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Existence and Attractivity Results for a Class of Degenerate Functional-Parabolic Problems.

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1. Introduction.

We want to study existence, uniqueness and asymptotical behaviour of the solutions of the following problem:

$$
(1.1)\qquad \begin{cases} \partial_t u(t,x) = \Delta u^m(t,x) + a(x)u(t,x) - b(x)u^2(t,x) - \\ \qquad\qquad - u(t,x)\displaystyle\int_{-\infty}^{t} ds\, k(t-s,x)u(s,x) \quad \text{in } (0,\infty)\times\Omega \\ u(t,x) = 0 \qquad \text{in } (0,\infty)\times\partial\Omega \\ u(t,x) = u_C(t,x) \qquad \text{in } (-\infty,0]\times\Omega\,. \end{cases}
$$

Here $\Omega \subset \mathbb{R}^n$ is an open bounded subset with smooth boundary $\partial\Omega$, $m > 1$ and k, b, u_C are given nonnegative functions. We shall always consider nonnegative solutions of (1.1).

In particular, we are interested in attractivity properties of the stationary solutions of (1.1)-namely, of the solutions of the elliptic

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problem

$$(1.2) \qquad \begin{cases} \Delta u^m(x) + a(x)u(x) - \big(b(x) + \hat{k}(x)\big)u^2(x) = 0 & \text{in } \Omega \\ u(x) = 0 \quad \text{on } \partial\Omega\,, \end{cases}$$

where $\hat{k}(x) := \int\limits_0^\infty ds\, k(s, x)$ $(x \in \Omega)$. As we shall discuss in the following, the support and attractivity properties of solutions of (1.2) are closely related to the structure of the set (supposed nonempty)

$$(1.3) \qquad P := \{x \in \Omega\colon a(x) > 0\}\,.$$

The integro-partial differential equation in (1.1) generalizes the classical Volterra's population equation [24], in that it includes space dependence and nonlinear diffusion effects. Such effects were advocated in population dynamics in [11]; their investigation is a rapidly growing subject. In the case $a = k = b = 0$, the equation reduces to the well-known porous medium equation.

The semilinear case (formally, $m = 1$) was dealt with in [19]: under suitable hypotheses, a unique nontrivial nonnegative solution of (1.2) proved to exist, which attracts in the supremum norm any (nonnegative) solution of (1.1) not identically vanishing in Ω. A major assumption was that the delay effect be « not too large » in a suitable sense. This is not needed if we study the asymptotical stability of stationary solutions (i.e., if we are interested in « local » results) [18]. Related work, concerning the case of Neumann homogeneous boundary conditions, can be found in [17, 22, 28].

In the present situation, the interplay between nonlinear diffusion and the source term $a(x)u$ gives the set of solutions of (1.2) a richer structure than in the case $m = 1$ (concerning this point, see [14]). Since we are interested in « global » attractivity results, we require the delay term to be suitably small. Then we generalize to the present case the above referred results for the semilinear case (see Sections 2, 4); the main tools of the proof are monotonicity results [7].

Existence, uniqueness and nonnegativity of solutions to (1.1) are proved in the quickest way suitable for our purposes, using the contraction principle (Section 3). Other approaches are possible, based on monotonicity or compactness results; we point out that all of these methods apply to a wide class of degenerate functional-parabolic problems. Here we limit ourselves to prove a compactness lemma, whence existence results follow via Schauder's fixed point theorem. Related work, although rather different in spirit, can be found in [9, 10, 26, 27].

2. Mathematical framework and results.

The following assumptions will be under throughout:

$$(A)\quad \begin{cases} (1)\ a, b \text{ are Hölder continuous in } \Omega; \\ (2)\ b \text{ is strictly positive in } \bar{\Omega}; \\ (3)\ k \text{ satisfies the following requirements:} \\ \quad (\alpha)\ \text{measurability with respect to } (t, x) \text{ in } (0, \infty)\times\Omega; \\ \quad (\beta)\ \text{Hölder continuity with respect to } x \in \Omega, \text{ uniformly in the compact subsets of } (0, \infty); \\ \quad (\gamma)\ \text{summability from } (0, \infty) \text{ to } L^\infty(\Omega); \\ \quad (\delta)\ \text{nonnegativity in } \Omega, \text{ for almost every } t \in (0, \infty); \\ (4)\ u_C \text{ belongs to the Banach space } C_b(-\infty, 0; L^\infty(\Omega)) \text{ of bounded continuous maps from } (-\infty, 0] \text{ to } L^\infty(\Omega). \text{ For any } t \in (-\infty, 0],\ u_C(t) \text{ belongs to the positive cone } L^\infty_+(\Omega) := \\ \quad := \{u \in L^\infty(\Omega) : u \geqslant 0 \text{ a.e. in } \Omega\}. \end{cases}$$

Let $Q_T := (0, T]\times\Omega$, $\Sigma_T := (0, T]\times\partial\Omega$ $(T > 0)$. By a solution of problem (1.1) on $[0, T]$ we mean any $u \in C([0, T]; L^1(\Omega)) \cap L^\infty(Q_T)$ such that

$$\int_\Omega u(t)\sigma(t) - \iint_{Q_t} (u\partial_t\sigma + u^m\Delta\sigma) = \int_Q u_C(0)\sigma(0) + \iint_{Q_t} ug\sigma \tag{2.1}$$

for any $\sigma \in C^2(\bar{Q}_t)$, $\sigma \geqslant 0$, $\sigma = 0$ on Σ_t $(t \in [0, T])$; here

$$g = g(s, x, u(s, x)) := a(x) - b(x)u(s, x) - \tag{2.2}$$

$$-\int_{-\infty}^{0} dr k(s - r, x)u_C(r, x) + -\int_0^s dr k(s - r, x)u(r, x) \quad (0 \leqslant s \leqslant t \leqslant T;\ x \in \Omega)\,.$$

A solution of (1.1) on $[0, T]$ for any $T > 0$ is said to be global. Upper and lower solutions of (1.1) are similarly defined.

By definition, a solution of (1.2) (i.e., a stationary solution of (1.1)) is any $u \in L^\infty(\Omega)$ such that

$$-\int_\Omega u^m \Delta\sigma = \int_\Omega u(a - bu - \hat{k}u)\sigma \tag{2.3}$$

for any $\sigma \in C^2(\bar{\Omega})$, $\sigma \geqslant 0$, $\sigma = 0$ on $\partial\Omega$. Due to assumptions ($A1$), ($A3$), for any solution u of (1.2) $w := u^m$ is a classical solution of the problem

$$\begin{cases} \Delta w(x) + a(x) w^{1/m}(x) - \big(b(x) + \hat{k}(x)\big) w^{2/m}(x) = 0 & \text{in } \Omega \\ w(x) = 0 \quad \text{on } \partial\Omega . \end{cases}$$

Set $[u_1, u_2] := \{u \in L^\infty_+(\Omega) \colon u_1 \leqslant u \leqslant u_2\}$ (here $\leqslant$ denotes the ordering induced in $L^\infty(\Omega)$ by $L^\infty_+(\Omega)$; $u_1, u_2 \in L^\infty_+(\Omega)$). An interval $[u_1, u_2] \subseteq \subseteq L^\infty_+(\Omega)$ is said to be L^p-attractive if there exists a subset $S \subseteq C_b(-\infty, 0; L^\infty_+(\Omega))$ such that: (i) $[u_1, u_2] \subset S$; (ii) for any $u_C \in S$, the corresponding solution $u(t, u_C)$ of (1.1) exists in $L^p(\Omega)$ for any $t \geqslant 0$ and satisfies

$$\operatorname{dist}\big(u(t, u_C), [u_1, u_2]\big) \to 0 \text{ in } L^p(\Omega) \text{ as } t \to \infty \quad \big(p \in [1, \infty]\big) .$$

The L^p-instability of a solution to (1.2) is defined in an obvious way.

Following [14], we say that $u \in L^\infty_+(\Omega)$ is strongly positive in an open subset $\Omega' \subseteq \Omega$ if $u \geqslant v$ in Ω' for some v continuous, strictly positive in Ω'.

Concerning (nonnegative) solutions of problem (1.1), the following existence and uniqueness result will be proven.

THEOREM 2.1. Let assumption (A) be satisfied. Then for any $u_C \in C_b(-\infty, 0; L^\infty_+(\Omega))$ there exists a unique global nonnegative solution of problem (1.1).

Now consider the set P defined in (1.3); denote by P_i ($i \in I \subseteq \mathbb{N}$) any of its connected components. The following theorem was proven in [14].

THEOREM 2.2. Let (A) and the assumption

($H1$) the set P is nonempty

be satisfied. Then:

(i) nontrivial nonnegative solutions of (1.2) exist;

(ii) any nonnegative solution of (1.2) is either positive or identically vanishing in P_i ($i \in I \subseteq \mathbb{N}$). Moreover, there exist solutions of (1.2) positive in P.

It is convenient to point out some ideas underlying the proof of the above theorem, which play an important rôle in the following. The proof of claim (i) is as follows: assumption ($A2$) ensures that « large » upper solutions of (1.2) exist; by ($H1$), arbitrary small lower solutions can be constructed in any open subset of P (see Lemma 4.1). Thus the claim follows by monotonicity methods [7]. The proof of (ii) is a rather technical consequence of the maximum principle.

As Theorem 2.2 proves, the set P determines to a large extent the support properties of nontrivial nonnegative solutions of (1.2). However, different situations are possible if $P \not\equiv \Omega$—in particular, if P is disconnected.

If this happens and $m > 2$, solutions of (1.2) having a free boundary may exist (equivalently, we say that such solutions have a dead core [8]). Thus we can construct nontrivial nonnegative solutions of (1.2), which vanish in different connected components P_i, so are not ordered in $L^\infty_+(\Omega)$. It follows that the set of such solutions doesn't have a minimal element (although it has a maximal one, due to ($A2$)), and its structure can be fairly complicated.

Similar remarks obviously hold for the problem

$$(2.4) \qquad \begin{cases} \Delta v^m(x) + a(x)v(x) - b(x)v^2(x) = 0 & \text{in } \Omega \\ v(x) = 0 \quad \text{on } \partial\Omega . \end{cases}$$

Let us denote by $\tilde{v}$ its maximal solution, which is positive in the set P. We shall need a further assumption, namely

($H2$) the set $P := \{x \in \Omega \colon a(x) - \hat{k}(x)\tilde{v}(x) > 0\}$ is nonempty.

Observe that $\tilde{P} \subseteq P$; moreover, ($H2$) is easily seen to hold whenever

$$(H3) \qquad b(x) \geqslant \hat{k}(x), \quad b \not\equiv \hat{k} \qquad (x \in \Omega) .$$

The following result will be proven.

THEOREM 2.3. Let assumptions (A), ($H2$) be satisfied. Then there exists an interval $[u, v] \subseteq L^\infty_+(\Omega)$ such that:

(i) $u, v \in C^\alpha(\bar{\Omega})$ $(\alpha \in (0, 1))$;

(ii) $u > 0$ in $\tilde{P}$;

(iii) $[u, v]$ contains any nonnegative solution of (1.2), which is positive in $\tilde{P}$.

The couple (u, v) mentioned in Theorem 2.3 solves a system, which is the limit of a family of « approximating problems » (see $(4.1)_p$, $(4.2)_p$, (4.3)). If the set $\tilde{P}$ is disconnected, to each connected component $\tilde{P}_i$ we can associate an interval $[u^i, v] \subseteq L^\infty_+(\Omega)$ such that $[u^i, v] \supseteq [u, v]$ ($i \in \tilde{I} \subseteq \mathbf{N}$; see Section 4). However, such an interval may contain nonnegative solutions of (1.2) vanishing in $\tilde{P}_k$, for some $k \neq i$. Such solutions reveal to be unstable [14, 16]—which is a good reason for considering solutions « with largest support ». In fact, the following holds.

THEOREM 2.4. Let assumptions (A), $(H2)$ be satisfied. Then the interval $[u, v]$, whose existence was asserted in Theorem 2.3,

(i) is invariant with respect to solutions of (1.1);

(ii) contains any stable nonnegative solution of (1.2);

(iii) is L^p-attractive ($p \in [1, \infty)$) if $n > 1$, or L^∞-attractive if $n=1$, with respect to solutions of (1.1), such that $u_C(0)$ is strongly positive in an open subset of $\tilde{P}_i$ for every $i \in \tilde{I} \subseteq \mathbf{N}$.

The above results become sharper if the interval $[u, v]$ reduces to a unique element; sufficient conditions for this to happen are given below.

THEOREM 2.5. Let $\tilde{P} = \Omega$ and assumptions (A), $(H3)$ be satisfied. Then the unique solution of (1.2) positive in Ω attracts any solution of (1.1), such that $u_C(0)$ is strongly positive in an open subset of Ω.

Theorem 2.5 extends the attractivity results for the semilinear case mentioned in Section 1 (see [19, Theorem 2]); in the above statement, attractivity is meant in the sense of Theorem 2.4.

If $\tilde{P} \subset \Omega$, more complicated situations can arise; in this respect, let us state the following result.

THEOREM 2.6. Let assumption (A) be satisfied; let u_1, u_2 be two nontrivial nonnegative solutions of (1.2) such that $u_1 \geqslant u_2$. Then $u_1 = u_2$ in any connected component of $\operatorname{supp} u_1$ where $u_2 \not\equiv 0$.

The proof is an easy consequence of [14, Theorem 5] and we shall omit it. If $P = \Omega$, Theorems 2.6 and 2.2—(ii) imply that there exists a unique nontrivial nonnegative solution of (1.2)—which actually is positive in Ω. However, this need not be true if $P \subset \Omega$ (in particular, if $\tilde{P} \subset \Omega$; see [14] for details).

3. Existence, uniqueness and nonnegativity.

Let us first prove Theorem 2.1.

PROOF OF THEOREM 2.1. Define

$$u^+ := \max\left\{|u_C(0)|_\infty, \left|\frac{a}{b}\right|_\infty\right\};$$

it is easy to see that u^+ is an upper solution, $u \equiv 0$ a lower solution of problem (1.1). Set

$$\{0, u^+\} := \{z \in C([0, T]; L^1(\Omega)) : z(t) \in [0, u^+]\ \forall t \in [0, T]\} \quad (T > 0);$$

it is easily checked that $\{0, u^+\}$ is a closed subset in $C([0, T]; L^1(\Omega))$. For any $z \in \{0, u^+\}$ consider the problem:

$$(3.1)\quad \begin{cases} \partial_t u(t, x) - \Delta u^m(t, x) + Mu(t, x) = \\ \qquad\qquad = z(t, x)\left[g(t, x, z(t, x)) + M\right] & \text{in } Q_T \\ u(t, x) = 0 & \text{in } \Sigma_T \\ u(0, x) = u_C(0, x) & \text{in } \Omega\,, \end{cases}$$

where M is any constant larger than

$$M_0 := \left(2|b|_\infty + |k|_{L^1(0,\infty;\, L^\infty(\Omega))}\right) u^+ + |k|_{L^1(0,\infty;\, L^\infty(\Omega))} |u_C|_{C_b(-\infty,0;\, L^\infty(\Omega))}\,.$$

Since the right-hand side of the differential equation in (3.1) belongs to $L^1(Q_T)$, there exists a unique solution $u \in C([0, T]; L^1(\Omega))$ of the same problem. Define

$$N : \{0, u^+\} \to C([0, T]; L^1(\Omega))\,, \qquad z \to Nz := u\,.$$

Due to the above choice of the constant M, it is easily seen that

$$0 \leqslant \iint_{Q_t} z[g(s,\cdot, z(s,\cdot)) + M]\sigma \leqslant \iint_{Q_t} u^+ M\sigma \qquad (t \in [0, T])$$

for any test function σ chosen as in Section 2. It follows that u^+ is an upper, $u \equiv 0$ a lower solution of problem (3.1); then

$$N(\{0, u^+\}) \subseteq \{0, u^+\}$$

by known comparison results [1].

Now observe that, for any $z, \hat{z} \in C([0, T]; L^1(\Omega))$, the following inequality holds:

$$\int_0^t ds \left| \int_0^s dr\, k(s-r,\cdot)[z(r,\cdot) - \hat{z}(r,\cdot)]\right|_{L^1(\Omega)} \leqslant$$

$$\leqslant \int_0^t ds \int_0^s dr |k(s-r,\cdot)|_{L^\infty(\Omega)} |z(r,\cdot) - \hat{z}(r,\cdot)|_{L^1(\Omega)} \leqslant$$

$$\leqslant \int_0^t ds |k(s,\cdot)|_{L^\infty(\Omega))} \int_0^t ds |z(s,\cdot) - \hat{z}(s,\cdot)|_{L^1(\Omega)} \qquad (t \in [0, T]) .$$

It follows easily that N is a contraction in $C([0, T]; L^1(\Omega))$ for any $T > 0$ sufficiently small. Then there exists a unique local solution of (1.1) in $\{0, u^+\}$, which can be prolonged to $[0, \infty)$ by standard arguments. This completes the proof.

Similar results holds for the problem

$$(3.2) \qquad \begin{cases} \partial_t u(t,x) = \Delta\varphi(u)(t,x) + u(t,x) g(t,x,u(t,x)) & \text{in } Q_T \\ u(t,x) = 0 & \text{in } \Sigma_T \\ u(0,x) = u_C(0,x) & \text{in } \Omega \end{cases}$$

with g defined as in (2.2) (namely, for problem (1.1) with u^m replaced by $\varphi(u)$), whenever φ is locally Lipschitz continuous, nondecreasing and such that $\varphi(0) = 0$.

As already remarked, the existence of solutions to problem (3.2) can be also proven by monotonicity or compactness methods. These have been used in [3, 13, 23] for the semilinear case; under the above assumptions on φ, due to the comparison results in [1], the arguments carry over to the present situation. As for compactness, suffice it to prove the following Proposition 3.1: then existence follows easily by Schauder's fixed point theorem.

Let us make the following assumption:

(φ) $\varphi \in C^1(\mathbb{R} \sim \{0\})$; $\varphi'(u) \geqslant c|u|^{p-1}$ (with $p > 0$ if $n \leqslant 2$, or $p > (n-2)/n$ if $n > 2$; $c > 0$); $\varphi(0) = 0$.

As is well-known, the operator A defined as follows:

$$\begin{cases} D(A) := \{u \in L^1(\Omega) : \varphi(u) \in W_0^{1,1}(\Omega), \, \Delta\varphi(u) \in L^1(\Omega)\} \\ Au := \Delta\varphi(u) \qquad (u \in D(A)) \end{cases}$$

is m-accretive in $L^1(\Omega)$ since φ is nondecreasing [6]. Denote by $\{S(t)\}_{t\geqslant 0}$ the corresponding nonlinear semigroup. The following proposition generalizes the results of [2] (see also [5, 25]).

Proposition 3.1. Let assumption (φ) be satisfied. Then the map $S(t)$: $L^1(\Omega) \to L^1(\Omega)$ is compact for any $t > 0$.

Proof. According to [4], it suffices to prove that

(*a*) $(I + \lambda A)^{-1}$: $L^1(\Omega) \to L^1(\Omega)$ is compact for any $\lambda > 0$;

(*b*) for any bounded subset $Z \subseteq D(A)$ and $\bar{t} > 0$, the map $t \to S(t)u$ $(u \in Z)$ is equicontinuous at $t = \bar{t}$.

We shall only discuss the case $n > 2$.

(*a*) Let us prove that the level sets

$$L_m := \{u \in D(A) : |u|_{L^1(\Omega)} \leqslant m, \, |\Delta\varphi(u)|_{L^1(\Omega)} \leqslant m\} \qquad (m > 0)$$

are relatively compact in $L^1(\Omega)$ [4]. Since

$$D := \{w \in W_0^{1,1}(\Omega) : \Delta w \in L^1(\Omega)\} \subseteq W_0^{1,q}(\Omega)$$

for any $q \in [1, n/(n-1)]$ [21] and $W_0^{1,q}(\Omega)$ is compactly embedded in $L^k(\Omega)$ for any $k \in [1, nq/(n-q)]$ if $q < n$, $\varphi(u)$ belongs to a compact subset of $L^k(\Omega)$ (for any $k \in [1, n/(n-2)]$ whenever $u \in L_m$ $(m > 0)$). As assumption (φ) holds, (a) easily follows.

(b) Consider the problem whose semigroup solution is $S(t)u_1$, namely

$$\begin{cases} \partial_t u(t,x) = \Delta\varphi(u)(t,x) & \text{in } Q_T \\ u(t,x) = 0 & \text{in } \Sigma_T \\ u(0,x) = u_1(x) & \text{in } \Omega\,. \end{cases}$$

It can be shown by a standard approximation psocedure that

$$\int_0^T dt \int_\Omega dx\, t\varphi'(u)\, u_t^2 \leqslant \int_Q dx\, H(u_1)\,,$$

where

$$H(u) := \int_0^u \varphi(v)\, dv\,.$$

Thus the conclusion follows by assumption (φ) as in [2].

4. Support and attractivity properties.

In Section 2 we mentioned that arbitrarily small lower solutions of (1.2) can be constructed in any open subset of P (supposed non-empty). We give for convenience of the reader the proof of this claim [14], which will be used several times in the following.

LEMMA 4.1. Let assumptions (A), $(H1)$ be satisfied. Then, for any $\bar{x} \in P$ and any neighbourhood $U \subseteq P$ of $\bar{x}$, there exist nontrivial nonnegative lower solutions u_ϱ of (1.2) $(\varrho \in (0, \bar{\varrho});\ \bar{\varrho} > 0)$, such that $u_\varrho(\bar{x}) > 0$, $\operatorname{supp} u_\varrho \subseteq U$ and $|u_\varrho|_\infty \xrightarrow[\varrho \to 0]{} 0$.

PROOF. Fix any open ball $B \subseteq U$ which contains $\bar{x}$; denote by λ_0 the first eigenvalue, by ξ_0 the corresponsing eigenfunction of the Laplacian in B with homogeneous Dirichlet boundary conditions (namely, $\Delta\xi_0 + \lambda_0\xi_0 = 0$ in B, $\xi_0 > 0$ in B, $|\xi_0|_2 = 1$, $\xi_0 = 0$ on ∂B).

Define

$$u_\varrho(x) := \begin{cases} [\varrho\xi_0(x)]^{1/m} & x \in B \\ 0 & x \in \Omega \sim B \end{cases} \qquad (\varrho > 0)\,.$$

For any $\sigma \in C^2(\bar{\Omega})$, $\sigma \geqslant 0$ in Ω, $\sigma = 0$ on $\partial\Omega$ we have

$$\int_\Omega [u_\varrho^m \Delta\sigma + u_\varrho(a - bu_\varrho - \hat{k}u_\varrho)\sigma] =$$

$$= -\int_{\partial B} \partial_n(u_\varrho^m)\sigma + \int_B [\Delta u_\varrho^m + u_\varrho(a - bu_\varrho - \hat{k}u_\varrho)]\sigma \geqslant$$

$$\geqslant \int_B [-\lambda_0 u_\varrho^{m-1} + (a - bu_\varrho - \hat{k}u_\varrho)] u_\varrho \sigma$$

(where ∂_n denotes the outward normal derivative at ∂B).

Now let $\bar{a} := \min_{x \in \bar{B}} a(x) > 0$. It is easy to see that $\bar{\varrho} > 0$ exists, such that

$$[-\lambda_0 u_\varrho^{m-1} + (a - bu_\varrho - \hat{k}u_\varrho)](x) \geqslant \frac{\bar{a}}{4}$$

for any $x \in B$ and $\varrho \in (0, \bar{\varrho})$. This proves the claim.

Now consider the following families of problems in Ω:

$$(4.1)_p \qquad \Delta v_{p+1}^m + av_{p+1} - \hat{k}u_p v_{p+1} - bv_{p+1}^2 = 0 \quad (p = 0, 1, 2, \ldots;\ u_0 \equiv 0)$$

$$(4.2)_p \qquad \Delta u_p^m + au_p - \hat{k}v_p u_p - bu_p^2 = 0 \quad (p = 1, 2, \ldots),$$

endowed with homogeneous Dirichlet boundary conditions. The following result will be proven.

LEMMA 4.2. Let assumptions (A), $(H2)$ be satisfied. Then for any integer $p \geqslant 1$ there exists a couple (u_p, v_p) such that:

(i) $u_p, v_p \in C^\alpha(\bar{\Omega})$ $(\alpha \in (0, 1))$;

(ii) v_p is the maximal solution of $(4.1)_p$ in the interval $[0, |a/b|_\infty]$;

(iii) u_p is minimal among solutions of $(4.2)_p$, which are positive in the set $\tilde{P}$;

(iv) $u_p \leqslant u_{p+1} \leqslant v_{p+1} \leqslant v_p$;

(v) $u_p \leqslant \bar{u} \leqslant v_p$ for any solution $\bar{u}$ of (1.2) positive in $\tilde{P}$.

PROOF. – Observe that $(4.1)_0$ coincides with the differential equation in (2.4); choose $v_1 = \tilde{v}$. Since ($H2$) holds, it follows by monotonicity arguments that solutions of $(4.2)_1$ positive in $\tilde{P}$ exist (see Section 2 and Lemma 4.1]; the existence of $u_1 > 0$ in $\tilde{P}$, minimal among them, follows by [14, Theorem 4].

Due to the inequality

$$\Delta v_1^m + a v_1 - \hat{k} v_1^2 - b v_1^2 = -\hat{k} v_1^2 \leqslant 0 ,$$

v_1 is an upper solution of $(4.2)_1$; since $v_1 > 0$ in $P \supseteq \tilde{P}$, the asserted minimality of u_1 implies $u_1 \leqslant v_1$. Then the inequality

$$a - \hat{k} u_1 \geqslant a - \hat{k} v_1$$

and ($H2$) entail the existence of solutions of $(4.1)_1$, which are positive in $\tilde{P}$; let us denote by v_2 the maximal one. It follows from the inequality

$$\Delta v_2^m + a v_2 - b v_2^2 = \hat{k} u_1 v_2 \geqslant 0$$

that v_2 is a lower solution for $(4.1)_0$; since v_1 is maximal in P, thus in $\tilde{P}$, this proves that $v_2 \leqslant v_1$.

Now the inequality

$$a - \hat{k} v_2 \geqslant a - \hat{k} v_1$$

implies the existence of a minimal solution u_2 of $(4.2)_2$, which is positive in $\tilde{P}$. Since

$$\Delta u_2^m + a u_2 - \hat{k} v_1 u_2 - b u_2^2 = \hat{k}(v_2 - v_1) u_2 \leqslant 0 ,$$

u_2 is an upper solution for $(4.2)_1$; hence $u_2 \geqslant u_1$ as u_1 is minimal among solutions positive in $\tilde{P}$.

Let $\bar{u}$ denote any solution of (1.2) positive in $\tilde{P}$. The inequality

$$\Delta \bar{u}^m + a\bar{u} - b\bar{u}^2 = \hat{k}\bar{u}^2 \geqslant 0$$

clearly implies $\bar{u} \leqslant v_1$; since

$$\Delta \bar{u}^m + a\bar{u} - \hat{k} v_1 \bar{u} - b\bar{u}^2 = \hat{k}(\bar{u} - v_1)\bar{u} \leqslant 0 ,$$

this in turn implies $\bar{u} \geqslant u_1$. The inequality $u_2 \leqslant \bar{u} \leqslant v_2$ follows easily by similar arguments.

The above procedure can be iterated at any order, thus proving claims (ii)-(v). Claim (i) is an immediate consequence of the regularity assumptions in ($\varDelta$) (observe that these allow to use strong differential inequalities, as we did; see Section 2). The proof is complete.

Now we can prove Theorem 2.3.

PROOF OF THEOREM 2.3. Define $u := \lim u_p$, $v := \lim v_p$; due to Lemma 4.2, the limits exist pointwise and in $L^p(\Omega)$ for any $p \geqslant 1$. Then the couple (u, v) satisfies (in the weak sense) the problem in Ω:

$$\begin{cases} \varDelta v^m + av - \hat{k}uv - bv^2 = 0 \\ \varDelta u^m + au - \hat{k}vu - bu^2 = 0 \end{cases} \tag{4.3}$$

with $u = v = 0$ on $\partial\Omega$. As pointed out in Section 2, standard regularity arguments show that (u^m, v^m) is a classical solution of (4.3), which implies claim (i). Claims (ii) and (iii) are an immediate consequence consequence of Lemma 4.2. This completes the proof.

Suppose that $\tilde{P}$ is disconnected and consider any connected component $\tilde{P}_i$. According to the above remarks, there exists a solution u_1^i of $(4.2)_1$, which is minimal among solutions positive in $\tilde{P}_i$. Follow the iteration procedure used in the proof of Lemma 4.2, with initial steps $u_0 \equiv 0 \Rightarrow v_1 = \tilde{v} \Rightarrow u_1^i \Rightarrow \ldots$; take at any step solutions of $(4.2)_p$, which are minimal among solutions positive in $\tilde{P}_i$. It should now be clear, how to get the interval $[u^i, v] \supseteq [u, v]$ mentioned in Section 2 ($i \in \tilde{I} \subseteq \mathbb{N}$).

The proof of Theorem 2.4 is similar to that given for the semilinear case in [19], or for a degenerate parabolic system in [15]—thus we omit it. Let us only mention that the main idea is to compare solutions of (1.1) with those of the parabolic problems

$$\begin{cases} \partial_t w_{p+1} = \varDelta w_{p+1}^m + aw_{p+1} - \hat{k}u_p w_{p+1} - bw_{p+1}^2 & \text{in } (0, \infty) \times \Omega \\ w_{p+1} = 0 & \text{in } (0, \infty) \times \partial\Omega \\ w_{p+1} = w_{p+1}^0 & \text{in } \{0\{ \times \Omega \end{cases}$$

($p = 0, 1, 2, \ldots$; $u_0 \equiv 0$) and, respectively,

$$\begin{cases} \partial_t z_p = \varDelta z_p^m + az_p - \hat{k}v_p z_p - bz_p^2 & \text{in} (0, \infty) \times \Omega \\ z_p = 0 & \text{in } (0, \infty) \times \partial\Omega \\ z_p = z_p^0 & \text{in } \{0\} \times \Omega\,. \end{cases}$$

Then the conclusion follows by the attractivity results in [7].
Finally, let us prove Theorem 2.5.

PROOF OF THEOREM 2.5. Due to Theorems 2.3 and 2.4, the conclusion follows if we prove that $u \equiv v$ in Ω under the present assumptions. For this purpose, observe that the functions [20]

$$\psi := u^{m-1}, \qquad \chi := v^{m-1}$$

satisfy the problem

$$\begin{cases} \Delta\psi + \dfrac{1}{m-1}\dfrac{|\nabla\psi|^2}{\psi} = \dfrac{m-1}{m}\{-a + \hat{k}\chi^{1/(m-1)} + b\psi^{1/(m-1)}\} & \\ \Delta\chi + \dfrac{1}{m-1}\dfrac{|\nabla\chi|^2}{\chi} = \dfrac{m-1}{m}\{-a + \hat{k}\psi^{1/(m-1)} + b\chi^{1/(m-1)}\} & \text{in } \Omega \\ \psi = \chi = 0 \quad \text{on } \partial\Omega . & \end{cases}$$

Observe that now $0 < \psi \leqslant \chi$ by Theorem 2.3. It follows from the above that

$$\begin{cases} \Delta(\psi - \chi) + \dfrac{1}{m-1}\left\{\dfrac{|\nabla\psi|^2}{\psi} - \dfrac{|\nabla\chi|^2}{\chi}\right\} - \\ \qquad\qquad - \dfrac{m-1}{m}(b - \hat{k})\{\psi^{1/(m-1)} - \chi^{1/(m-1)}\} = 0 \quad \text{in } \Omega \\ \psi - \chi = 0 \quad \text{on } \partial\Omega . \end{cases}$$

Due to assumption ($H3$), the conclusion follows by the maximum principle.

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