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Extension of *CR*-forms and related problems

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Extension of CR-Forms and Related Problems.

ALESSANDRO PEROTTI (*)

Introduction.

Let D be a bounded domain of $\mathbf{C}^n$, $n\geqslant 3$, whose boundary contains a real hypersurface S of class C^1, connected, with boundary ∂S, and such that $A = \partial D \setminus S$ is a non empty piecewise C^1 real hypersurface. Assume that there exists a family $\{V_j\}_{j\in\mathbf{N}}$ of $(n-3)$-complete open sets such that $\overline{V}_{j+1}\subset V_j$, $\left(\bigcap_{j\in\mathbf{N}} V_j\right)\cap\overline{D}=\overline{A}$.

In particular these conditions are satisfied when A is contained in the zero-set of a pluriharmonic function, a situation which is considered in [11].

We show (Theorem 1) that every locally Lipschitz CR-form of type $(p, 0)$ on $\mathring{S}$ extends, in a unique way, by a $(p, 0)$-form holomorphic on D and continuous on $D\cup\mathring{S}$.

This result is obtained employing the techniques used in [11], and is based on the existence of primitives of Martinelli-Bochner-Koppelman integral kernel adapted to the sets V_j.

The result shown here sharpens what obtained in [11], [13], [14], where the extension problem is posed only for CR-functions, and on more particular domains.

Furthermore, we consider CR-forms of type (p, q) on S, with $q>0$. In this case extendibility depends on the Levi convexity of S, as shown by Andreotti and Hill [4] and Kohn and Rossi [8] when S is the boundary of a compact region. However, under the following assumptions we can obtain a jump theorem.

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Let D be a bounded domain of $\mathbf{C}^n$, $n \geqslant 2$, of the type considered above, with A of class C^1. Let $2 \leqslant s \leqslant n-2$ be a fixed integer. Assume that there exists a family $\{V_j\}_{j\in\mathbf{N}}$ of $(n-s-2)$-complete open sets such that $\overline{V}_{j+1} \subset V_j$, $\left(\bigcap_{j\in\mathbf{N}} V_j\right) \cap \overline{D} = \overline{A}$.

Then we can show (Theorem 2) that if $0 \leqslant p \leqslant n$, $1 \leqslant q \leqslant s-1$, every regular CR-form of type (p, q) on S is the jump across S between two $\bar\partial$-closed forms defined on D and on $\mathbf{C}^n \setminus \left(\overline{D} \cup \bigcap_{j\in\mathbf{N}} \overline{V}_j\right)$.

Finally, we give some applications of the jump theorem under pseudoconvexity assumptions on S.

We obtain some results about the $\bar\partial_b$-problem and the Cauchy problem for $\bar\partial$-operator. In particular, extension theorems for CR-forms of type (p, q) are obtained (Theorems 3, 4). We prove these results for forms of class C^m, $m \leqslant +\infty$. In the case of C^∞ forms, these problems have been considered by Andreotti and Hill in [4], under weaker conditions for S.

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1. Preliminaries.

1) We recall the Martinelli-Bochner-Koppelman formula (see [6] Ch. 1 and [1] Ch. 1).

Let Δ be the diagonal of $\mathbf{C}^n \times \mathbf{C}^n$. For $(z, \zeta) \in \mathbf{C}^n \times \mathbf{C}^n \setminus \Delta$, we consider the differential form

$$U(z, \zeta) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} (-1)^{j-1} \frac{\bar z_j - \bar\zeta_j}{|z-\zeta|^{2n}} (d\bar z_1 - d\bar\zeta_1) \wedge \ldots \wedge \widehat{(d\bar z_j - d\bar\zeta_j} \wedge \wedge \ldots \wedge (d\bar z_n - d\bar\zeta_n) \wedge dz_1 \wedge \ldots \wedge dz_n .$$

$U(z, \zeta) \in C^\infty_{(n,n-1)}(\mathbf{C}^n \times \mathbf{C}^n \setminus \Delta)$, and we have the decomposition

$$U(z, \zeta) = \sum_{q=0}^{n-1} U_{0,q}(z, \zeta)$$

in forms $U_{0,q}(z, \zeta)$ of type $(n, n-q-1)$ with respect to z and type $(0, q)$ with respect to ζ.

Let $\mu_q(z, \zeta)$ be the form such that $U_{0,q} = \mu_q \wedge dz_1 \wedge \ldots \wedge dz_n$. For

$0 \leqslant p \leqslant n$ and $0 \leqslant q \leqslant n-1$, we consider the forms

$$U_{p,q}(z,\zeta) = (-1)^{p(n-1)}\mu_q(z,\zeta)\wedge \sum_{|I|=p}{}' \sigma(I)\, dz[I]\wedge d\zeta_I\,,$$

where $dz[I] = dz_1\wedge\ldots\wedge\widehat{dz_{i_1}}\wedge\ldots\wedge\widehat{dz_{i_p}}\wedge\ldots\wedge dz_n$, $\sigma(I)$ is the sign determined by $dz_I\wedge dz[I] = \sigma(I)\, dz_1\wedge\ldots\wedge dz_n$, and the sum is taken on increasing multiindices. $U_{p,q}$ is C^∞ on $\mathbf{C}^n\times\mathbf{C}^n\setminus\Delta$, of type $(n-p, n-q-1)$ in z and (p,q) in ζ. We set $U_{p,-1}\equiv U_{p,n}\equiv 0$.

REMARK. The forms $U_{p,q}$ introduced above differ in sign from the corresponding forms defined in [1]. This is due to the fact that they are considered as forms on the product manifold $\mathbf{C}^n\times\mathbf{C}^n$, and not as double forms.

Let D be a bounded domain of $\mathbf{C}^n$ with piecewise C^1 boundary. The orientation of D is defined by the form $dx_1\wedge\ldots\wedge dx_n\wedge dy_1\wedge\ldots\wedge dy_n$, where $z_\alpha = x_\alpha + iy_\alpha$ $(\alpha = 1,\ldots,n)$, and ∂D has the orientation induced from D.

For $0\leqslant p, q\leqslant n$, let f be a continuous (p,q)-form on $\bar{D}$ such that $\bar\partial f$ (defined in the weak sense) is also continuous on $\bar{D}$. Then the Martinelli-Bockner-Koppelman formula holds:

$$\int_{\partial D} f(z)\wedge U_{p,q}(z,\zeta) - \int_D \bar\partial f(z)\wedge U_{p,q}(z,\zeta) + \bar\partial\int_D f(z)\wedge U_{p,q-1}(z,\zeta) =$$
$$= \begin{cases} (-1)^q f(\zeta) & \text{if } \zeta\in D\,,\\ 0 & \text{if } \zeta\notin \bar{D}\,.\end{cases}$$

The form $U(z,\zeta)$ is $\bar\partial$-closed on $\mathbf{C}^n\times\mathbf{C}^n\setminus\Delta$ (see [6] 1.7) Since the component of $\bar\partial U$ of type $(n, n-q)$ in z and $(0,q)$ in ζ is $\bar\partial_z U_{0,q} + \bar\partial_\zeta U_{0,q-1}$, the condition $\bar\partial U = 0$ is equivalent to the property

$$\bar\partial_z U_{0,q} = -\bar\partial_\zeta U_{0,q-1} \quad \text{for } 0\leqslant q\leqslant n$$

(indices z, ζ mean differentiation with respect to z and ζ respectively).

This property obviously holds for μ_q, and so for $U_{p,q}$:

$$\bar\partial_z U_{p,q} = -\bar\partial_\zeta U_{p,q-1} \quad \text{for } 0\leqslant p, q\leqslant n\,. \tag{1}$$

In particular, $\bar\partial_z U_{p,0} = \bar\partial_\zeta U_{p,n-1} = 0$.

2) In order to apply the integral representation formula which we have just mentioned, we need some primitives of the kernel $U(z, \zeta)$.

We recall that an open set D of $\mathbf{C}^n$ is called *q-complete* if there exists an exhaustion function for D which is strongly q-plurisubharmonic (i.e. its Levi form has at least $n-q$ positive eigenvalues at every point of D). Such an open set is *q-cohomologically complete*, i.e. $H^p(D, \mathcal{F}) = 0$ for every $p > q$ and every coherent analytic sheaf $\mathcal{F}$ on D ([3]).

PROPOSITION 1. *For fixed* $0 \leqslant s \leqslant n-2$, *let* $V \subseteq \mathbf{C}^n$, $n > 1$, *be a* $(n-s-2)$*-complete open set. Then we can find forms* $\eta_{p,q}(z, \zeta)$ $(0 \leqslant p \leqslant n, 0 \leqslant q \leqslant s)$, *of class* C^∞ *on* $V \times (\mathbf{C}^n \setminus \overline{V})$, *of type* $(n-p, n-q-2)$ *in* z *and* (p, q) *in* ζ, *such that*

$$U_{p,0}(z, \zeta) = \bar{\partial}_z \eta_{p,0}(z, \zeta)$$

$$U_{p,q}(z, \zeta) = \bar{\partial}_z \eta_{p,q}(z, \zeta) + \bar{\partial}_\zeta \eta_{p,q-1}(z, \zeta) \qquad \textit{for } 1 \leqslant q \leqslant s.$$

PROOF. Set $U_p(z, \zeta) := \sum_{q=0}^{n-1} U_{p,q}(z, \zeta) \in C^\infty_{(n,n-1)}(\mathbf{C}^n \times \mathbf{C}^n \setminus \Delta)$. From (1) we have that U_p is $\bar{\partial}$-closed on $\mathbf{C}^r \times \mathbf{C}^n \setminus \Delta$. Let $\{B_\alpha\}_{\alpha \in J}$ be a locally finite family of Stein open subsets of $\mathbf{C}^n \setminus \overline{V}$ which covers $\mathbf{C}^n \setminus \overline{V}$. For fixed $\alpha \in J$, $V \times B_\alpha$ is $(n-s-2)$-complete, therefore $(n-s-2)$-cohomologically complete. Then we can find for any $0 \leqslant p \leqslant n$ a form $\eta_p^\alpha \in C^\infty_{(n,n-2)}(V \times B_\alpha)$ such that $\bar{\partial}\eta_p^\alpha = U_p$ on $V \times B_\alpha$.

Let $\{\varphi_\alpha\}_{\alpha \in J}$ be a C^∞ partition of unity subordinate to the covering $\{B_\alpha\}_{\alpha \in J}$.

Set $\eta_p(z, \zeta) := \sum_{\alpha \in J} \varphi_\alpha(\zeta) \eta_p^\alpha(z, \zeta) \in C^\infty_{(n,n-2)}(V \times (\mathbf{C}^n \setminus \overline{V}))$.

Then we have $\bar{\partial}\eta_p = U_p + \sum_{\alpha \in J} \bar{\partial}_\zeta \varphi_\alpha \wedge \eta_p^\alpha$.

Let $\eta_p = \sum_{q=0}^{n-2} \eta_{p,q}$ be the decomposition of η_p in forms $\eta_{p,q}$ of type $(n-p, n-q-2)$ in z and (p, q) in ζ. By comparison of types, we obtain in particular $\bar{\partial}_z \eta_{p,0} = U_{p,0}$ and the proposition is proved for $s = 0$.

Now take $s \geqslant 1$. We have $\bar{\partial}(\eta_p^\alpha - \eta_p^\beta) = 0$ on $V \times (B_\alpha \cap B_\beta)$, and then we can find $\gamma_p^{\alpha\beta} \in C^\infty_{(n,n-3)}(V \times (B_\alpha \cap B_\beta))$ such that $\bar{\partial}\gamma_p^{\alpha\beta} = \eta_p^\alpha - \eta_p^\beta$.

Set $\gamma_p(z,\zeta) := \sum_{\alpha,\beta\in J} \bar{\partial}_\zeta\varphi_\alpha(\zeta)\wedge\varphi_\beta(\zeta)\gamma_p^{\alpha\beta}(z,\zeta) \in C^\infty_{(n,n-2)}(V\times(\mathbf{C}^n\setminus\bar{V}))$. Then

$$\bar{\partial}(\eta_p+\gamma_p) = U_p + \sum_{\alpha\in J}\bar{\partial}_\zeta\varphi_\alpha\wedge\eta_p^\alpha - \sum_{\alpha,\beta\in J}\bar{\partial}_\zeta\varphi_\alpha\wedge\bar{\partial}_\zeta\varphi_\beta\wedge\gamma_p^{\alpha\beta} -$$
$$-\sum_{\alpha,\beta\in J}\bar{\partial}_\zeta\varphi_\alpha\wedge\varphi_\beta(\eta_p^\alpha-\eta_p^\beta) = U_p - \sum_{\alpha,\beta\in J}\bar{\partial}_\zeta\varphi_\alpha\wedge\bar{\partial}_\zeta\varphi_\beta\wedge\gamma_p^{\alpha\beta}.$$

If $\gamma_p = \sum_{q=1}^{n-2}\gamma_{p,q}$ is the decomposition in forms $\gamma_{p,q}$ of type $(n-p, n-q-2)$ in z and (p,q) in ζ, by comparison of types we obtain

$$\bar{\partial}_z\eta_{p,0} = U_{p,0}\,; \qquad \bar{\partial}_z(\eta_{p,1}+\gamma_{p,1}) + \bar{\partial}_\zeta\eta_{p,0} = U_{p,1}$$

and the proposition is proved for $s=1$.

If $s>1$, it is sufficient to solve the equation $\bar{\partial}\tau_p^{\alpha\beta\delta} = \gamma_p^{\alpha\beta}+\gamma_p^{\beta\delta}+\gamma_p^{\delta\alpha}$ on $V\times(B_\alpha\cap B_\beta\cap B_\delta)$ and then consider the form

$$\tau_p(z,\zeta) := \sum_{\alpha,\beta,\delta\in J}\bar{\partial}_\zeta\varphi_\alpha(\zeta)\wedge\bar{\partial}_\zeta\varphi_\beta(\zeta)\wedge\varphi_\delta(\zeta)\tau_p^{\alpha\beta\delta}(z,\zeta)\,.$$

Then we have

$$\bar{\partial}(\eta_p+\gamma_p+\tau_p) = U_p + \sum_{\alpha,\beta,\delta}\bar{\partial}_\zeta\varphi_\alpha\wedge\bar{\partial}_\zeta\varphi_\beta\wedge\bar{\partial}_\zeta\varphi_\delta\wedge\tau_p^{\alpha\beta\delta}\,,$$

and therefore

$$\bar{\partial}_z\eta_{p,0} = U_{p,0}\,; \qquad \bar{\partial}_z(\eta_{p,1}+\gamma_{p,1}) + \bar{\partial}_\zeta\eta_{p,0} = U_{p,1}\,;$$
$$\bar{\partial}_z(\eta_{p,2}+\gamma_{p,2}+\tau_{p,2}) + \bar{\partial}_\zeta(\eta_{p,1}+\gamma_{p,1}) = U_{p,2}\,.$$

By induction we can prove the proposition for any $0\leqslant s\leqslant n-2$. ■

REMARK. Given a sequence $\{V_j\}_{j\in\mathbf{N}}$ of open sets satisfying the hypothesis of Proposition 1, with $\bar{V}_{j+1}\subset V_j$, the forms $\eta^j_{p,q}$ defined on $V_j\times(\mathbf{C}^n\setminus\bar{V}_j)$ can be constructed in such a way that $\eta^j_{p,q}(z,\zeta) = \eta^{j+1}_{p,q}(z,\zeta)$ if $z\in V_{j+1}$ and dist $(\zeta, V_j) > 1/j$. In fact, we can construct the covering $\mathcal{U}^{j+1}$ of $\mathbf{C}^n\setminus\bar{V}_{j+1}$ recursively by taking a locally finite covering $\mathcal{U}^{j+1}_1$ of $\bar{V}_j\setminus\bar{V}_{j+1}$ with balls of radius less than $1/4(j+1)$, and then setting $\mathcal{U}^{j+1} := \mathcal{U}^j\cup\mathcal{U}^{j+1}_1$, where $\mathcal{U}^j$ is the covering of $\mathbf{C}^n\setminus\bar{V}_j$ already constructed. Then on the set $\{\zeta\in\mathbf{C}^n\colon \text{dist}(\zeta, V_j) > 1/j\}$ the coverings $\mathcal{U}^j$ and $\mathcal{U}^{j+1}$ coincide, and if $B\in\mathcal{U}^j$ is such that

$B \cap \{\zeta \in \mathbf{C}^n: \mathrm{dist}(\zeta, V_j) > 1/j\} \neq \emptyset$, then we have $B \cap B_1 = \emptyset$ for every $B_1 \in \mathcal{U}_1^{j+1}$. Therefore, the partition of unity subordinate to $\mathcal{U}^{j+1}$ can be taken equal to that subordinate to $\mathcal{U}^j$ on the set

$$\{\zeta \in \mathbf{C}^n: \mathrm{dist}(\zeta, V_j) > 1/j\} .$$

2. Extension of *CR*-forms of type $(p, 0)$.

1) Let D be a bounded domain of $\mathbf{C}^n$, $n \geqslant 3$, with the following properties:

I) ∂D contains a real hypersurface S of class C^1, connected with boundary ∂S;

II) $A := \partial D \setminus S \neq \emptyset$ is a piecewise C^1 real hypersurface;

III) there exists a family $\{V_j\}_{j\in\mathbf{N}}$ of $(n-3)$-complete open sets such that $\overline{V}_{j+1} \subset V_j$, $\left(\bigcap_{j\in\mathbf{N}} V_j\right) \cap \overline{D} = \overline{A}$.

Among the open sets of this type there are those considered in [11] where A is contained in the zero-set of a pluriharmonic function.

REMARK. It can be shown (the proof is not trivial) that the complement of a $(n-2)$-complete open set cannot have compact components. This implies that if D verifies properties I), II), III), and j is so large that $S \setminus V_j$ is connected, then the component of $\mathbf{C}^n \setminus (\overline{D} \cup \overline{V}_j)$ whose boundary contains $S \setminus V_j$ is unbounded.

In order to obtain an extension theorem for *CR*-forms of type $(p, 0)$ (the weak solutions of the tangential Cauchy-Riemann equation), we shall need the following result, that in the case of functions is proved in [11].

PROPOSITION 2. *Let Σ be an oriented C^1 real hypersurface of $\mathbf{C}^n$. Let f be a locally Lipschitz CR-form of type $(p, 0)$ on Σ. For any C^1 $(n+r)$-chain C_{n+r} of Σ and any $(n-p, r-1)$-form θ, of class C^∞ on a neighbourhood of C_{n+r}, the following formula holds:*

$$\int_{C_{n+r}} f \wedge \bar{\partial}\theta = (-1)^p \int_{\partial C_{n+r}} f \wedge \theta .$$

PROOF. We can repeat the proof given in [11], using the kernel $U_{p,0}$ in place of the Martinelli-Bochner kernel $U_{0,0}$. ■

2) Now we are able to prove the extension theorem for CR-forms of type $(p, 0)$.

THEOREM 1. *Let $D \subseteq \mathbf{C}^n$, $n \geq 3$, be a bounded domain that verifies conditions* I), II), III), *and $0 \leq p \leq n$. Then every locally Lipschitz CR-form f on $\mathring{S}$ of type $(p, 0)$ extends, in a unique way, by a $(p, 0)$-form F, holomorphic on D and continuous on $D \cup \mathring{S}$.*

PROOF. Let $j \in \mathbf{N}$ be a fixed integer.

Let D' be an open set with C^1 boundary such that $D \setminus V_j \subset D' \subset D$ and $\bar{D}' \cap \bar{A} = \emptyset$. We set $S' := S \cap \bar{D}'$ and $A' := \partial D' \setminus S'$.

Suppose we have found the extension F. Then the Martinelli-Bochner-Koppelman formula applied on D' gives

$$F(\zeta) = \int_{S'} f(z) \wedge U_{p,0}(z, \zeta) + \int_{A'} F(z) \wedge U_{p,0}(z, \zeta) \quad \text{if } \zeta \in D' .$$

For $z \in D \cap V_j$ and $\zeta \in \mathbf{C}^n \setminus \bar{V}_j$, we have

$$F(z) \wedge U_{p,0}(z, \zeta) = (-1)^p d_z\big(F(z) \wedge \eta^j_{p,0}(z, \zeta)\big) ,$$

where $\eta^j_{p,0}$ is the C^∞ form on $V_j \times (\mathbf{C}^n \setminus \bar{V}_j)$ given by Proposition 1. Therefore

$$(*) \quad F(\zeta) = \int_{S'} f(z) \wedge U_{p,0}(z, \zeta) - (-1)^p \int_{\partial S'} f(z) \wedge \eta^j_{p,0}(z, \zeta) \quad \text{if } \zeta \in D \setminus \bar{V}_j .$$

Since $(*)$ holds for every $j \in \mathbf{N}$, the uniqueness of the extension follows.

Now we prove existence. Let $F(\zeta)$ be the $C^\infty (p, 0)$-form defined on $\mathbf{C}^n \setminus (S \cup \bar{V}_j)$ by $(*)$. First we show that F is holomorphic on $\mathbf{C}^n \setminus (S \cup \bar{V}_j)$:

$$\bar{\partial} F(\zeta) = (-1)^{p-1} \int_{S'} f(z) \wedge \bar{\partial}_\zeta U_{p,0}(z, \zeta) - \int_{\partial S'} f(z) \wedge \bar{\partial}_\zeta \eta^j_{p,0}(z, \zeta) =$$

$$= (-1)^p \int_{S'} f(z) \wedge \bar{\partial}_z U_{p,1}(z, \zeta) - \int_{\partial S'} f(z) \wedge \bar{\partial}_\zeta \eta^j_{p,0}(z, \zeta) .$$

REMARK. By definition of integration with respect to z (see [6] Ch. 1), if $f(z)$ and $\alpha(z,\zeta)$ are differential forms and C is a chain of dimension $\dim C = \deg f + \deg \alpha(\cdot,\zeta)$, we have

$$\bar\partial_\zeta\Big(\int_C f(z)\wedge\alpha(z,\zeta)\Big) = (-1)^{\deg\alpha(\cdot,\zeta)}\int_C f(z)\wedge\bar\partial_\zeta\alpha(z,\zeta)\,.$$

We go back to our proof.

From Proposition 2 and Proposition 1 we obtain

$$\bar\partial F(\zeta) = \int_{\partial S'} f(z)\wedge[U_{p,1}(z,\zeta) - \bar\partial_\zeta\eta^j_{p,0}(z,\zeta)] =$$
$$= \int_{\partial S'} f(z)\wedge\bar\partial_z\eta^j_{p,1}(z,\zeta) = (-1)^p\int_{\partial(\partial S')} f(z)\wedge\eta^j_{p,1}(z,\zeta) = 0\,.$$

We set

$$F_1(\zeta) := \int_{S'} f(z)\wedge U_{p,0}(z,\zeta) \qquad \text{for } \zeta\in\mathbf{C}^n\setminus S\,,$$

$$F_2(\zeta) := (-1)^p\int_{\partial S'} f(z)\wedge\eta^j_{p,0}(z,\zeta) \qquad \text{for } \zeta\in\mathbf{C}^n\setminus\overline{V}_j$$

and denote by F_i^+, F_i^- ($i = 1, 2$) their restrictions to $D\setminus\overline{V}_j$ and $\mathbf{C}^n\setminus(\overline{D}\cup\overline{V}_j)$ respectively. Since f is locally Lipschitz, $F_1^\pm$ extend continuously to $S\setminus\overline{V}_j\subset S'$, and we have $F_1^+ - F_1^- = f$ on $S\setminus\overline{V}_j$.

Moreover, $F_2^+ = F_2^-$ on $S\setminus\overline{V}_j$ and therefore F extends continuously to $(D\setminus\overline{V}_j)\cup(S\setminus\overline{V}_j)$ and $F = f + F_1^- - F_2^-$ on $S\setminus\overline{V}_j$.

Now take the integer j as in the remark in section 2.1. Let W be a bounded Stein neighbourhood of $\overline{D}$ and $\zeta\in\mathbf{C}^n\setminus(\overline{V}_j\cup\overline{W})$ fixed. On W we can find a primitive ψ of $U_{p,0}(\cdot,\zeta)$. Then from Proposition 2 we have

$$\int_{S'} f(z)\wedge U_{p,0}(z,\zeta) = (-1)^p\int_{\partial S'} f(z)\wedge\psi(z)$$

and therefore

$$F(\zeta) = (-1)^p\int_{\partial S'} f(z)\wedge[\psi(z) - \eta^j_{p,0}(z,\zeta)]\,.$$

Since $\bar\partial_z[\psi - \eta^j_{p,0}(\cdot, \zeta)] = 0$ on $V_j \cap W$, there exists ψ' such that $\psi - \eta^j_{p,0}(\cdot, \zeta) = \bar\partial_z \psi'$, and then

$$\int_{\partial S'} f \wedge [\psi - \eta^j_{p,0}(\cdot, \zeta)] = \int_{\partial S'} f \wedge \bar\partial_z \psi' = (-1)^p \int_{\partial(\partial S')} f \wedge \psi' = 0 .$$

Therefore $F \equiv 0$ on $\mathbf{C}^n \setminus (\bar V_j \cup \bar W)$ and the remark in 2.1 implies that $F_1^- - F_2^- = 0$ on $S \setminus \bar V_j$, and $F = f$ on $S \setminus \bar V_j$.

Thus we have found an extension F_j of f on $D \setminus \bar V_j$, for any $j \in \mathbf{N}$ sufficiently large. If $j' > j$, the $(p, 0)$-form $F_j - F_{j'}$ has components which are holomorphic on $D \setminus \bar V_j$ and vanish on $S \setminus \bar V_j$. Therefore $F_{j'} - F_j = 0$ on $D \setminus \bar V_j$. In fact, if g is such a component, the function obtained extending g by zero on a connected neighbourhood of a point of $S \setminus \bar V_j$ is holomorphic in the weak sense, and therefore zero by uniqueness of analytic continuation.

By the same reasoning we can obtain again the uniqueness of the extension. ■

3. Applications of the extension theorem.

1) Let $\varphi_1, \ldots, \varphi_m$ be pluriharmonic C^2 functions on $\mathbf{C}^n$. Let D be a domain verifying I) and II) and such that $A \subset \bigcup_{i=1}^m \{\varphi_i = 0\}$ and $D \subset \bigcap_{i=1}^m \{\varphi_i > 0\}$. This situation was considered in [11] for $m = 1$ and [13] for $m = 2$.

Set $\psi_j := \prod_{i=1}^m \varphi_i - 1/j$ and $V_j := \{z \in \mathbf{C}^n : \varphi_j(z) < 0\}$. For $z_0 \in \partial V_j$, the holomorphic tangent space to ∂V_j at z_0 is

$$T_{z_0}(\partial V_j) = \left\{ w \in \mathbf{C}^n : \partial \psi_j(z_0)(w) = \sum_{i=1}^m \partial \varphi_i(z_0)(w) \prod_{h \neq i} \varphi_h(z_0) = 0 \right\},$$

and the Levi form of ψ_j is given by

$$\mathscr{L}_{\psi_j, z_0}(w) = \sum_{i=1}^m \sum_{k \neq i} \overline{\partial \varphi_i(z_0)(w)}\, \partial \varphi_k(z_0)(w) \prod_{h \neq i,k} \varphi_h(z_0) .$$

Consider the sets

$$V_{z_0, i} := \{z \in \mathbf{C}^n : \varphi_i(z) < \varphi_i(z_0)\} \qquad (i = 1, \ldots, m).$$

The subspace of $T_{z_0}(\partial V_j)$

$$E_{z_0} := \{w \in \mathbf{C}^n : \partial\varphi_i(z_0)(w) = 0 \text{ for } i = 1, \ldots, m\} = \bigcap_{i=1}^{m} T_{z_0}(\partial V_{z_0,i})$$

has dimension not less than $n - m$, and $\mathcal{L}_{\psi_j, z_0} E_{z_0} = 0$. Therefore the Levi form of ψ_j restricted to $T_{z_0}(\partial V_j)$ has at most $n - 1 - (n - m) = m - 1$ negative eigenvalues at each point of ∂V_j. The set V_j is then (weakly) $(m-1)$-pseudoconvex, and so it is $(m-1)$-complete (see [16]).

If $m \leqslant n - 2$ the family $\{V_j\}_{j \in \mathbf{N}}$ satisfies condition III) of the extension theorem, which can then be applied on D.

2) Theorem 1 can also be applied to deduce the well known theorem on global extension of CR-forms of type $(p, 0)$ defined on the boundary of a bounded domain of $\mathbf{C}^n$ (see [7] Th. 2.3.2′ and [1] Th. 3.2).

COROLLARY 1. *Let U be a bounded domain of $\mathbf{C}^n$, $n \geqslant 3$, with ∂U of class C^1 and connected. Then every locally Lipschitz CR-form of type $(p, 0)$ on ∂U extends, in a unique way, by a $(p, 0)$-form holomorphic on U and continuous on $\overline{U}$.*

PROOF. Let $z_0 \in \partial U$ and $r_0 > 0$ such that $D := U \setminus B(z_0, r_0) \neq \emptyset$ and $S := \partial U \setminus B(z_0, r_0)$ is connected. We set $A := \partial B(z_0, r_0) \cap U$ and $V_j := B(z_0, r_0 + 1/j)$, $j \in \mathbf{N}$. Since the set D verifies conditions I), II), III) of Theorem 1, we obtain the extension on D.

Since z_0 and r_0 are arbitrary, we have the extension on the whole U. ∎

REMARK. Corollary 1 holds also for $n = 2$ and for CR-forms of type $(p, 0)$ only continuous on ∂U (see [1]).

3. The extension theorem can also be applied to the following situation, that generalizes the result contained in [14] Th. 1.

Let D be a domain of $\mathbf{C}^n$, $n \geqslant 3$, verifying I) and II). Suppose A relatively open in a hypersurface M defined by a C^∞ function ϱ on a Stein open set U of $\mathbf{C}^n$. Suppose also that there exists $r > 0$ such that ϱ is strongly $(n-3)$–plurisubharmonic on the set $\{z \in U : 0 < \varrho(z) < r\}$.

Then the extension theorem holds on D. In fact, the open sets $V_j := \{z \in U : \varrho(z) < 1/j\}$ $(j > 1/r)$ verify condition III) of Theorem 1. To see this, consider the functions $\varphi_j := \varphi - \varepsilon_j \log(-\varrho + 1/j)$, where

φ is a strongly plurisubharmonic exhaustion function for U. If $\varepsilon_j > 0$ is small enough, we obtain, after restricting U if necessary, that φ_j is a strongly $(n-3)$-plurisubharmonic exhaustion function for V_j.

4. « Jump « theorem for *CR*-forms of type (p, q).

1) Now we consider *CR*-forms of type (p, q), with $q > 0$. As we shall see later, in this case it is not possible to obtain an extension theorem as for $(p, 0)$-forms without imposing a pseudoconvexity condition on S.

However, we can prove a « jump » theorem, i.e. a *CR*-form can be written as the difference between two $\bar\partial$-closed forms, defined on the two sides of the hypersurface (additive Riemann-Hilbert problem).

In the case when the *CR*-forms are defined on the boundary of a compact set, this result is proved in [1] Th. 2.10-2.11.

In the following we deal with bounded domains of $\mathbf{C}^n$, $n \geqslant 2$, satisfying conditions I) and II) of 2.1, where A is of class C^1 and has the following property:

III′) for a fixed $2 \leqslant s \leqslant n-2$, there exists a family $\{V_j\}_{j\in\mathbf{N}}$ of $(n-s-2)$-complete open sets such that $\overline{V}_{j+1} \subset V_j$, $\left(\bigcap\limits_{j\in\mathbf{N}} V_j\right) \cap \overline{D} = \overline{A}$.

THEOREM 2. *Let $D \subseteq \mathbf{C}^n$, $n \geqslant 2$, be a bounded domain satisfying properties* I), II), III′). *Let $0 \leqslant p \leqslant n$, $1 \leqslant q \leqslant s-1$ or $q = n-1$. Consider a (p, q)-form f of class C^1 on a neighbourhood of S, and suppose f is CR on $\mathring{S}$ for $q \neq n-1$. Then there exist two C^∞ forms of type (p, q) F^+ on D and F^- on $\mathbf{C}^n \setminus \left((\overline{D} \cup \left(\bigcap\limits_{j\in\mathbf{N}} \overline{V}_j\right)\right)$, continuous up to $\mathring{S}$, with $\bar\partial F^+ = \bar\partial F^- = 0$ and $F^+ - F^- = f$ on $\mathring{S}$.*

PROOF. Let $j \in \mathbf{N}$ be a fixed integer.

Let $\tilde f$ be a C^1 extension of f on a neighbourhood of $\overline{D}$. From Martinelli-Bochner-Koppelman formula we get

$$(1)\qquad \int_S f(z) \wedge U_{p,q}(z, \zeta) + \int_A \tilde f(z) \wedge U_{p,q}(z, \zeta) - \int_D \bar\partial \tilde f(z) \wedge U_{p,q}(z, \zeta) + \\ + \bar\partial \int_D \tilde f(z) \wedge U_{p,q-1}(z, \zeta) = \begin{cases} (-1)^q \tilde f(\zeta) & \text{if } \zeta \in D \\ 0 & \text{if } \zeta \notin \overline{D} \end{cases}$$

and by Proposition 1, for $\zeta \notin \overline{V}_j$ we have

$$(2)\qquad \int_A \tilde{f}\wedge U_{p,q} = \int_A \tilde{f}\wedge[\bar\partial_z \eta^j_{p,q} + \bar\partial_\zeta \eta^j_{p,q-1}] =$$
$$= -(-1)^{p+q}\int_{\partial S} f\wedge \eta^j_{p,q} - (-1)^{p+q}\int_A \bar\partial_z \tilde{f}\wedge\eta^j_{p,q} + \int_A \tilde{f}\wedge\bar\partial_\zeta \eta^j_{p,q-1}\,.$$

Then, if $1 \leqslant q \leqslant s-1$ we set

$$F^\pm := (-1)^q \int_S f\wedge U_{p,q} - (-1)^p \int_{\partial S} f\wedge \eta^j_{p,q} + (-1)^q \int_A \tilde{f}\wedge \bar\partial_\zeta \eta^j_{p,q-1} +$$
$$+ (-1)^q \bar\partial_\zeta \int_D \tilde{f}\wedge U_{p,q-1}$$

on $D\setminus\overline{V}_j$ and on $\mathbf{C}^n\setminus(\overline{D}\cup\overline{V}_j)$ respectively.

Therefore we have

$$\bar\partial F^\pm = (-1)^{p-1}\int_S f\wedge\bar\partial_\zeta U_{p,q} - (-1)^q \int_{\partial S} f\wedge \bar\partial_\zeta \eta^j_{p,q} =$$
$$= (-1)^p \int_S f\wedge\bar\partial_z U_{p,q+1} - (-1)^q \int_{\partial S} f\wedge\bar\partial_\zeta \eta^j_{p,q} =$$
$$= (-1)^q \int_{\partial S} f\wedge[U_{p,q+1} - \bar\partial_\zeta \eta^j_{p,q}] = (-1)^q \int_{\partial S} f\wedge \bar\partial_z \eta^j_{p,q+1} = 0\,,$$

since f is CR.

If $q = n-1$, we consider the forms

$$F^\pm := (-1)^{n-1}\int_S f\wedge U_{p,n-1} + (-1)^{n-1}\int_A \tilde{f}\wedge U_{p,n-1} + (-1)^{n-1}\bar\partial_\zeta \int_D \tilde{f}\wedge U_{p,n-2}\,,$$

which are $\bar\partial$-closed since $\bar\partial_\zeta U_{p,n-1} = 0$.

From (1) and (2) we now obtain

$$\begin{cases} F^+ = \tilde{f} + (-1)^p \displaystyle\int_A \bar\partial_z \tilde{f}\wedge \eta^j_{p,q} + (-1)^q \int_D \bar\partial_z \tilde{f}\wedge U_{p,q} \\[2ex] F^- = (-1)^p \displaystyle\int_A \bar\partial_z \tilde{f}\wedge \eta^j_{p,q} + (-1)^q \int_D \bar\partial_z \tilde{f}\wedge U_{p,q} \end{cases}$$

(the first integral is missing for $q = n-1$).

Since the integral $\int_D \bar\partial_z \tilde f \wedge U_{p,q}$ is absolutely convergent for every $\zeta \in \mathbf{C}^n$, F^+ and F^- extend continuously up to $S \setminus \bar V_j$, and we have

$$F^+|_{S\setminus \bar V_j} - F^-|_{S\setminus \bar V_j} = \tilde f|_{S\setminus \bar V_j} = f|_{S\setminus \bar V_j}\,.$$

By the remark following Proposition 1, $F^\pm$ define, as $j \in \mathbf{N}$ varies, two C^∞ forms, $\bar\partial$-closed on D and on $\mathbf{C}^n \setminus \left(\bar D \cup \bigcap_{j\in\mathbf{N}} \bar V_j\right)$ respectively, continuous up to $\mathring{S}$, and such that $F^+ - F^- = f$ on $\mathring{S}$. ■

Remark. If $S \in C^\infty$ and $f \in C^m_{(p,q)}(S)$ $(m \geq 2)$, then $F^\pm$ extend up to $\mathring{S}$ as forms of class $C^{(m,\lambda)}$, with $\lambda \in (0,1)$ (i.e. the coefficients of $F^\pm$ are C^m and their derivatives of order m are λ-Hölder).

This follows from Proposition 0.10 of [2] applied to the integral $\int_{D_\varepsilon} \bar\partial_z \tilde f \wedge U_{p,q}$, where $\{D_\varepsilon\}_{\varepsilon>0}$ is an increasing family of open sets with C^∞ boundary such that $\bigcup_{\varepsilon>0} D_\varepsilon = D$ and $\bigcup_{\varepsilon>0} (S \cap \bar D_\varepsilon) = \mathring{S}$.

5. Applications.

1) A first application of Theorem 2 allows to obtain an extension theorem for CR-forms of type (p, q).

Let D be a domain which satisfies I) and II), with S contained in a smooth and strictly pseudoconvex hypersurface Σ and A of class C^1. Assume that $\bar A$ has a fundamental system of Stein neighbourhoods $\{V_j\}_{j\in\mathbf{N}}$ with boundaries ∂V_j transversal to Σ.

Theorem 3. *Let $0 \leq p \leq n$ and $1 \leq q \leq n-3$. Let f be a (p,q)-form of class C^m on S $(2 \leq m \leq +\infty)$ and W a neighbourhood of $\bar A$. If f is CR on $\mathring{S}$, then there exists a (p,q)-form F of class C^{m-2} on $D \cup \mathring{S}$, $\bar\partial$-closed on D, and such that $F|_{S\setminus W} = f_{S\setminus W}$.*

In the proof of this theorem we need the following approximation lemma (for a proof see [15] p. 244 or [4] p. 785):

Lemma. *Let $V \subseteq \mathbf{C}^n$ be an open set and $G := \{z \in V\colon g(z) < 0$ and $h(z) < 0\}$ where g, h are C^∞ on V and $dg(z) \neq 0$ if $g(z) = 0$, $dh(z) \neq 0$ if $h(z) = 0$, $dg \wedge dh(z) \neq 0$ if $g(z) = h(z) = 0$. We suppose that $\bar G$ is a compact connected region of $\mathbf{C}^n$. Let W be a neighbourhood of the set*

$\{z \in V: g(z) = h(z) = 0\}$. *Then there exists a domain* $G' \subset G$ *defined on* V *by a* C^∞ *function* F *such that* $\partial G' \setminus \partial G \subset W$ *and*

$$\mathscr{L}_{F,z} \geqslant \alpha(z)\, \mathscr{L}_{g,z} + \beta(z)\, \mathscr{L}_{h,z}$$

for every $z \in \partial G'$, *where* $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$ *and*

$$\operatorname{supp} \alpha \cap \partial G' \subset W \cup \{g = 0\}\,, \qquad \operatorname{supp} \beta \cap \partial G' \subset W \cup \{h = 0\}\,.$$

PROOF OF THEOREM 3. Let $\Sigma = \{\varrho = 0\}$, where ϱ is a strongly plurisubharmonic function on a neighbourhood U of $\bar{D}$. Let $\varrho': U \to \mathbb{R}$ be a C^∞ function such that $\varrho' \leqslant \varrho$ on U, $\varrho' = \varrho = 0$ on $S \setminus W$ and $\varrho' < \varrho = 0$ on ∂S, and with the same convexity properties as ϱ.

Take $V_j \subseteq W$. We may suppose that V_j is defined by a strongly plurisubharmonic function ψ on a neighbourhood of V_j.

Let ψ' be a C^∞ function on $\mathbb{C}^n$ such that $\psi' = \psi$ on a small neighbourhood V of $(\partial V_j \setminus D) \cap \{\varrho' > 0\}$ and $\psi' < 0$ on the component of $\{\varrho' \leqslant 0\} \setminus V$ which contains D.

The open set $D' := \{\varrho' < 0\} \cap \{\psi' < 0\}$ contains $\bar{D} \setminus \mathring{S}$, and $\partial D' \supset S \setminus W$. Applying the lemma to D' we can obtain a C^∞ domain $D'' \subset D'$, strictly pseudoconvex, such that $D'' \supset \bar{D} \setminus \mathring{S}$ and $\partial D'' \supset S \setminus W$.

Since condition III$'$) of 4.1 is verified for $s = n - 2$, we can apply Theorem 2 on D and obtain two $\bar\partial$-closed forms F^+ on D and F^- on $\mathbb{C}^n \setminus \bar{D}$, of class $C^{(m,\lambda)}$ up to $\mathring{S}$ $(0 < \lambda < 1)$, such that $F^+ - F^- = f$ on $\mathring{S}$.

The form $F^-|_{\mathbb{C}^n \setminus \overline{D''}}$ is C^m up to the boundary $\partial D''$. Let $\tilde{F}^-$ be a C^m extension to $\mathbb{C}^n$, and let $\beta := \bar\partial \tilde{F}^- \in C^{m-1}_{(p,q+1)}(\mathbb{C}^n)$. Then we have $\bar\partial \beta = 0$ and $\operatorname{supp} \beta \subseteq \overline{D''}$. According to Theorem 4.3 of [1] (see also [9]), we can find a $u \in C^{m-2}_{(p,q)}(\mathbb{C}^n)$ with $\operatorname{supp} u \subseteq \overline{D''}$ and $\bar\partial u = \beta$.

Let $F := F^+ - \tilde{F}^- + u \in C^{m-2}_{(p,q)}(D \cup \breve{S})$. Then $\bar\partial F = -\bar\partial \tilde{F}^- + \bar\partial u = 0$, and we have $F|_{S \setminus W} = (F^+ - F^-)|_{S \setminus W} = f|_{S \setminus W}$. ■

REMARKS. (1) In particular, Theorem 3 can be applied when S is strictly pseudoconvex and A is contained in the zero-set of a pluriharmonic function.

(2) If S is q-pseudoconvex, a theorem analogous to Theorem 4.3 of [1] holds for forms of certain types depending on q (see [12]). This can be applied as before to obtain the extension.

(3) If CR-forms are C^∞, Theorem 3 is a particular case of a more general theorem which can be deduced from results of Andreotti and

Hill [4] and which holds under weaker convexity assumptions on S. These results are based on a difficult cohomology vanishing theorem, while in the preceding theorem only integral representation formulas are used.

Let D be a domain verifying conditions I) and II) of 2.1, with $S \subset \Sigma := \{z \in U: \varrho(z) = 0\}$, U open set of $\mathbf{C}^n$. Suppose that $\mathscr{L}_{\varrho,z}$ has at least $r+1$ positive eigenvalues for z in a neighbourhood W of S.

Let $\psi: U \to \mathbf{R}$ be strongly $(n-r-1)$-plurisubharmonic on a neighbourhood of the set $\{\psi = 0\}$, such that $d\psi \neq 0$ on $\{\psi = 0\}$, $\psi < 0$ on $\bar{D}$ and $\{\psi = 0\} \cap \Sigma \subset W$. Let $D' := \{\varrho < 0\} \cap \{\psi < 0\}$ be such that $D' \subset\subset U$ and $d\varrho \wedge d\psi \neq 0$ on $\Sigma \cap \{\psi = 0\}$.

From the results of [4] we can obtain the following theorem.

THEOREM. *If $0 \leqslant p \leqslant n$ and $0 \leqslant q \leqslant r-1$, every CR-form of type (p, q) of class C^∞ on $\breve{S}$ extends on $D \cup \breve{S}$ by a $\bar{\partial}$-closed C^∞ form.*

PROOF. We can apply the lemma to D' and obtain a domain $D'' = \{F = 0\}$ contained in D' such that $D'' \supset \bar{D} \setminus S$, $\partial D'' \supset S$ and $\mathscr{L}_{F,z|T_z(\partial D'')}$ has at least r positive eigenvalues at each point $z \in \partial D''$. In fact, $\mathscr{L}_{\varrho,z}$ is positive definite on a r-dimensional subspace of $T_z(\partial D'')$, for every $z \in \partial D'' \cap W$. The same holds for $\mathscr{L}_{\psi,z}$, for z in a neighbourhood of $\{\psi = 0\}$.

Let $V \subset U$ be an open neighbourhood of D'' such that $D \cup \mathring{S} \subset V$, $\partial D'' \setminus \mathring{S} \subset \partial V$. Now the theorem follows from Theorem 6 of [4] part I and Theorem 6 of [4] part II, since $\mathring{S} = \{z \in V: F(z) = 0\}$ and we have $D'' = V^- := \{z \in V: F(z) < 0\} \supset D$. ∎

Now suppose that S is a C^∞ real hypersurface defined by $\varrho = 0$ and let x_0 be a point of strict pseudoconvexity. Then there exixts an open neighbourhood U of x_0 such that $\bar{U} \cap \{\varrho \leqslant 0\}$ is biholomorphic to the intersection of a strictly convex set with a halfspace, and so it verifies the conditions considered in Remark (1).

Thus we obtain a local extension theorem for CR-forms, which generalizes the local extension theorem for CR-functions of H. Lewy [10]:

COROLLARY 2. *There exists a neighbourhood S' of x_0, relative to S, such that every CR (p, q)-form of class C^m $(2 \leqslant m \leqslant +\infty)$ on a neighbourhood of S' in S $(0 \leqslant p \leqslant n,\ 0 \leqslant q \leqslant n-3)$, extends by a $\bar{\partial}$-closed form on an open set D contained in the convex side, such that $\partial D \supset S'$.* ∎

REMARK. For C^∞ forms, also this result is a particular case of a theorem of Andreotti and Hill (see Theorem 6 of [4] part I and Theorem 2 of [4] part II), which assures local extendibility of C^∞ CR-forms of type (p, q) near a point $x_0 \in S$ where the Levi form has at least $r+1$ positive eigenvalues, for $0 \leqslant q \leqslant r-1$.

2) Now we consider the problem of extension of CR-forms outside D.

Suppose that S is strictly pseudoconvex and A is contained in the zero-set of a pluriharmonic function φ.

THEOREM 4. *For $0 \leqslant p \leqslant n$, $1 \leqslant q \leqslant n-3$, let f be a CR-form of type (p, q) of class C^m on S $(2 \leqslant m \leqslant +\infty)$. Then*

(a) if $m < +\infty$ and W is an open neighbourhood of $\bar{A}$, there exists a (p, q)-form F of class C^{m-1} on $(\mathbf{C}^n \setminus \bar{D}) \cup \mathring{S}$ which is $\bar\partial$-closed on $\mathbf{C}^n \setminus \bar{D}$ and such that $F|_{S \setminus W} = f|_{S \setminus W}$;

(b) if $m = +\infty$, there exists a (p, q)-form F, C^∞ on $(\mathbf{C}^n \setminus \bar{D}) \cup \mathring{S}$, such that $\bar\partial F = 0$ on $\mathbf{C}^n \setminus \bar{D}$ and $F|_{\mathring{S}} = f|_{\mathring{S}}$.

PROOF. Since condition III′) is verified for $s = n-2$, we can apply Theorem 2 on D and obtain two $\bar\partial$-closed forms F^+ on D and F^- on $\mathbf{C}^n \setminus \bar{D}$, of class C^m up to $\mathring{S}$, such that $F^+ - F^- = f$ on $\mathring{S}$.

Let $m < +\infty$. Let $\lambda > 0$ so small that the set $D' := \{z \in D: -\varphi(z) + \lambda |z|^2 > 0\}$ contains $D \setminus W$. Applying the lemma to D' we get a strictly pseudoconvex domain D'' with C^∞ boundary which contains $D \setminus W$.

Let $u \in C^m_{(p,q-1)}(\bar{D}'')$ be such that $\bar\partial u = F^+$ (see [15] Th. 3). Let $\tilde{u}$ be a C^m extension of u to $\mathbf{C}^n$. Set $F := \bar\partial \tilde{u} - F^-$. Then $F \in C^{m-1}_{(p,q)}\cdot((\mathbf{C}^n \setminus \bar{D}) \cup \mathring{S})$ and $\bar\partial F = 0$, $F|_{S \setminus W} = (F^+ - F^-)|_{S \setminus W} = f|_{S \setminus W}$.

Now take $m = +\infty$. According to Theorem 2 of [13], we can find $u \in C^\infty_{(p,q-1)}(D \cup \mathring{S})$ such that $\bar\partial u = F^+$. Let $\tilde{u}$ be a C^∞ extension of u to $\mathbf{C}^n$.

Then $F := \bar\partial \tilde{u} - F^-$ is the desired extension of f. ■

3) Under the same assumptions, we consider the inhomogeneus $\bar\partial_b$-problem on $\mathring{S}$:

$$\bar\partial_b u = f \tag{1}$$

where f is a $(p, q+1)$-form on $\mathring{S}$ and u is a (p, q)-form on $\mathring{S}$.

COROLLARY 3. *Let $0 \leqslant q \leqslant n-4$, and let f be a CR-form of type $(p, q+1)$ of class C^m on $\breve{S}$ $(3 \leqslant m \leqslant +\infty)$. Then* *(a) if $q > 0$ and $m = +\infty$, there exists a solution $u \in C^{\infty}_{(p,q)}(\mathring{S})$ of* (1);

(b) if $q = 0$ or $m < +\infty$ and W is an open neighbourhood of $\bar{A}$, there exists a (p, q)-form $u \in C^{m-2}_{(p,q)}(S \setminus \overline{W})$ such that $\bar{\partial}_b u = f|_{S\setminus W}$.

PROOF. For $j \in \mathbf{N}$ sufficiently large, consider the set $D_j := \{z \in D: \varphi(z) > 1/j\}$. Let $S_j := S \cap \bar{D}_j$.

According to Theorem 3, we can find a form $\tilde{f}_j \in C^{m-2}_{(p,q+1)}(D_j \cup \mathring{S}_j)$ such that $\bar{\partial}\tilde{f}_j = 0$ on D_j and $\tilde{f}_{j|\mathring{S}_{j-1}} = f|\mathring{S}_{j-1}$.

As in the proof of Theorem 4, we can construct a smooth strictly pseudoconvex domain D'_j such that $\bar{D}_{j-1} \subset \overline{D'_j} \subset D_j \cup \mathring{S}_j$. Let $u_j \in C^{m-2}_{(p,q)}(\overline{D'_j})$ be such that $\bar{\partial} u_j = \tilde{f}_j$ on D'_j. Then $\bar{\partial}_b(u_{j|\mathring{S}_{j-1}}) = f|\mathring{S}_{j-1}$.

Since $\bar{\partial}(u_j - u_{j+1|D'_j}) = 0$, if $q > 0$ and $m = +\infty$ we can find a $v_j \in C^{\infty}_{(p,q-1)}(\overline{D'_j})$ such that $u_j - u_{j+1|D'_j} = \bar{\partial} v_j$. Let $\tilde{v}_j$ be a C^{∞} extension of v_j to $\mathbf{C}^n$. Replace u_{j+1} by $u'_{j+1} := u_{j+1} + \bar{\partial}\tilde{v}_j$. Then $u'_{j+1} \in C^{\infty}_{(p,q)}(\overline{D'_{j+1}})$ and $u'_{j+1|D'_j} = u_j$, $\bar{\partial}_b(u'_{j+1|\mathring{S}_j}) = f_{|\mathring{S}_j}$, and therefore we can glue the forms together and obtain the desired form u.

Now take $q = 0$ or $m < +\infty$. For j sufficiently large, we have $\mathring{S}_{j-1} \supset S \setminus \overline{W}$. Then $u_{j|S\setminus\overline{W}} \in C^{m-2}_{(p,q)}(S \setminus \overline{W})$ and $\bar{\partial}_b(u_{j|S\setminus\overline{W}}) = f_{|S\setminus\overline{W}}$: ■

REMARK. Results similar to this have been obtained by Boggess [5] using an explicit integral formula for the solutions.

4) Finally, under the same hypotheses we consider the general Cauchy problem for $\bar{\partial}$:

$$\bar{\partial} u = f\,, \qquad u_{|\mathring{S}} = g \tag{2}$$

where f is a $(p, q+1)$-form on $D \cup \mathring{S}$, g is a (p, q)-form on $\mathring{S}$ and u is a (p, q)-form on $D \cup \mathring{S}$.

COROLLARY 4. *Let $0 \leqslant q \leqslant n-3$, $f \in C^{m}_{(p,q+1)}(D \cup \mathring{S})$ and $g \in C^{m}_{(p,q)}(\mathring{S})$ $(2 \leqslant m \leqslant +\infty)$ such that $\bar{\partial} f = 0$ on D and $\bar{\partial}_b b = f_{|\mathring{S}}$. Then*

(a) if $q = 0$ or $q > 1$ and $m = +\infty$, there exists a solution $u \in C^{m}_{(p,q)}(D \cup \mathring{S})$ of (2);

(b) if $q = 1$ or $q > 0$ and $m < +\infty$, and W is an open neighbourhood of $\bar{A}$, there exists a form $u \in C^{m-2}_{(p,q)}((D \cup S)\setminus\overline{W})$ such that $\bar{\partial} u = f$ on $D \setminus \overline{W}$, $u_{|S\setminus\overline{W}} = g_{|S\setminus\overline{W}}$.

For $q = 0$ the solution is unique.

Proof. Let $m = +\infty$. According to Theorem 2 of [13], we can find $w \in C^{\infty}_{(p,q)}(D \cup \overset{\circ}{S})$ such that $\bar\partial w = f$ on D. Then $\bar\partial_b(g - w_{|\overset{\circ}{S}}) = 0$.

If $q = 0$, from Theorem 1 we can obtain an holomorphic extension $h \in C^{\infty}_{(p,0)}(D \cup \overset{\circ}{S})$ of $g - w_{|\overset{\circ}{S}}$ (see Proposition 0.10 of [2] and the remark following Theorem 2).

We set $u := w + h$. Then we have $\bar\partial u = \bar\partial w = f$ on D and $u_{|\overset{\circ}{S}} = w_{|\overset{\circ}{S}} + h_{|\overset{\circ}{S}} = g$.

If $q > 1$, from Corollary 3 we get a form $v \in C^{\infty}_{(p,q-1)}(\overset{\circ}{S})$ such that $\bar\partial_b v = g - w_{|\overset{\circ}{S}}$. Let $\tilde v$ be a C^∞ extension of v on $D \cup \overset{\circ}{S}$, and $u := w + \bar\partial \tilde v$. Then $\bar\partial u = \bar\partial w = f$ and $u_{|\overset{\circ}{S}} = g$.

Now take $m < +\infty$ or $q = 1$. For $j \in \mathbf{N}$, let D_j, S_j and D'_j be as in the proof of Corollary 3.

If $q > 0$, take j such that $D_j \supset D \setminus W$, and let D' be a smooth strictly pseudoconvex domain such that $D_j \subset D' \subset D$. Let $W \in C^m_{(p,q)}(\overline{D'})$ be such that $\bar\partial w = f_{|D'}$. Then $\bar\partial_b(g_{|\overset{\circ}{S}_j} - w_{|\overset{\circ}{S}_j}) = 0$. According to Theorem 3, we can find a $\bar\partial$-closed extension $v \in C^{m-2}_{(p,q)}((D \cup \overset{\circ}{S}) \setminus \overline{W})$ of $(g - w)_{S \setminus \overline{W}}$.

Then $u := w + v \in C^{m-2}_{(p,q)}((D \cup \overset{\circ}{S}) \setminus \overline{W})$, and $\bar\partial u = f$ on $D \setminus \overline{W}$, $u|_{S \setminus \overline{W}} = g|_{S \setminus \overline{W}}$.

Now suppose $q = 0$. For any j, let $w_j \in C^m_{(p,0)}(\overline{D}'_j)$ be such that $\bar\partial w_j = f$ on D'_j.

Then $\bar\partial_b((g - w_j)|_{\overset{\circ}{S}_{j-1}}) = 0$, and from Theorem 1 we get an holomorphic extension $h_j \in C^m_{(p,0)}(D_{j-1} \cup \overset{\circ}{S}_{j-1})$ of $(g - w_j)|_{|\overset{\circ}{S}_{j-1}}$.

Let $u_j := w_j + h_j \in C^m_{(p,0)}(D_{j-1} \cup \overset{\circ}{S}_{j-1})$. Then $\bar\partial u_j = f$ on D_{j-1} and $u_{j|\overset{\circ}{S}_{j-1}} = g_{|\overset{\circ}{S}_{j-1}}$.

We have $\bar\partial(u_j - u_{j+1|D_{j-1}}) = 0$ and $(u_j - u_{j+1})_{|\overset{\circ}{S}_{j-1}} = 0$. Then $u_{j+1|D_{j-1} \cup \overset{\circ}{S}_{j-1}} = u_j$, and setting $u_{|D_{j-1} \cup \overset{\circ}{S}_{j-1}} := u_j$ we obtain the solution $u \in C^m_{(p,0)}(D \cup \overset{\circ}{S})$.

If $u_1, u_2 \in C^m_{(p,0)}(D \cup \overset{\circ}{S})$ are two solutions of (2), then $u_1 \equiv u_2$, since $\bar\partial(u_1 - u_2) = 0$ and $(u_1 - u_2)_{|\overset{\circ}{S}} = 0$. ■

Remark. For C^∞ forms, these results are contained in those of Andreotti and Hill (Proposition 4.1 of [4] part I).

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