume that* $\Gamma \in C^k$ *[resp.* $C^1$, *if* $k = 0$*]. Condition* (2.1) *is not required here. Then, there exists a linear continuous map* $G \in \mathcal{L}(W^k \times L^1(I; W^k); C(I; W^k))$ *such that* $u = G(u_0, f)$ *is a solution of problem* (2.2), *for each pair* $(u_0, f)$. *Moreover,*

$$(2.7) \qquad \|u\|_{k,I} \le c(\Omega, n, N, p, k)(\|u_0\|_k + |||f|||_{I,k}) \exp[\theta_k T] .$$

*Convention.* Whenever it is claimed that a property holds for $|\lambda| > \theta_k$, it is understand that in the definition of $\theta_k$ a suitable choice of $c$ is to be made.

It is worth noting that the theorems 2.1 and 2.2 are stated in a form which is not convenient for applications to non-linear problems. In fact, in many of the applications the coefficient $v$ and the solution $u$ belong to the same Sobolev space. A main point here is that the proofs work again if the coefficients $v$ (and $a$) belong to suitable Sobolev spaces, rather than to $C^k$. One has to use just Sobolev's embedding theorems (and Hölder's inequality) in order to deal with terms of the form $D^\alpha v \cdot D^\beta u$. The choice of the particular Sobolev spaces depends on the applications we have in mind. Since there are only slight modifications to be made on the proofs, it seems preferable to us to give the proofs for a specific case. We made the choice $v, a \in C^k$, in order to avoid a continuous and trivial recall to Sobolev's embedding theorems. We state (below) the corresponding results also for a specific case in which the coefficients belong to Sobolev spaces, since we are interested on it for applications to the Euler (and similar) equations. For convenience, we state this last results only in case that $k \ge 0$ (since $k \ge 2$ in the above applications).

THEOREM 2.1*. *Let* $k$ *be a non-negative integer, and let* $\Gamma \in C^{k+2}$. *Assume that* (2.1) *holds and that*

$$(H_k^*) \qquad \begin{cases} v, a \in W^k , & \text{if } k > 1 + (n/p) , \\ v \in W^{1,\infty} , \quad a \in W^1 , & \text{if } k \leqslant 1 \text{ and } p > n . \end{cases}$$

*Let* $0 \le l \le k$. *Then, equation* (2.4) *has a unique solution* $u \in W_l^k$ *for each* $f \in W_l^k$, *provided* $|\lambda| > \theta_k^*$. *Here,*

$$\theta_k^* \equiv c(\Omega, n, N, p, k)(\|v\|_k + \|a\|_k) , \qquad \text{if } k \ge 2 ,$$

*and*

$$\theta_0^* \equiv \theta_1^* \equiv c(\Omega, n, N, p)(\|v\|_{1,\infty} + \|a\|_1)\,, \qquad if\ k \le 1\,.$$

*Moreover,*

$$\|u\|_k \leqslant \frac{1}{|\lambda| - \theta_k^*} \|f\|_k\,.$$

*Furthermore, equation* (2.4) *has a (unique) solution* $u \in W_l^{k-1}$, *for each* $f \in W_l^{k-1}$. *Moreover,*

$$\|u\|_{k-1} \le \frac{1}{|\lambda| - \theta_k^*} \|f\|_{k-1}\,.$$

*Finally, the last assertion of theorem* 2.1 *holds again by replacing* $\theta_k$ *by* $\theta_k^*$ *in equation* (3.18).

THEOREM 2.2*. *Let* $k$ *be a non-negative integer, and let* $\Gamma \in C^{k+2}$. *Assume that* (2.1) *holds, and that*

$$(H_k^*) \qquad \begin{cases} v, a \in L^\infty(I;\, W^k) \cap C(I;\, W^{k-1})\,, & if\ k > 1 + (n/p)\,, \\ v \in L^\infty(I;\, W^{1,\infty}) \cap C(I;\, L^\infty)\,, \quad a \in L^\infty(I;\, W^1) \cap C(I;\, L^\infty)\,, & \\ & if\ k \le 1\,, \quad p > n\,. \end{cases}$$

*Then, the family of operators* $\{A_l^k(t)\}_{t\in I}$, *is* $(1, \theta_k^*)$*-stable in* $W_l^k$, *where by definition*

$$\theta_k^* \equiv c(\Omega, n, N, p, k)(\|v\|_{I,k} + \|a\|_{I,k})\,, \qquad if\ k \ge 2\,,$$

*and*

$$\theta_0^* \equiv \theta_1^* \equiv c(\|v\|_{I;1,\infty} + \|a\|_{I,1})\,, \qquad otherwhise\,.$$

*If, in addition,* $u_0 \in W^k$, *and* $f \in L^1(I;\, W^k)$, *then the Cauchy problem* $(2.2)_1$, $(2.2)_3$ *has a unique strong solution* $u \in C(I;\, W^k)$. *Moreover, if* $0 \le l \le k$, *and if* $u_0 \in W_l^k$, $f \in L^1(I;\, W_l^k)$, *the above solution* $u$ *belongs to* $C(I;\, W_l^k)$. *Finally,*

$$(2.8) \qquad \begin{cases} \|u\|_{I,k} \le (\|u_0\|_k + |||f|||_{I,k}) \exp[\theta_k^* T]\,, \\ \|u\|_{I,k-1} \le (\|u_0\|_{k-1} + |||f|||_{I,k-1}) \exp[\theta_k^* T]\,. \end{cases}$$

One also proves the following result.

COROLLARY 2.3*. *Let* $0 = l \le k$, *let* $(H_k^*)$ *holds, and assume that* $\Gamma \in C^k$ [*resp.* $C^1$, *if* $k = 0$]. *Condition* (2.1) *is not required here. There exists a linear operator* $G$ *from* $W^k \times L^1(I; W^k)$ *into* $C(I; W^k)$ *such that* $u = G(u_0, f)$ *is a solution of problem* (2.2) *for each pair* $(u_0, f)$. *Moreover the estimates* (2.8) *hold provided the right hand sides are multiplied by a suitable constant* $c(\Omega, n, N, p, k)$.

The following result will be usefull on dealing with nonlinear partial differential equations. Similar results hold in connection with theorem 2.2, and for the stationary problem.

COROLLARY 2.4*. *Assume that* $w, b, z_0, g$ *is another set of functions verifying the hypothesis required in corollary* 2.3*. *Let* $u$ *and* $z$ *be the solution of problem* (2.2) *for data* $v, a, u_0, f$ *and* $w, b, z_0, g$, *respectively. Then*

$$(2.9) \qquad \|z - u\|_{I,k-1} \le c\,\{\|z_0 - u_0\|_{k-1} + |||g - f|||_{I,k-1} +$$
$$+ (\|u_0\|_k + |||f|||_{I,k})(\|w - v\|_{I,k-1} + \|b - a\|_{I,k-1}) \exp[c\theta_k^* T] \exp[c\mu_k^* T]\}$$

*where* $\mu_k^* = c(\|w\|_{I,k} + \|b\|_{I,k})$, $\theta_k^* = c(\|v\|_{I,k} + \|a\|_{I,k})$, *and* $c$ *denotes different positive constants depending only on* $\Omega, n, N, p, k$.

REMARK. In all of the previous statements in which we do not assume (2.1) (hence, the uniqueness may fail) it is understood that the solution considered is that constructed in the corresponding proofs.

APPLICATIONS. In reference [6] we prove existence and regularity for the solution of the stationary, compressible, Navier-Stokes equations, and its convergence to the corresponding solution of the incompressible equations, as the Mach number goes to zero. A main tool in the proof is the theorem 2.1 in reference [8], which is a variant of the theorem 2.1 above.

An application of the last statement of theorem 2.1 is given by Kohn and Lowe in his interesting paper [18].

Finally, as an application of theorem 2.2*, we provide in section 6 a simple proof of the persistence property in Sobolev spaces for the solution of the Euler equations (6.1) in a bounded domain $\Omega \subset \mathbf{R}^n$, $n \geqq 2$.

REMARK 2.4. The proof of the existence of a solution of equation (2.4) in spaces $W^k$ is not an immediate consequence of the a priori estimate (3.13) together with an existence theorem in stronger spaces. Let us show the main obstacle. We consider, for convenience, the case in which $v \in C^k$ and $a \equiv 0$, and (just to fix the ideas) we recall the classical existence results of reference [19]. Let $f \in W^k$. The solution $u$ of problem (2.4) (provided by [19] or by any other existence result) is not sufficiently regular to justify the calculations leading to (3.13). This obstacle is not overcome by approximating $f$ (in the $W^k$ norm) with a sequence $f_n \in W^{m,2}$, for a fixed $m$ such that $W^{m,2} \hookrightarrow W^{k+1}$, since $v$ prevents the regularity of the solutions $u_n$. However, if one also approximates $v \in C^k$ (in the $C^k$ norm) by a sequence $v_n \in C^m$, then one gets solutions $u_n \in W^{m,2} \hookrightarrow W^{k+1}$, of problem $\lambda u_n + (v_n \cdot \nabla) u_n = f_n$, provided $\lambda > c[v_n]_m$. Moreover, the estimate $\|u_n\|_k \leqq [1/(\lambda - c[v_n]_m)]\|f_n\|_k$, $\forall n$, holds. However one can not pass to the limit as $n \to +\infty$, since $[v_n]_m \to +\infty$.

We point out that one can overcome the above obstacle (if $0 = l \leqq k$) by arguing as done for proving the point (iv) in theorem 3.9 below (the existence of the solution of the equation $\bar{\lambda} u + (v \cdot \nabla) u + au = f + (\bar{\lambda} - \lambda) u$ is shown here by arguing as done after equation (3.17), in the proof of theorem 3.8. This argument is used also in reference [8]).

In the evolution case, there is a weaker counterpart of the above obstacle. Again, the coefficient $v(t, x)$ is not sufficient regular for providing a solution $u(t, x)$ to which the calculations leading to the a priori estimate in Sobolev spaces applies rigorously. Nevertheless, in the evolution case, if one approximates the coefficient $v$ by regular coefficients $v_n$, one gets an estimate in the $C(I; W^k)$ norm, which is independent of $n$. A compactness argument shows the existence of a solution $u \in L^\infty(I; W^k)$. However, we lose the strong continuity on $I$ with values in $W^k$. We note that, in order to prove this last property by using the characteristics, quite hard arguments seems to be necessary. See Bourguignon and Brezis [10]. It could appear that all this question is artificial, since one should overcome it by assuming that $v$ is more regular. However, this last case is not sufficient to deal with many interesting nonlinear problems.

REMARK 2.5. It is worth noting that the results and proofs given here apply, with slight modifications, to the more general equation

$$D_t u + \sum_{i=1}^{n} a_i D_i u + au = f,$$

if the $N \times N$ symmetric matrices $a_i(t, x)$ verify the condition

$$\sum_{i=1}^{n} \nu_i a_i = 0 \quad \text{on } \Gamma,$$

*provided* $p = 2$.

REMARK. First order hyperbolic systems in domains with boundary have been studied by several authors. Since the main references are well known, it seems unecessary to provide them here. Let us just recall the references [1], [2], [5], [8], [12], [19], [20], [21], [22], [23], which are more or less connected to our paper.

## 3. The stationary problem (case $k \geq 0$).

We start this section by proving the following auxiliary result:

LEMMA 3.1. *Let $k \geq 1$, and assume that $v \in C^k$ satisfies the condition* (2.1). *Then,* $(v\cdot\nabla)u \in \mathring{W}^k$ *if* $u \in W_k^{k+1}$.

PROOF. By induction on $k$. If $k = 1$, the vectors $\nabla u_j$ and $\nu$ (for each $j = 1, \ldots, n$) have the same direction, since $u_j = 0$ on $\Gamma$. Hence, $v\cdot\nabla u_j = 0$. Assume now that the thesis holds for the value $k$, and let $u \in W_{k+1}^{k+2}$. By the induction hypothesis, one has $(v\cdot\nabla)u \in \mathring{W}^k$. On the other hand, $D_i[(v\cdot\nabla)u] = [(D_i v)\cdot\nabla]u + (v\cdot\nabla)D_i u$, $i = 1, \ldots, n$. The first term on the right hand side of this identity belongs to $\mathring{W}^k$. The same holds for the second one, by the induction hypothesis, since $D_i u \in W_k^{k+1}$. ■

LEMMA 3.2. *Let $t \in I$ be fixed. Under the hypothesis of lemma* 3.1, *and for each fixed $l = 0, \ldots, k$, the linear subspace $D_l^k(t)$ is dense in $W_l^k$, and $A_l^k(t)$ is a closed operator in $W_l^k$.*

PROOF. $\mathring{D}^k(t)$ is dense in $\mathring{W}^k$, since (lemma 3.1) $W_k^{k+1} \subset \mathring{D}^k(t)$. Moreover $D^k(t)$ is dense in $W^k$. Let now $u \in W_l^{k+1}$, for a fixed $l \geq 1$. Since $W_l^{k+1} = W^{k+1} \cap W_l^{l+1}$, one has $(v\cdot\nabla)u \in W^k \cap \mathring{W}^l = W_l^k$. This shows that $W_l^{k+1} \subset D_l^k(t)$. Consequently, this last subspace is dense in $W_l^k$. The closedness of the operators is quite immediate. ■

The following two lemmas underlay our proofs.

LEMMA 3.3. *Let* $p \in ]1, +\infty[$, $w = (w_1, \dots, w_N) \in C^1$, *and set* $\Lambda = (\delta + |w|^2)^{\frac{1}{2}}$, $\delta > 0$. *Then, for each* $x \in \Omega$, *one has*

$$\sum_{i=1}^{n} (D_i w)\cdot D_i(\Lambda^{p-2}w) = \Lambda^{p-2}|Dw|^2 + \frac{p-2}{4}\Lambda^{p-4}|\nabla(|w|^2)|^2 \,, \tag{3.1}$$

*and*

$$\sum_{i=1}^{n} (D_i w)\cdot D_i(\Lambda^{p-2}w) = \tag{3.2}$$

$$= \Lambda^{p-4}\Big\{[(p-1)|w|^2 + \delta]|Dw|^2 + (2-p)\Big[|w|^2|Dw|^2 - \sum_{i=1}^{n}(w\cdot(D_i w))^2\Big]\Big\}\,.$$

*In particular, for each* $p \in ]1, +\infty[$, *one has*

$$-\int \Delta w \cdot \Lambda^{p-2} w \geq 0\,, \qquad \forall w \in W_1^2\,. \tag{3.3}$$

PROOF. We left to the reader the proofs of (3.1) and (3.2) (1). If $w \in C^2(\bar{\Omega})$ vanishes on $\Gamma$, equation (3.3) follows upon integration by parts. Since the set $\{w \in C^2 \colon w|_\Gamma = 0\}$ is dense in $W_1^2$, (3.3) holds also for $w \in W_1^2$. Note that $\Lambda_n^{p-2}w_n \to \Lambda^{p-2}w$ strongly in $L^{p/(p-1)}$, if $w_n \to w$ in $L^p$ (by a well known Krasnoselskii's theorem). ∎

LEMMA 3.4. *Let* $w = (w_1, \dots, w_N) \in C^1$. *Then*

$$\Lambda^{p-2}(D_i w)\cdot w = \frac{1}{p} D_i \Lambda^p\,, \qquad i = 1, \dots, n\,. \tag{3.4}$$

*In particular, if* $v \in C^1(\bar{\Omega})$ *verifies* (2.1), *one has*

$$\int [(v\cdot\nabla)w]\cdot\Lambda^{p-2}w = -\frac{1}{p}\int(\operatorname{div} v)\Lambda^p\,, \qquad \forall w \in W^1\,. \tag{3.5}$$

PROOF. Left to the reader. ∎

(1) Recall the definitions (1.1) and (1.2).

LEMMA 3.5. *Assume that $v \in C^1$ verifies* (2.1), *that $a \in C^0$ and that $f \in L^p$. Let $u \in W^1$ be a solution of*

$$\lambda u + (v\cdot\nabla)u + au = f\,. \tag{3.6}$$

*Then, for* $|\lambda| > \tilde{\theta} \equiv c(p, N)([v]_1 + [a]_0)$ *one has*

$$(|\lambda| - \tilde{\theta})|u|_p \leq |f|_p\,. \tag{3.7}$$

*In particular, the solution $u$ of* (3.6), *if it exists, is unique.*

PROOF. The proof is done by multiplying both sides of (3.6) by $(\delta + |u|^2)^{(p-2)/2}u$, by integrating in $\Omega$, and by passing to the limit as $\delta \to 0^+$. ■

THEOREM 3.6. *Let the hypothesis* ($H_2$) *and* (2.1) *be satisfied. Then, for* $|\lambda| > \theta_2$, *equation* (3.6) *has a unique solution $u \in W^2_1$ for each $f \in W^2_1$. Moreover,*

$$(|\lambda| - \theta_2)\|u\|_2' \leq \|f\|_2' \tag{3.8}$$

*where, by definition,* $\|u\|_2' = |u|_p + |\Delta u|_p$. *In particular* (*in the time dependent case*) *the family* $\{A^2_1(t)\}$, $t \in I$, *is* $(1, \theta_2)$-*stable in* $W^2_1$, *with respect to the* (*equivalent*) *norm* $\|\ \|_2'$.

PROOF. Let $\varepsilon > 0$ if $\lambda > 0$, $\varepsilon < 0$ if $\lambda < 0$, and consider the elliptic Dirichlet problem

$$\begin{cases} -\varepsilon\Delta u_\varepsilon + \lambda u_\varepsilon + (v\cdot\nabla)u_\varepsilon + au_\varepsilon = f & \text{in } \Omega\,, \\ (u_\varepsilon)_\Gamma = 0\,. \end{cases} \tag{3.9}$$

In order to fix the ideas, assume that $\lambda > 0$. For a sufficiently large $\lambda$, the above problem has a unique solution $u_\varepsilon \in W^4_1$. Moreover (a crucial point!)

$$(\Delta u_\varepsilon)|_\Gamma = 0\,. \tag{3.10}$$

Hence $\Delta u_\varepsilon \in W^2_1$. Set $\Lambda = (\delta + |\Delta u_\varepsilon|^2)^{\frac{1}{2}}$, where $\delta > 0$. Equations (3.3) and (3.5) imply

$$(3.11)\qquad \begin{cases} -\displaystyle\int \Delta(\Delta u_\varepsilon)\cdot \Lambda^{p-2}\Delta u_\varepsilon \geq 0\,, \\ \displaystyle\int [(v\cdot\nabla)\Delta u_\varepsilon]\cdot \Lambda^{p-2}\Delta u_\varepsilon = -\frac{1}{p}\int (\operatorname{div} v)\,\Lambda^p\,, \end{cases}$$

for each $\varepsilon > 0$. Equation $(3.11)_2$ together with the identity $\Delta[(v\cdot\nabla)u] = (v\cdot\nabla)\Delta u + 2\nabla v\colon \nabla^2 u + (\Delta v\cdot\nabla)u$, yields

$$(3.12)\qquad \int \Delta[(v\cdot\nabla)u_\varepsilon]\cdot \Lambda^{p-2}\Delta u_\varepsilon = -\frac{1}{p}\int (\operatorname{div} v)\Lambda^p + \\ + 2\int (\nabla v\colon\nabla^2 u_\varepsilon)\cdot \Lambda^{p-2}\Delta u_\varepsilon + \int [(\Delta v\cdot\nabla)u_\varepsilon]\cdot \Lambda^{p-2}\Delta u_\varepsilon\,,$$

where $\nabla v\colon \nabla^2 u = \sum_{i,l=1}^{n} (D_l v_i)(D_i D_l u)$. By applying the operator $\Delta$ to both sides of equations $(3.9)_1$, by taking the scalar product in $\boldsymbol{R}^n$ with $\Lambda^{p-2}\Delta u_\varepsilon$, by integrating in $\Omega$, by taking in account $(3.11)_1$ and (3.12), it follows that

$$\lambda\int |\Delta u_\varepsilon|^2\Lambda^{p-2} - \frac{1}{p}\int (\operatorname{div} v)\Lambda^p \leqslant \\ \leqslant \int (2|\nabla v\colon\nabla^2 u_\varepsilon| + |(\Delta v\cdot\nabla)u_\varepsilon| + |\Delta(a u_\varepsilon)| + |\Delta f|)|\Delta u_\varepsilon|\Lambda^{p-2}\,.$$

Since $0 \leq |\Delta u_\varepsilon|\Lambda^{p-2} \leq \Lambda^{p-1}$, the Lebesgue's dominated convergence theorem applies, as $\delta \to 0^+$. Hence, the last inequality holds if $\Lambda$ is replaced by $|\Delta u_\varepsilon|$. In particular $(\lambda - \theta_2)\,|\Delta u_\varepsilon|_p \leq |\Delta f|_p$. Consequently, there exists a subsequence $u_\varepsilon$ weakly convergent in $W^2_1$ to a limit $u$. Since $\varepsilon\Delta u_\varepsilon \to 0$, in $L^p$, as $\varepsilon \to 0$, it follows that $u$ is a solution of (3.6), and verifies (3.7). Clearly, $(\lambda - \theta_2)(|\Delta u|_p + |u|_p) \leq (|\Delta f|_p + |f|_p)$ (use also (3.7)). ■

LEMMA 3.7. *Let $k \geq 0$, let $f \in W^k$, let assume that $(H_k)$ and* (2.1) *hold. If $u \in W^{k+1}$ is a solution of* (3.6) *then*

$$(3.13)\qquad (|\lambda| - \theta_k)\|u\|_k \leq \|f\|_k\,.$$

PROOF. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index, $|\alpha| = k$. By using an abbreviate notation, the application of the operator $D^\alpha$ to both sides of equation (3.6) yields

$$\lambda D^\alpha u + (v\cdot\nabla)D^\alpha u + \sum[(D^k v)(Du) + \ldots + (Dv)(D^k u)] + \\ + \sum[(D^k a)u + \ldots + a(D^k u)] = D^\alpha f\,. \tag{3.14}$$

Set $\Lambda = (\delta + |D^k u|^2)^{\frac{1}{2}}$, where $\delta$ is a positive parameter, and $|D^k u|^2 = \sum |D^\alpha u_j|^2$; this summation is extended to all $\alpha$ such that $|\alpha| = k$, and to all $j$, $1 \leq j \leq N$. By multiplying both sides of equation (3.14) by $\Lambda^{p-2}D^\alpha u$, by adding side by side for all $\alpha$ such that $|\alpha| = k$, and by integrating in $\Omega$, it follows that

$$\lambda\int \Lambda^{p-2}|D^k u|^2 \leq \frac{1}{p}[\operatorname{div} v]_0|\Lambda|_p^p + c([v]_k + [a]_k)\|u\|_k|\Lambda|_p^{p-1} + |D^k f|_p|\Lambda|_p^{p-1}\,.$$

Note that

$$\int\Big[\sum_{|\alpha|=k}(v\cdot\nabla)D^\alpha u\Big]\cdot\Lambda^{p-2}D^\alpha u = \frac{1}{p}\int(v\cdot\nabla)\Lambda^p\,.$$

By passing to the limit on the above inequality, as $\delta \to 0^+$, one gets

$$\lambda|D^k u|_p \leq c([v]_k + [a]_k)\|u\|_k + |D^k f|_p\,. \tag{3.15}$$

Clearly, (3.15) holds for every integer $k_0$ such that $0 \leq k_0 \leq k$. By adding side by side all these estimates one gets (3.13). ∎

THEOREM 3.8. *Let the hypothesis* $(H_1)$ *and* (2.1) *be satisfied. Then, for* $|\lambda| > \theta_1$, *the equation* (3.6) *has a unique solution* $u \in W_1^1$, *for each* $f \in W_1^1$. *Moreover,*

$$(|\lambda| - \theta_1)\|u\|_1 \leq \|f\|_1\,. \tag{3.16}$$

PROOF. Without loss of generality, we assume here that $\lambda > 0$. For the time being we assume that $(H_2)$ holds, and that $\bar\lambda > \theta_2$. Let $f_m \in W_1^2$ be a sequence such that $f_m \to f$ in $W_1^1$, and let $u_m \in W_1^2$ be the solution of the equation $\bar\lambda u_m + (v\cdot\nabla)u_m + au_m = f_m$. Equation (3.13) shows that $u_m$ is a Cauchy sequence in $W_1^1$. Hence, its limit $u$ is the solution of the equation $\bar{\lambda}u + (v\cdot\nabla)u + au = f$. Moreover,

$$(3.17) \qquad (\bar{\lambda} - \theta_1)\|u\|_1 \leq \|f\|_1 .$$

Let now $\lambda \in ]\theta_1, \theta_2]$ and fix a real $\bar{\lambda}$ such that $\bar{\lambda} > \theta_2$. Denote by $u = Tw$ the solution of the equation $\bar{\lambda}u + (v\cdot\nabla)u + au = f + (\bar{\lambda} - \lambda)w$, where $w \in W_1^1$. It follows from (3.17) that $T$ is a contraction in $W_1^1$. The fixed point $u = Tu$ is a solution of (3.6), and (3.16) holds.

Finally, if $v$ and $a$ do not verify $(H_2)$, we approximate them (in the $C^1$ norm) by two sequences $v_m$ and $a_m$, verifying $(H_2)$ and (2.1). The solution $u_m$ of the corresponding equations verifies the estimate (3.16). Hence, there exists a subsequence, which is weakly convergent to an element $u \in W_1^1$. Clearly, $u$ is a solution of (3.6), and (3.16) holds. ■

THEOREM 3.9. *Let $k \geq 1$, and let $l \in \{0, ..., k\}$. Assume that the conditions $(H_k)$ and* (2.1) *hold. Then, if $|\lambda| > \theta_k$, equation* (3.6) *has a unique solution $u \in W_l^k$, for each $f \in W_l^k$. Moreover,*

$$(3.18) \qquad (|\lambda| - \theta_k)\|u\|_k \leq c(\Omega, n, N, p, k)\|f\|_k .$$

*If $\Gamma \in C^{|k|+2}$, the above solution $u$ verifies the estimate* (3.13).

*Finally, without assuming* (2.1), *one has the following result. Assume that $\Gamma \in C^k$ [resp. $C^1$ if $k = 0$]. Let $l = 0$ and assume that the condition $(H_k)$ is verified. Then, there exists a linear continuous map $G \in \mathcal{L}(W^k)$ such that $u = Gf$ is a solution of equation* (3.6), *for each $f \in W^k$. Moreover, the estimate* (3.18) *holds.*

PROOF. *Step* (i). Here we prove the first statement (including (3.18)) of the above theorem, for $l = k$. The proof is done by induction on $k$. For $k = 1$ the result was proved in theorem 3.8. Let us establish it for $k = 2$. Assume that $f \in W_2^2$, and that $a, v \in C^2$. Theorem 3.6 shows that equation (3.6) has a unique solution $u \in W_1^2$, which verifies (3.18). Let us show that $u \in W_2^2$. By differentiating (3.6) with respect to $x_i$, $i = 1, ..., n$, we get

$$(3.19) \qquad \lambda D_i u + (v\cdot\nabla)D_i u + aD_i u + [(D_i v)\cdot\nabla]u = D_i f - (D_i a)u .$$

This is again a system of type (3.6) in the $nN$ variables $D_i u_j$, whose solution $D_i u$ belongs to $W^1$. On the other hand, $D_i f - (D_i a)u \in W_1^1$.

Hence, theorem 3.8 guarantees the existence of a (unique) solution in the space $W_1^1$, if $|\lambda| \geq c(\Omega, n, nN, p)\,([v]_2 + [a]_1)$. By lemma 3.5, the above two solutions coincide. This shows that $u \in W_2^2$.

Assume now that the thesis holds for values less than or equal to $k$, $k \geq 2$. Let $f \in W_{k+1}^{k+1}$, $a, v \in C^{k+1}$. Since $W_{k+1}^{k+1} \subset W_k^k$, the induction hypothesis shows the existence of a unique solution $u \in W_k^k$, verifying (3.18). Moreover, $D_i f - (D_i a)\, u \in W_k^k$, and the induction hypothesis, applied to equation (3.19), shows that $D_i u \in W_k^k$, for $|\lambda| \geq \theta \equiv$ $\equiv c(\Omega, n, nN, p, k)\big([v]_{k+1} + [a]_k\big)$. Furthermore,

$$\big(|\lambda| - \theta\big)\|Du\|_{k,p} \leq c(\Omega, n, nN, p, k)\big(\|Df\|_k + [a]_{k+1}\|u\|_k\big)\,.$$

Hence, $u \in W_{k+1}^{k+1}$, and $u$ verifies (3.18) for a suitable constant $c(\Omega, n, N, p, k)$.

*Step* (ii). Here, we prove the first statement of the theorem for $l = 0$, and also the statement in which (2.1) is not assumed. Let $B$ be an open ball such that $\bar{\Omega} \subset B$, and let $S \in \mathfrak{L}\big(C^k, C_0^k(\bar{B})\big)$, $T \in \mathfrak{L}\big(W^k, W_k^k(B)\big)$, be linear continuous maps such that $(Sv)|_\Omega = v$, $(Sa)|_\Omega = a$, $(Tf)|_\Omega = f$ (2). Hence, $S\ v$ is a continuation of $v$ from $\Omega$ to $B$, and so on. Set $\tilde{v} = Sv$, $\tilde{a} = Sa$, $\tilde{f} = Tf$. The part (i) of our proof shows the existence of a solution $\hat{u} \in W_k^k(B)$ of problem $\lambda\hat{u} +$ $+ (\tilde{v}\cdot\nabla)\hat{u} + \tilde{a}\hat{u} = \tilde{f}$. Clearly, $u = \hat{u}|_\Omega$ is a solution of (3.6). The reader can easily verify that (3.18) holds, since $\|u\|_k \leq \|\hat{u}\|_k$, and since the norms of the maps $T$ and $S$ are bounded by constants depending only on $\Omega, n, N, p, k$.

Note that the existence of the solution $u$ in $\Omega$ was established without using condition (2.1). Furthermore, the maps $S$ and $T$ exist if $\Gamma$ is assumed to be only a Lipschitz manifold, since the continuation of functions in Sobolev spaces, from $\Omega$ to $B$, can be done under this hypothesis, by a Calderon's result. Hence, the last assertion in theorem 3.9 is proved.

*Step* (iii). The first statement of the theorem holds in $W^k$ (by step (ii)) and in $W_l^l$ (by step (i)). Hence, it holds in $W_l^k = W^k \cap W_l^l$, for each $l \in \{0, \ldots, k\}$.

(2) Note that, for convenience, the same symbol $S$ denotes two different maps, since $v$ is a vector and $a$ is a matrix.

*Step* (iv). We prove now that (3.13) holds if $\Gamma \in C^{|k|+2}$. Obviously, it suffices to take in accont the case $l = 0$. In order to fix the ideas we assume that $\lambda > 0$. Let $f \in W^k$, and assume that $a, v \in C^{k+1}$, and that (2.1) holds. Let $f_m \in W^{k+1}$, $f_m \to f$ in $W^k$. If $\lambda > \theta_{k+1}$, the problem $\lambda u_m + (v \cdot \nabla) u_m + a u_m = f_m$, has a unique solution $u_m \in W^{k+1}$. Moreover, lemma 3.7 shows that the estimate (3.13) holds for the couple $u_m, f_m$. This estimate proves that $u_m$ is a Cauchy sequence in $W^k$. It easily follows that the limit $u$ is the solution of (3.6), and that $u$ verifies (3.13).

Now, we want to replace the above condition $\lambda > \theta_{k+1}$ by the weaker assumption $\lambda > \theta_k$. Let $v, a$ be as above, assume that $\theta_{k+1} \geqslant \lambda > \theta_k$, and let $u \in W^k$ be the solution of equation (3.6), whose existence is guaranteed by the first part of theorem 3.9. Fix $\bar{\lambda} > \theta_{k+1}$. Since $\bar{\lambda} u + (v \cdot \nabla) u + a u = f + (\bar{\lambda} - \lambda) u$, the result proved above shows that

$$(\bar{\lambda} - \theta_k)\|u\|_k \leq \|f\|_k + (\bar{\lambda} - \lambda)\|u\|_k .$$

Hence, $u$ verifies (3.13).

Finally, let $v, a \in C^k$, and assumed that $v$ verifies (2.1). Let $v_m, a_m \in C^{k+1}$, be such that $v_m$ verifies (2.1), and that $v_m \to v$, $a_m \to a$ in $C^k$ as $m \to +\infty$. Standard techniques show that such a sequence $v_m$ exists. Let $\lambda > \theta_k$. We may assume that $\lambda > \theta_k^{(m)} \equiv c([v_m]_k + [a_m]_k)$. Let $u_m$ be the solution of $\lambda u_m + (v_m \cdot \nabla) u_m + a_m u_m = f$. By the above result, $(\lambda - \theta_k^{(m)})\|u_m\|_k \leq \|f\|_k$. It easily follows that $u_m \to u$ weakly in $W^k$, that $u$ is a solution of (3.6), and that (3.13) holds. ∎

COROLLARY 3.10. *Under the assumption of theorem* 3.9 *the family of operators* $\{A_l^k(t)\}_{t \in I}$ *is* $(1, \theta_k)$*-stable in* $W_l^k$.

Let us now consider the case $k = 0$. We start by defining the operator $A^0$.

DEFINITION 3.11. *We define* $A^0$ *as the closure in* $L^p$ *of the operator* $A_1^1\colon D_1^1 \to W_1^1$.

One easily verifies that $A_1^1$ is preclosed in $L^p$. In fact, if $u_n \in W_1^1$, $(v \cdot \nabla) u_n \in W_1^1$, $u_n \to 0$ in $L^p$ and $(v \cdot \nabla) u_n + a u_n \to f$ in $L^p$, then $\int f \cdot \varphi = \lim \int [(v \cdot \nabla) u_n + a u_n] \cdot \varphi = 0$, as $n \to +\infty$, for every $\varphi \in \mathfrak{D}(\Omega)$. Hence $f = 0$, which shows that $A_1^1$ is preclosed. Let us now solve the equation $\lambda u + A^0 u = f$, for $\lambda > \theta_1$ and $f \in L^p$. Let $f_n \in W_1^1$ be a sequence convergent to $f$ in $L^p$. By Lemma 3.5 it follows that

$(\lambda-\theta_1)|u_n-u_m|_p \le |f_n-f_m|_p$. Hence $u_n \to u$ in $L^p$. It readily follows that $u \in D^0$, that $\lambda u + A^0 u = f$, and that $(\lambda-\theta_1)|u|_p \le |f|_p$. Moreover, one easily verifies that the solution $u \in D^0$ is unique. The reader should note that $D^0 \subset \{u \in L^p\colon \mathcal{A}u \in L^p\}$.

Remark. As above, one verifies that the operator $A^1\colon D^1 \to W^1$ is preclosed in $L^p$. Since $A^1_1 \subset A^1$ and $\lambda + A^0$ maps $D^0$ onto $L^p$, for a suitable $\lambda$, it follows that $A^0$ is also the closure of $A^1$ in $L^p$.

The following result is now obvious.

Lemma 3.12. *The statements in theorem* 3.9 *and in Corollary* 3.10 *holds for* $k=0$.

## 4. The evolution problem (case $k \ge 0$).

Proof of theorem 2.2 (case $k \ge 0$). The first part of the theorem (stability) was proved above. Now we prove the second part of the theorem 2.2 by showing that the evolution operator $U(t, s)$ associated with $\{A^k_l(t)\}$ is strongly continuous in $W^k_l$, for each fixed pair $l, k$ such that $0 \le l \le k$. We prove this result by using the theorem 5.2 of Kato [14]. For convenience, the symbol $K$ after the reference number to an equation, an assumption, or a result, means that we refer to the reference numbers on [14]. We set, in theorem 4.1-$K$, $X = W^{k-1}_l$, $Y = W^k_l$ where $k \ge 1$. From corollary 3.10 it follows that $A^{k-1}_l$ is $(1, \theta_k)$-stable in $X$, and that $A^k_l$ is $(1, \theta_k)$-stable in $Y$. Note that $A^k_l$ is the part of $A^{k-1}_l$ in $Y$. In particular, assumptions (i)-$K$ and (ii)-$K$ hold. The condition (iii)-$K$ is easily verified; the inclusion $Y \subset D^{k-1}_l(t)$ was proved in lemma 3.2. Moreover, the assumption (iv)-$K$ in theorem 5.1-$K$, and the assumption (v)-$K$ in theorem 5.2-$K$ hold (without resort to an equivalent norm in $Y$). Hence, theorem 5.2-$K$ shows that the evolution operator $U(t, s)$ is strongly continuous in $Y$, jointly in $t, s$. Here, there are no exceptional values of $t$, as follows from remarks 5.3-$K$ and 5.4-$K$. In fact, our families of operators are reversible (for that reason, we have been considering the time interval $[-T, T]$ instead of $[0, T]$).

The strong continuity of $U(t, s)$ in $L^p$ follows together with that in $W^1_1$, since the assumptions done are the same in both cases.

The estimate (2.6) follows from the formulae $u(t) = U(t, 0)u_0 + {}$ $+\int_0^t U(t, s) f(s)\, ds$, together with $(e)$ in theorem 5.1-$K$. ■

A REMARK ON THE PROOF OF THEOREM 2.2*. Let $k \geq 1$ be fixed. Under the hypothesis $(H_k)$ assumed in theorem 2.2, the theorem 2.1 furnishes the $(1, \theta_k)$-stability in $Y = W^k$ and in $X = W^{k-1}$, since $(H_k)$ implies $(H_{k-1})$. On the contrary, in theorem 2.2* the hypothesis $(H_k^*)$ does not imply $(H_{k-1}^*)$, if $2 + (n/p) \geq k > 1 + (n/p)$. For that reason, we establish in theorem 2.1* an independent estimate for $\|u\|_{k-1}$ under the hypothesis $(H_k^*)$.

PROOF OF COROLLARY 2.3. The proof is similar to that done for the stationary case, in part (ii) of the proof of theorem 3.9. Now, we extend the coefficients $v, a \in L^\infty(I; C^k) \cap C(I; C^{k-1})$ to coefficients $\tilde{v}, \tilde{a} \in L^\infty(I; C_0^k(\bar{B})) \cap C(I; C_0^{k-1}(\bar{B}))$, and we extend the data $f \in L^1(I; W^k)$ and $u_0 \in W^k$ to data $\tilde{f} \in L^1(I; W_k^k(B))$ and $\tilde{u}_0 \in W_k^k(B)$. The extension maps are linear and continuous between the corresponding function spaces. Now, the existence of the solution $\hat{u} \in C(I; W_k^k(B))$ of the evolution problem $D_t\hat{u} + (\tilde{v}\cdot\nabla)\hat{u} + \tilde{a}\hat{u} = \tilde{f}$, $\hat{u}|_{t=0} = \tilde{u}_0$, is guaranted by theorem 2.2. The solution referred to, in corollary 2.3, is just the restriction of $\hat{u}$ to $I \times \Omega$.

Finally, the estimate (2.7) follows from (2.6), since $\|u\|_{k,I} \leq \|\hat{u}\|_{k,I}$ and since the norms of $\tilde{u}_0, \tilde{f}, \tilde{a}$, and $\tilde{v}$, are bounded by positive constants $c(\Omega, n, N, p, k)$ times the norms of $u_0, f, a$, and $v$, respectively. Obviously, the norms of functions labeled by $\sim$ or by $\wedge$ always concern the domain $B$ (and not $\Omega$).

The same device is used on proving the estimates stated in corollary 2.3*. ■

PROOF OF COROLLARY 2.4*. The construction of the solutions $u$ and $z$ shows that $u(t) = \hat{u}(t)|_\Omega$, $z(t) = \hat{z}(t)|_\Omega$, where $\hat{u}$ and $\hat{z}$ are the solutions of the problems $D_t\hat{u} + (\tilde{v}\cdot\nabla)\hat{u} + \tilde{a}\hat{u} = \tilde{f}$, $\hat{u}|_{t=0} = \tilde{u}_0$, and $D_t\hat{z} + (\tilde{w}\cdot\nabla)\hat{z} + \tilde{b}\hat{z} = \tilde{g}$, $\hat{z}|_{t=0} = \tilde{z}_0$. Hence,

$$
(4.1) \qquad \begin{cases} D_t(\hat{z} - \hat{u}) + (\tilde{w}\cdot\nabla)(\hat{z} - \hat{u}) + \tilde{b}(\hat{z} - \hat{u}) = \\ \qquad\qquad = (\tilde{g} - \tilde{f}) - [(\tilde{w} - \tilde{v})\cdot\nabla]\,\hat{u} - (\tilde{b} - \tilde{a})\hat{u}\,, \quad \text{in } I \times \Omega\,, \\ (\hat{z} - \hat{u})|_{t=0} = \tilde{z}_0 - \tilde{u}_0 \quad \text{on } I \times \Gamma\,. \end{cases}
$$

By applying $(2.8)_2$ to the solution $\hat{z} - \hat{u}$ of problem (4.1) we show that

$$
\begin{aligned} \|\hat{z} - \hat{u}\|_{I,k-1} \leq \{ &\|\tilde{z}_0 - \tilde{u}_0\|_{k-1} + |||\tilde{g} - \tilde{f}|||_{I,k-1} + \\ &+ |||[(\tilde{w} - \tilde{v})\cdot\nabla]\hat{u}|||_{I,k-1} + |||(\tilde{b} - \tilde{a})\hat{u}|||_{I,k-1}\} \exp[\tilde{\mu}_k^* T]\,, \end{aligned}
$$

where $\tilde{\mu}_k^* = c(\|\tilde{w}\|_{I,k} + \|\tilde{b}\|_{I,k}) \leq c\mu_k^*$. Recall that the symbol $c$ may denote different constants. Now, we estimate the right hand side of the above inequality by taking in account that $\|\tilde{z}_0 - \tilde{u}_0\|_{k-1} \leq \leq c\|z_0 - u_0\|_{k-1}$ (since $\tilde{z}_0 - \tilde{u}_0 = \widetilde{z_0 - u_0}$), and that a similar argument applies to the terms $\tilde{g} - \tilde{f}$, $\tilde{w} - \tilde{v}$, and $\tilde{b} - \tilde{a}$. Moreover,

$$|||[(\tilde{w} - \tilde{v})\cdot\nabla]\hat{u}|||_{I,k-1} \leq c|||w - v|||_{I,k-1}\|\hat{u}\|_{k,I},$$

and (by $(2.8)_1$) $\|\hat{u}\|_{k,I} \leq c \exp(c\theta_k^* T)(\|u_0\|_k + |||f|||_{I,k})$. The term $(\tilde{b} - \tilde{a})\hat{u}$ is treated in a similar way. Finally, $\|z - u\|_{k-1,I} \leq \|\hat{z} - \hat{u}\|_{k-1,I}$. ■

## 5. The case $k < 0$ (stationary and evolution problem).

In this section, we consider the case $k < 0$. The proofs are done by using the corresponding results for $k \geq 0$, together with duality arguments. Since the method is the same for the stationary and for the evolution case, we fix our attention on this last one, by proving the theorem 2.2 for $k < 0$. For convenience, we will denote the negative integers by $-k$, where $k > 0$. Let $a^*$ be the transpose of the matrix $a$, and consider the formal adjoint $\mathcal{B}(t)$ of $\mathcal{A}(t)$, i.e. the operator

$$\mathcal{B}(t)\varphi = -(v\cdot\nabla)\varphi - (\operatorname{div} v)\varphi + a^*\varphi,$$

acting in the distributional sense.

DEFINITION. For each $t \in I$, we denote by $B_k^k(t)$, $k \geq 1$, the operator $\mathcal{B}(t)$ with domain

$$D_k^{k,q}(t) = \{\varphi \in W_k^{k,q}: (v\cdot\nabla)\varphi \in W_k^{k,q}\},$$

where $q = p/(p-1)$.

Since $\mathcal{B}(t)$ belongs to the class of operators defined by equation (2.3), and since $q \in ]1, +\infty[$, all the results proved in the preceeding sections apply to the operators $B_k^k(t)$. In particular, if $|\lambda| > \theta_k$, one has

$$\lambda \in \varrho(B_k^k(t)), \quad \text{and } \|R(\lambda, B_k^k(t))\| \leq \frac{1}{|\lambda| - \theta_k},$$

where the resolvent operator acts now on the Banach space $W_k^{k,q}$.

Note that the constant $c$, appearing on the definition of $\theta_k$, depends now on $q$ instead of $p$. However, since $q = p/(p-1)$, we may use again the symbol $\theta_k$.

Recalling that $B_k^k(t)$ is closed and densely defined, we introduce the following definition:

DEFINITION. *Let $k \geq 1$. For each $t \in I$, we denote by $A_{-k}(t)$, the adjoint of the operator $B_k^k(t)$: $D_k^{k,q}(t) \to W_k^{k,q}$. In symbols*

$$A_{-k}(t) = (B_k^{k,q}(t))^* . \tag{5.1}$$

By the way, note that $A_{-k}(t)$ is the restriction of $\mathcal{A}(t)$ to the set $\{u \in W^{k,p} \colon \mathcal{A}(t) u \in W^{-k,p}\}$.

If $|\lambda| > \theta_k$, one has $(\lambda + B_k^k(t))^* = \lambda + A_{-k}(t)$. On the other hand, a well known result on Functional Analysis shows that $[(\lambda + B_k^k(t))^*]^{-1} = [(\lambda + B_k^k(t))^{-1}]^*$. Hence

$$R(\lambda, A_{-k}(t)) = \big(R(\lambda, B_k^k(t))\big)^* ,$$

moreover

$$\|R(\lambda, A_{-k}(t))\|_{\mathcal{L}(W^{-k})} = \|R(\lambda, B_k^k(t))\|_{\mathcal{L}(W_k^{k,q})} \leq 1/(|\lambda| - \theta_k) . \tag{5.2}$$

This shows that the family $\{A_{-k}(t)\}$ is $\theta_k$-stable in $W^{-k}$.

Let now $k \geq 1$. For each fixed $t \in I$, one has

$$B_k^k(t) \subset B_{k-1}^{k-1}(t) \in \mathcal{L}(W_k^{k,q};\, W_{k-1}^{k-1,q}) . \tag{5.3}$$

On the other hand, as shown in the previous sections, the domain of $B_k^k(t)$ is dense in $W_k^{k,q}$. Hence $B_k^k(t)$ can be defined (by density) as an element of $\mathcal{L}(W_k^{k,q};\, W_{k-1}^{k-1,q})$. By duality, one gets from (5.3)

$$A_{-k}(t) \supset A_{-k+1}(t) \in \mathcal{L}(W^{-k+1};\, W^{-k}) . \tag{5.4}$$

Since this last operator is the adjoint of $B_{k-1}^{k-1}(t) \in \mathcal{L}(W_k^{k,q},\, W_{k-1}^{k-1,q})$, which is a continuous map on $I$, it follows from (5.4) that the restriction of $A_{-k}(t)$ to $W^{-k+1}$ defines a continuous map from $I$ into $\mathcal{L}(W^{-k+1};\, W^{-k})$. By setting $X = W^{-k}$, $Y = W^{-k+1}$ $(k > 0)$ in theorem 4.1 [14], one shows that the evolution operator $U(t, s)$ is strongly continuous in $W^{-k}$, jointly in $t, s$. This proves theorem 2.2, in the « negative case ». ∎

## 6. Persistence property and Euler equations.

Persistence property means that the solution $u(t)$ at time $t$ belongs to the same function space $X$ as does the initial state, and describes a *continuous trajectory* in $X$. A rigorous proof of this property could be in many cases a difficult task. Here, we study the persistence property under the effect of quite general external forces $f$, namely $f \in L^1(I; X)$. However, the method applies to many other equations, as for instance to the generalized Euler equations studied by H. Beirão da Veiga [3], or to the Euler equations for nonhomogeneous fluids, see H. Beirão da Veiga and A. Valli [4]. For the reader's convenience we illustrate this method by considering the Euler equations

$$(6.1) \qquad \begin{cases} D_t u + (u\cdot\nabla)u + \nabla\pi = f & \text{in } I\times\Omega\,, \\ \operatorname{div} u = 0 & \text{in } I\times\Omega\,, \\ u\cdot\nu = 0 & \text{on } I\times\Gamma\,, \\ u_{|t=0} = u_0(x) & \text{in } \Omega\,, \end{cases}$$

in a bounded domain $\Omega \subset \boldsymbol{R}$, $n \geq 2$. Without loss of generality, we assume in the following statement that $I = ]-\infty, +\infty[$.

THEOREM 6.1. *Let $k > 1 + (n/p)$, where $p \in ]1, +\infty[$, and $n \geq 2$. Assume that $\Gamma \in C^k$, $u_0 \in W^k$, $u_0\cdot\nu = 0$ on $\Gamma$, $\operatorname{div} u_0 = 0$ in $\Omega$, $f \in L^1(I; W^k)$. Then, there exists a local solution $u \in C(J; W^k)$ of problem (6.1), where $J = [-\tau, \tau]$ and $\tau = c(\Omega, n, p, k)(\|u_0\|_k + |||f|||_{I,k})^{-1}$. Moreover, $\|u\|_{J,k} \leq c'(\Omega, n, p, k)(\|u_0\|_k + |||f|||_{I,k})$.*

The exisistence of a local solution $u \in C(I^*, W^k)$ for problem (6.1) is well known, if the external forces are regular. See Ebin and Marsden [11] where $f \equiv 0$, and Bourguignon and Brezis [10] where $X = W^{s,p}$ and $f \in C(I; W^{s+1,p})$. However the proofs given by these authors are harder then the one suggested here (specially that in reference [11]). A simple proof of the existence of a local solution $u \in L^\infty(I^*; W^k)$, under the assumptions of theorem 6.1, is given by Temam [24].

We notice that in reference [9] we establish also the well-posedness of system (6.1) in Sobolev spaces $W^{k,p}$, by using Kato's perturbation theory. See [9], theorems 5.2 and 5.3. In reference [25] this result is extended to non-homogeneous inviscid fluids.

PROOF OF THEOREM 6.1. In the sequel we assume that the vector field $\nu$, defined on $\Gamma$, is extended to a neighbourhood of $\Gamma$, as a $C^{k-1}$ vector field. The results do not depend on the particular extension of $\nu$.

For convenience, we work here in the interval $I = [0, +\infty[$. Set $J = [0, \tau]$, $\tau > 0$, and define the convex set

$$\begin{aligned} \boldsymbol{K} = \{w \in L^\infty(J;\, W^k) \cap C(J;\, W^{k-1}) \colon w|_{t=0} = u_0,\ \operatorname{div} w(t) = 0 \text{ in } \Omega\,, \\ w(t)\cdot\nu = 0 \text{ on } \Gamma,\ \forall t \in J,\ \|w\|_{J,k} \leqq 4A,\ \|w\|_{J,k-1} \leqq B\}\,. \end{aligned}$$

The values of the positive constants $\tau, A, B$ will be fixed later on. $\boldsymbol{K}$ is a closed, convex, bounded subset of the Banach space $C(J;\, W^{k-1})$. In fact, if $w_n \in \boldsymbol{K}$, $w_n \to w$ as $n \to +\infty$, it follows, by the weak*-compactness of the bounded subsets of $L^\infty(J;\, W^k)$ and by the lower semi-continuity of the norm respect to the weak*-convergence, that $w \in L^\infty(J;\, W^k)$ and that $\|w\|_{J,k} \leqq 4A$.

Now we define a map $S$ on $\boldsymbol{K}$ as follows. Let $v \in \boldsymbol{K}$ and let $\pi$ be the solution of the problem

$$\text{(6.2)} \qquad \begin{cases} -\Delta\pi = \sum\limits_{i,j} (D_i v_j)(D_j v_i) - \operatorname{div} f \text{ in } \Omega\,, \\[2ex] \dfrac{\partial\pi}{\partial\nu} = \sum\limits_{i,j} (\partial\nu_j/\partial x_i) v_i v_j + f\cdot\nu \quad \text{on } \Gamma\,, \end{cases}$$

for each $t \in J$. Note that $\operatorname{div}[(v\cdot\nabla)v] = \Sigma(D_i v_j)(D_j v_i) + v\cdot\nabla(\operatorname{div} v)$ in $\Omega$, and that $[(v\cdot\nabla)v]\cdot\nu = v\cdot\nabla(v\cdot\nu) - \Sigma(\partial\nu_j/\partial x_i)v_i v_j$ on $\Gamma$. Since $\operatorname{div} v = 0$ in $\Omega$ and $v\cdot\nabla(v\cdot\nu) = 0$ on $\Gamma$, the compatibility condition for the Neumann boundary value problem (6.2) is verified, and the solution $\pi$ exists and is determined up to an additive constant. However, we are interested only on $\nabla\pi$.

Theorem 2.2* guarantees the existence and the uniqueness of a solution $u$ of the evolution problem

$$\text{(6.3)} \qquad \begin{cases} D_t u + (v\cdot\nabla)u = f - \nabla\pi & \text{in } J\times\Omega\,, \\ u_{|t=0} = u_0\,, & \text{in } \Omega\,. \end{cases}$$

We set $Sv = u$, $\forall v \in \boldsymbol{K}$. By applying well known regularity results to the elliptic boundary value problem (6.2), one gets

$$\|\nabla\pi\|_{J,k} \leqq c\tau A^2 + c|||f|||_{J,k}\,.$$

We recall that the symbol $c$ may denote different positive constants, even in the same equation. From $(2.8_1)$ it follows that

$$\|u\|_{J,k} \leqq c\left(\|u_0\|_k + |||f|||_{I,k} + \tau A^2\right) \exp\left[cA\tau\right]. \tag{6.4}$$

On the other hand, from $(6.3_1)$ and from the above estimates, it follows in particular that $|||D_t u|||_{J,k-1} \leqq c\tau A\|u\|_{J,k} + c|||f|||_{I,k} + c\tau A^2$. Hence

$$\|u\|_{J,k-1} \leqq c\left(\|u_0\|_k + |||f|||_{I,k}\right) + c\tau A^2 + c\tau A\|u\|_{J,k}. \tag{6.5}$$

Now we will use the Helmholtz decomposition of $L^p$, namely $L^p = X_p \oplus G_p$. We refer to [13], for definition and results. A similar argument (in $L^2$ spaces) is used in reference [17], in order to study the system (6.1). We denote by $P$ and by $Q = I - P$ the projections associated with the above decomposition of $L^p$. It is well known that the restrictions of $P$ and $Q$ to $W^l$ are continuous from $W^l$ into $W^l$, $l \geqq 0$. In particular, the norm of the linear map $P$ is bounded in $W^k$ and in $W^{k-1}$ by a constant $c = c(\Omega, n, p, k)$. Hence, from (6.4), it follows that

$$\|Pu\|_{J,k} \leqq c_1\left(\|u_0\|_k + |||f|||_{I,k} + \tau A^2\right) \exp\left[c_2 A\tau\right],$$

where, by definition, $(Pu)(t) = P(u(t))$, $\forall t \in J$. We fix $A = c_1\left(\|u_0\|_k + |||f|||_{I,k}\right)$, and we assume that $\tau$ verify the conditions

$$c_2 A\tau \leqq \log 2, \qquad c_1 A\tau \leqq 1. \tag{6.6}$$

It readily follows that $\|Pu\|_{J,k} \leqq 4A$, $\forall v \in K$. On the other hand, since $\tau A \leqq 1/c_1$, since $\|u\|_{J,k} \leqq cA$, and since $\|Pu\|_{J,k-1} \leqq c\|u\|_{J,k-1}$, one easily verifies (by using (6.5)) that $\|Pu\|_{J,k-1} \leqq c_3 A$, for a suitable constant $c_3$. By defining $B = c_3 A$, one has $T(K) \subset K$ where, by definition, $T = PS$.

Let us show that $T$ is a strict contraction, respect to the $C(J; W^{k-1})$ norm. Set $u = S(v)$, $u' = S(v')$, and denote by (6.2′) and (6.3′) the equations (6.2) and (6.3) with $v, \pi, u$ replaced by $v', \pi', u'$, respectively. From the equations (6.3) and (6.3′), and from (2.9), we deduce that $\|u' - u\|_{J,k-1} \leqq c \exp[cA\tau]\left(|||\nabla(\pi' - \pi)|||_{J,k-1} + \exp[cA\tau]A\|v' - v\|_{J,k-1}\right)$. On the other hand, by subtracting the respective sides of equations

(6.2) and (6.2′), one easily verifies that

$$\|\nabla(\pi' - \pi)\|_{k-1} \leqq cA\|v' - v\|_{k-1}, \qquad \forall t \in J .$$

Hence,

$$\|\boldsymbol{P}u' - \boldsymbol{P}u\|_{J,k-1} \leqq c\|u' - u\|_{J,k-1} \leqq c_4 \exp[c_5 A\tau] A\tau \|v' - v\|_{J,k-1} .$$

Consequently, if

$$c_5 A\tau \leqq \log 2 , \qquad 4c_4 A\tau \leqq 1 , \tag{6.7}$$

one has $\|Tv' - Tv\|_{J,k-1} \leqq \frac{1}{2}$, $\forall v, v' \in \boldsymbol{K}$.

Let $v = \boldsymbol{P}u = \boldsymbol{P}Sv$ be the fixed point of $T$. If we prove that $\boldsymbol{P}u = u$, then $v = u$, and equation (6.3) shows that $u$ is a solution of the Euler equations (6.1). Let us show that $Qu(t) = u(t)$, $\forall t \in J$. Since $Qw = 0$ means that $\operatorname{div} w = 0$ in $\Omega$, and that $w \cdot \nu = 0$ on $\Gamma$, equation (6.2) shows that $Q\big((v\cdot\nabla)v + \nabla\pi - f\big) = 0, \forall t \in J$. Hence, by applying the operator $Q$ to both sides of equation ($6.3_1$), and by recalling that $u = v + Qu$, it readily follows that $D_t(Qu) + Q[(v\cdot\nabla)Qu] = 0$, $\forall t \in J$. By multiplying both sides scalarly in $L^2$ by $Qu$, one gets$(\frac{1}{2})D_t\|Qu(t)\|^2_{L^2} = 0$, a.e. in $J$. Since $Qu(0) = Qu_0 = 0$, it follows that $Qu(t) = 0$, $\forall t \in J$. Hence $\boldsymbol{P}u = u$.

Finally, we remark that the conditions imposed on $\tau$ in the above proof, namely (6.6) and (6.7), follow from the assumption on the value of $\tau$ made in theorem 6.1. ■

## REFERENCES


[1] C. Bardos - D. Brezis - H. Brezis, *Perturbations singulières et prolongements maximaux d'opérateurs positifs*, Arch. Rat. Mech. Analysis, **53** (1973), pp. 69-100.

[2] C. Bardos - J. Rauch, *Maximal positive boundary value problems as limits of singular perturbation problems*, Trans. Amer. Math. Soc., **270** (1982), pp. 377-408.

[3] H. Beirão da Veiga, *On an Euler type equation in Hydrodynamics*, Ann. Mat. Pura Appl., **125** (1980), pp. 279-295.

[4] H. Beirão da Veiga - A. Valli, *Existence of $C^\infty$ solutions of the Euler equations for non-homogeneous fluids*, Comm. Partial Diff. Eq., **5** (1980), pp. 95-107.

[5] H. Beirão da Veiga, *Homogeneous and non-homogeneous boundary value problems for first order linear hyperbolic systems arising in fluid mechanics*, Comm. in Partial Differential Equations, part I: **7** (1982), pp. 1135-1149; part II: **8** (1983), pp. 407-432.

[6] H. Beirão da Veiga, *An $L^p$-theory for the n-dimensional, stationary, compressible, Navier-Stokes equations, and the incompressible limit for compressible fluids. The equilibrium solutions*, Comm. Math. Phys., **109** (1987), pp. 229-248.

[7] H. Beirão da Veiga, *On a stationary transport equation*, Ann. Univ. Ferrara, **32** (1986), pp. 79-91.

[8] H. Beirão da Veiga, *Existence results in Sobolev spaces for a stationary transport equation*, U.T.M. 203, June 1986, Univ. Trento, to appear in the volume dedicated by « Ricerche di Matematica », to the memory of Professor C. Miranda.

[9] H. Beirão da Veiga, *Kato's perturbation theory and well-posedness for the Euler equations in bounded domains*, Arch. Rat. Mech. Anal., to appear.

[10] J. P. Bourguignon - H. Brezis, *Remarks on the Euler equation*, J. Funct. Analysis, **15** (1974), pp. 341-363.

[11] D. G. Ebin - J. E. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math., **92** (1970), pp. 102-163.

[12] K. O. Friedrichs, *Symmetric positive linear differential equations*, Comm. Pure Appl. Math., **11** (1958), pp. 333-418.

[13] D. Fujiwara - H. Morimoto, *An $L_r$-theorem of the Helmholtz decomposition of vector fields*, J. Fac. Sci. Univ. Tokyo, **24** (1977), pp. 685-700.

[14] T. Kato, *Linear evolution equations of « hyperbolic » type*, J. Fac. Sci. Univ. Tokyo, **17** (1970), pp. 241-258.

[15] T. Kato, *Linear evolution equations of « hyperbolic » type II*, J. Math. Soc. Japan, **25** (1973), pp. 648-666.

[16] T. Kato, *Quasi-linear equations of evolution, with applications to partial differential equations*, in « Spectral theory and differential equations », Lecture Notes in Mathematics, vol. **448**, Springer-Verlag (1975).

[17] T. Kato - C. Y. Lai, *Nonlinear evolution equations and the Euler flow*, J. Funct. Analysis, **56** (1984), pp. 15-28.

[18] R. V. Kohn - B. D. Lowe, *A variational method for parameter identification*, to appear.

[19] P. D. Lax - R. S. Phillips, *Local boundary conditions for dissipative symmetric linear differential operators*, Comm. Pure Appl. Math., **13** (1960), pp. 427-455.

[20] A. Majda - S. Osher, *Initial boundary value problems for hyperbolic equations with uniformly characteristic boundary*, Comm. Pure Appl. Math., **28** (1975), pp. 607-675.

[21] J. RAUCH, *Symmetric positive systems with boundary characteristic of constant multiplicity*, Trans. Amer. Math. Soc., **291** (1985), pp. 167-187.

[22] J. RAUCH - F. MASSEY, *Differentiability of solutions to hyperbolic initial-boundary value problems*, Trans. Amer. Math. Soc., **189** (1974), pp. 303-318.

[23] S. SCHOCHET, *Singular limits in bounded domains for quasilinear symmetric hyperbolic systems having a vorticity equation*, to appear.

[24] R. TEMAM, *On the Euler equations of incompressible perfect fluids*, J. Funct. Analysis, **20** (1975), pp. 32-43.

[25] H. BEIRÃO DA VEIGA, *An well-posedness theorem for nonhomogeneous inviscid fluids via a perturbation theorem*, to appear.

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