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Rheological models and hysteresis effects

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Rheological Models and Hysteresis Effects.

A. VISINTIN (*)

SUMMARY - Several classes of rheological models are obtained by means of combinations in parallel and in series of (possibly infinite) elastic, viscous and plastic elements. Emphasis is given to models of elasto-plasticity exhibiting hysteresis effects, and mathematical representations are provided. Examples are discussed; one of these corresponds to the classical Preisach model used for ferromagnetism. A mathematical representation is introduced for a model of local fracture, too. In the approximation of small deformations, related dynamical problems are then formulated as systems of (possibly infinite) variational equations or inequalities; existence and uniqueness results are proved.

Introduction.

Here we study the mathematical properties of some classes of linear and non-linear rheological models, in particular of elastic and plastic type. These fulfil the property of rate-independence, *i.e.* their characteristic equations are invariant for application of any increasing homeomorphism to the time variable; in several cases this allows to represent the constitutive relation by means of a Volterra (*i.e.* causal) memory functional (section 2).

These developments extend those of Krasnosel'skiĭ and co-workers (see [7]). We also give several examples, and in particular we construct a rheological model corresponding to the classical Preisach model

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for ferromagnetic hysteresis (section 3). Finally we introduce (what the author supposes) a new rheological model with memory, aimed to represent local fracture and, more generally, irreversibility.

Then we study some related dynamical problems, assuming that the deformations are so small that the deformation tensor can be linearized. The use of the previous rheological models leads to the formulation of systems of (possibly infinite) variational equations or inequalities; for these we prove existence and uniqueness results (sections 4, 5). The mathematical techniques we use are not new (see [3, 9], *e.g.*); however here we deal with especially large classes of constitutive relations, including elasto-plasticity with strain-hardening.

In this paper we confine ourselves to continuum mechanics; however rheological models have the analogue in several other fields; for instance they correspond to circuital models in electromagnetism. Moreover the coupling of Maxwell's equations with constitutive laws representable by means of circuital models leads to mathematical problems similar to those studied here.

1. Rheological models.

In the mathematical description of the mechanical properties of a material, *i.e.* in the stress-strain relation, a distinction has to be done between the spheric components of the symmetrical Piola-Kirchoff stress tensor and of the strain tensor

$$\sigma' = \tfrac{1}{3}\Big(\sum_{i=1}^{3}\sigma_{ii}\Big)I\,; \quad \varepsilon' := \tfrac{1}{3}\Big(\sum_{i=1}^{3}\varepsilon_{ii}\Big)I \qquad (I\text{: identity } 3^2\text{-tensor})$$

and the deviatoric components of the same tensors

$$\sigma'' := \sigma - \sigma'\,; \qquad \varepsilon'' := \varepsilon - \varepsilon'.$$

Under the assumption of infinitesimal deformations, the relation between σ' and ε' is usually assumed to be elastic

$$\sigma' = \alpha(\varepsilon')\,;$$

since σ' and ε' are spheric, α is essentially a function $\mathbf{R}^+ \to \mathbf{R}^+$, or more generally a graph, and is maximal monotone. However also other relations can be considered between σ' and ε' (see [5], pag. 481).

For σ'' and ε'' several types of constitutive relations have been studied; these laws are usually represented by means of so-called ***rheological models***, which are constructed combining a more restricted class of ***basic rheological elements*** in series and in parallel (see [4, 5], *e.g.*).

The use of rheological models for the formulation of constitutive laws provides a concrete (though artificial) model, and consequently guarantees that these laws are consistent with the fundamental physical principles.

For being quite rigorous, one should speak of combinations in series and parallel just in the linear case, hence for infinitesimal deformations. Anyway we shall use these concepts for defining constitutive relationships also for the case of finite deformations.

We shall consider two classes of basic elements: *elastic elements*, characterized by a *rheological equation of state* of the form

$$\sigma \in \alpha(\varepsilon)\,, \tag{1.1}$$

and *viscous elements*, characterized by

$$\sigma \in \beta(\dot{\varepsilon})\,, \tag{1.2}$$

the dot denoting the time derivative. Here α and β denote cyclically maximal monotone graphs, namely subdifferentials of convex functions; hence α^{-1} and β^{-1} have the same property and (1.1), (1.2) can be written equivalently in the form

$$\varepsilon \in \alpha^{-1}(\sigma)\,, \qquad \dot{\varepsilon} \in \beta^{-1}(\sigma)\,.$$

As we shall see, this fact is especially convenient for the computation of the rheological equations corresponding to combinations in series and in parallel of these elements. Of course (1.1) and (1.2) are relations between tensors, not between matrices; however the above properties of α and β have tensorial invariance.

(1.1) includes the following particular cases:

(i) *Linear elasticity*:

$$\sigma = \lambda : \varepsilon (i.e.\ \sigma_{ij} = \sum_{l,m=1}^{3} \lambda_{ijlm}\varepsilon_{lm}\,, \qquad i, j = 1, 2, 3) \tag{1.3}$$

with λ positive definite and symmetric, in the sense that $\lambda_{ijlm} = \lambda_{jilm} = \lambda_{ijml}$ (the property of symmetry for 3^4-tensors will be understood in this sense everywhere in this paper).

(ii) ***Hencky's law*** for plastic deformations

$$\sigma \in (\partial I_K)^{-1}(\varepsilon)\,, \tag{1.4}$$

where K is a non-empty, closed, convex subset of the space of deviatoric tensors, $K \ni 0$; I_K is the indicator function of K, *i.e.*

$$I_K(\xi) = 0 \quad \text{if } \xi \in K\,, \qquad I_K(\xi) = +\infty \quad \text{if } \xi \notin K\,;$$

thus I_K is a proper, lower semi-continuous, convex function; ∂I_K denotes its subdifferential.

(iii) Law of *locking materials*:

$$\sigma \in \partial I_K(\varepsilon) \tag{1.5}$$

where K is as in (ii).

(1.2) contains the following particular cases:

(iv) *Linear viscosity*:

$$\sigma = \mu\,\vdots\,\dot{\varepsilon}\,, \tag{1.6}$$

with μ positive definite and symmetric 3--tensor.

(v) *Rigid perfect plasticity*:

$$\sigma = (\partial I_K)^{-1}(\dot{\varepsilon})\,, \tag{1.7}$$

where K is as in (ii) and represents the yield criterium.

Besides these classical models, we shall consider elements characterized by *memory laws* of the form

$$\sigma(t) = [\mathcal{G}(\varepsilon(\cdot), \sigma_0)](t) \tag{1.8}$$

or

$$\varepsilon(t) = [\mathcal{F}(\sigma(\cdot), \varepsilon^0)](t)\,; \tag{1.9}$$

here $\varepsilon(\cdot)$ and $\sigma(\cdot)$ denote the *hystory* of ε and σ in $[0, T]$, ε^0 and σ^0 are the respective initial values; $\mathcal{F}$ and $\mathcal{G}$ are Volterra (*i.e.* causal) functionals. This will be made precise in section 5. We shall confine ourselves to single-valued $\mathcal{F}$ and $\mathcal{G}$; in particular they can represent elasto-plasticity with strain-hardening, visco-elasticity with long memory and so on.

As for parallel and serial combinations of the latter elements, limitations arise since the representation (1.8) has not always an inverse of the form (1.9) and conversely.

We remind some basic properties of combinations in series and in parallel:

(*a*) If two or more either elementary or composed rheological models are coupled in series, then they experience the same stress, which is also the stress of the global model; moreover the strain of the composed model is the sum of their strains (which are different, in general):

$$\sigma = \sigma_1 = \sigma_2 = \dots; \qquad \varepsilon = \varepsilon_1 + \varepsilon_2 + \dots .$$

(*b*) For combinations in parallel these properties of stress and strain are interchanged:

$$\varepsilon = \varepsilon_1 = \varepsilon_2 = \dots; \qquad \sigma = \sigma_1 + \sigma_2 + \dots .$$

Thus a duality property appears between combinations in parallel and in series. The coupling in series (in parallel respect.) of two either elementary or composed models A_1, A_2 will be denoted by the ***rheological formula*** $A_1 - A_2$ ($A_1|A_2$, respect.). These rules are extended in a natural way to combinations in parallel and in series of infinite elements. Let $(\mathcal{T}, \mathcal{A}, \mu)$ be a measure space with μ finite non-negative measure; let $\{A_\varrho\}_{\varrho\in\mathcal{T}} \in \mathcal{A}$ be a family of rheological elements. Their arrangement in series will be denoted by the rheological formula $\sum_{\varrho\in\mathcal{T}} A_\varrho$ and corresponds to the constitutive relations

$$\text{(1.10)} \qquad \begin{cases} \tilde{\sigma} = \sigma_\varrho \qquad \mu_\varrho\text{-a.e. in } \mathcal{T} \\ \tilde{\varepsilon} = \displaystyle\int_{\mathcal{T}} \varepsilon_\varrho \, d\mu_\varrho ; \end{cases}$$

here $\tilde{\sigma}$, $\tilde{\varepsilon}$ are the stress and strain of the composed model; σ_ϱ, ε_ϱ are the stress and strain of its elements. We write $\tilde{\varepsilon}$ in order to distinguish it from the application $\varepsilon: \varrho \mapsto \varepsilon_\varrho$ and likewise for $\tilde{\sigma}$; this convention will be followed henceforth.

Similarly the combination in parallel of $\{A_\varrho\}_{\varrho\in\mathfrak{I}}$ will be denoted by the rheological formula $\prod_{\varrho\in\mathfrak{I}} A_\varrho$ and corresponds to the constitutive relations

$$\left\{\begin{aligned} &\tilde{\varepsilon} = \varepsilon_\varrho \qquad \mu_\varrho\text{ - a.e. in } \mathfrak{I} \\ &\tilde{\sigma} = \int_{\mathfrak{I}} \sigma_\varrho \, d\mu_\varrho \,. \end{aligned}\right. \tag{1.11}$$

2. Elasto-plastic models and hysteresis functionals.

Here we represent elasto-plastic effects by means of rheological models obtained by combining in series and in parallel elastic and perfectly plastic elements. These basic elements are rate-independent, that is their constitutive relations (see (2.1), 2.2) below) are invariant for the application of any increasing diffeomorphism to the time variable; consequently the same property holds also for their combinations. Here we confine ourselves to the uniaxial (*i.e.* one-dimensional) case. By E (P, respect.) we shall denote an elastic (plastic, respect.) element, corresponding to constitutibe laws of the form

$$\varepsilon = \alpha(\sigma) \qquad \text{or} \qquad \sigma = \beta(\varepsilon)\,, \tag{2.1}$$

$$\dot{\varepsilon} \in \partial I_K(\sigma)\,, \tag{2.2}$$

with α and $\beta := \alpha^{-1}$ continuous functions $\mathbf{R} \to \mathbf{R}$ and I_K indicator function of a non-empty, closed, convex set $K \subset \mathbf{R}$; we assume that $\alpha(0) = 0$ and $K \ni 0$.

We distinguish several arrangements:

1*st model* - Elastic and plastic elements in series: E-P.

This model corresponds to the constitutive law

$$\dot{\varepsilon} \in \alpha(\sigma)^{\cdot} + \partial I_K(\sigma)\,; \tag{2.3}$$

for the sake of simplicity, we detail the case in which α is linear:

$$\varepsilon = \lambda\sigma \ (\lambda\text{: positive constant});$$

however the following mathematical developments can be easily extended to the case of a nonlinear α. Now we shall show that σ can be expressed as a Volterra (*i.e.* causal) functional of ε:

$$\sigma(t) = [\mathcal{G}(\varepsilon, \sigma_0)]\,(t) \tag{2.4}$$

where $\varepsilon := \varepsilon(\cdot)$ denotes the function $t \mapsto \varepsilon(t)$ and $\sigma^0 \in \boldsymbol{K}$.

Following [7], at first we construct $\mathcal{G}$ for any continuous piecewise linear input function ε. For a generic $t \in]0, T]$, let $t_0 = 0 < t_1 < \ldots < < t_N = t$ be such that ε is linear in $[t_{n-1}, t_n]$ for $n = 1, \ldots, N$; we set recursively $\sigma^n :=$ projection onto $\boldsymbol{K}$ of $\sigma^{n-1} + (1/\lambda)\,[\varepsilon(t_n) - \varepsilon(t_{n-1})]$ for $n = 1, \ldots, N$, and then $[\mathcal{G}(\varepsilon, \sigma_0)]\,(t) = \sigma^N$. One can show that the functional $\mathcal{G}(\cdot, \sigma^0)$ is Lipschitz-continuous from the set of continuous piecewise linear functions $[0, T] \to \mathbf{R}$ into $C^0([0, T])$ (theorem 2 of [7]); hence $\mathcal{G}(\cdot, \sigma^0)$ can be extended univocally into a Lipschitz-continuous functional from $C^0([0, T])$ into itself, which we still denote by $\mathcal{G}$. In fig. 1 we sketch the more general case corresponding to a non-linear α.

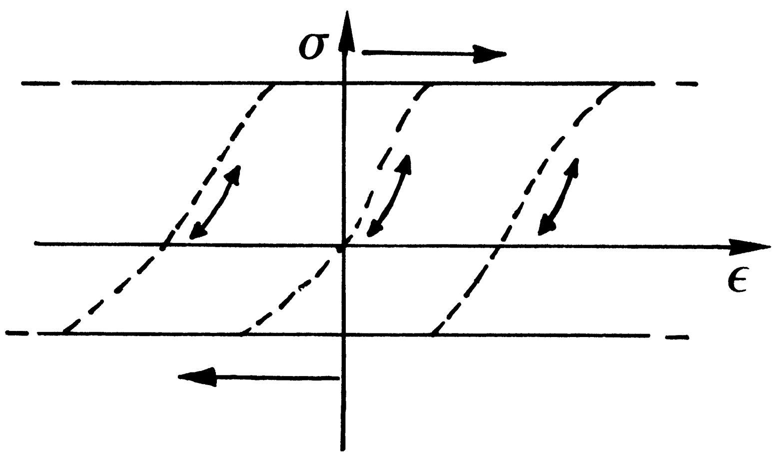

Figure 1

$\mathcal{G}$ fulfils the following properties:

(2.5) $$\mathcal{G}: D := C^0([0, T]) \times K \to C^0([0, T]; K)\,,$$

(2.6) $$\forall (\varepsilon, \sigma^0) \in D, \quad [\mathcal{G}(\varepsilon, \sigma^0)](0) = \sigma^0$$

(2.7) $$\begin{cases} \forall \bar{t} \in]0, T], \ \forall (\varepsilon_i, \sigma^0) \in D (i = 1, 2), \quad \text{if } \varepsilon_1 = \varepsilon_2 \text{ in } [0, \bar{t}], \text{ then} \\ [\mathcal{G}(\varepsilon_i, \sigma^0)](\bar{t}) = [\mathcal{G}(\varepsilon_2, \sigma^0)](\bar{t}) \ \text{(Causality)} \end{cases}$$

(2.8) $$\begin{cases} \forall (\varepsilon, \sigma^0) \in D, \ \forall \bar{t}, \ t \in [0, T] (\bar{t} < t), \ \text{setting } \sigma^{\bar{t}} := [\mathcal{G}(\varepsilon, \sigma^0)](\bar{t}), \\ [\mathcal{G}(\varepsilon, \sigma^0)](t) = [\mathcal{G}(\varepsilon(\bar{t} + \cdot), \sigma^{\bar{t}})](t - \bar{t}) \ \text{(Semi-group property)} \end{cases}$$

(2.9) $$\begin{cases} \forall (\varepsilon, \sigma^0) \in D, \ \forall s: [0, T] \to [0, T] \text{ continuous non-decreasing,} \\ \hfill \forall t \in [0, T], \\ [\mathcal{G}(\varepsilon \circ s, \sigma^0)](t) = [\mathcal{G}(\varepsilon, \sigma^0)](s(t)) \ \text{(Rate-independence)} \end{cases}$$

(2.10) $$\begin{cases} \forall (\varepsilon, \sigma^0), (\varepsilon_n, \sigma^0) \in D (n = 1, 2, \dots), \text{ if } \varepsilon_n \to \varepsilon \text{ strongly in } C^0([0, T]) \\ \text{then} \\ \mathcal{G}(\varepsilon_n, \sigma^0) \to \mathcal{G}(\varepsilon, \sigma^0) \text{ strongly in } C^0([0, T]) \ \text{(Continuity).} \end{cases}$$

This last property is named « vibrostability » by Krasnosel'skiĭ (see [6]). Moreover

(2.11) $$\begin{cases} \forall (\varepsilon_1, \sigma^0), (\varepsilon_2, \sigma^0) \in D, \ \forall \bar{t}, t \in [0, T] (\bar{t} < t), \text{ if } \varepsilon_1 = \varepsilon_2 \text{ in } [0, \bar{t}] \text{ and} \\ \varepsilon_1, \varepsilon_2 \text{ are each one non-increasing or non-decreasing in } [\bar{t}, t], \\ \text{then} \\ \{[\mathcal{G}(\varepsilon_1, \sigma^0)](t) - [\mathcal{G}(\varepsilon_2, \sigma^0)](t)\} \cdot [\varepsilon_1(t) - \varepsilon_2(t)] \geq 0 \\ \hfill (\text{« Piecewise monotonicity »}). \end{cases}$$

Concerning the possible generalization of the above construction to the multiaxial case, the author found technical difficulties in proving the Lipschitz-continuity of $\mathcal{G}(\cdot, \sigma^0)$, defined on piecewise linear functions $t \mapsto \varepsilon(t) \in \mathbf{R}^9_s$.

2nd model - Elastic and plastic elements in parallel: E|P.
This corresponds to the law

(2.12) $$\sigma \in \beta(\varepsilon) + (\partial I_K)^{-1}(\dot{\varepsilon})\,,$$

with the same notations as before. If β is injective, then (2,12) can be expressed in the form

$$\varepsilon(t) = [\mathcal{F}(\sigma, \varepsilon^0)]\,(t)\,, \tag{2.13}$$

with $\mathcal{F}$ Volterra functional (see fig. 2); this formula is similar to (2.4) and can be justified by means of the same procedure.

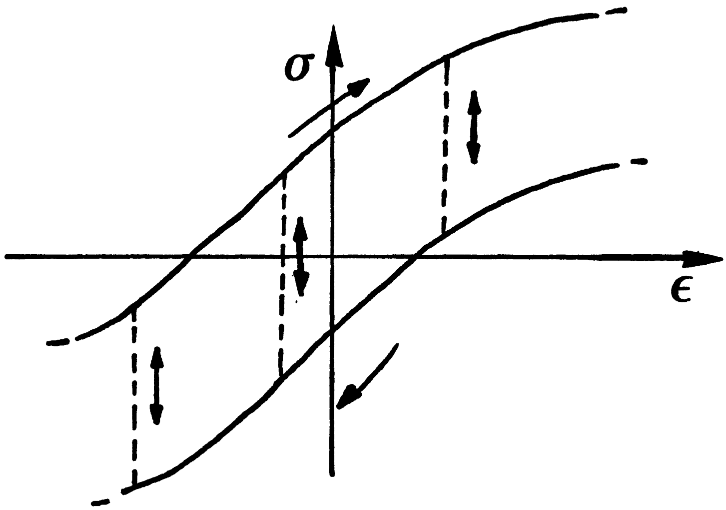

Figure 2

The domain D of $\mathcal{F}$ is a proper subset of $C^0([0, T]) \times \mathbf{R}$, the couple $(\sigma(0), \varepsilon^0)$ being confined to a region $\mathcal{S}$ corresponding to attainible states. Thus

$$\mathcal{F}: D \to C^0([0, T]); \qquad \forall (\sigma, \varepsilon^0) \in D, \qquad (\sigma(t), [\mathcal{F}(\sigma, \varepsilon^0)]\,(t)) \in \mathcal{S};$$

$\mathcal{F}$ fulfills the same properties (2.6), ..., (2.11) as $\mathcal{G}$ does.

3rd model - Parallel combination of elastic and plastic elements in series: $\prod_{\varrho \in \mathcal{T}} (E_\varrho\text{-}P_\varrho)$, the measure space $(\mathcal{T}, \mathcal{A}, \mu)$ being as in section 1 (see also [7]).

As we saw, E_ϱ-P_ϱ corresponds to the constitutive relation (cf.

the $1st$ model above)

$$\dot{\varepsilon} \in \alpha_\varrho(\sigma)^\cdot + \partial I_{K_\varrho}(\sigma) \qquad \mu_\varrho\text{ - a.e. in } \mathcal{T},$$

or also, if α_ϱ is injective,

$$\sigma = \mathcal{G}_\varrho(\varepsilon, \sigma^0)\,.$$

The composed model corresponds to

$$(2.14) \qquad \begin{cases} \dot{\tilde{\varepsilon}} = \alpha_\varrho(\sigma_\varrho)^\cdot + \partial I_{K_\varrho}(\sigma_\varrho) & \mu_\varrho\text{ - a.e. in } \mathcal{T}, \\ \tilde{\sigma} = \int_{\mathcal{T}} \sigma_\varrho \, d\mu_\varrho. \end{cases}$$

or also, if α_ϱ is injective μ_ϱ - a.e. in $\mathcal{T}$,

$$(2.15) \qquad \tilde{\sigma} = \int_{\mathcal{T}} \mathcal{G}_\varrho(\tilde{\varepsilon}, \sigma_\varrho^0)\, d\mu_\varrho =: \tilde{\mathcal{G}}(\tilde{\varepsilon}, \{\sigma_\varrho^0\}_{\varrho\in\mathcal{T}})\,,$$

with $\tilde{\mathcal{G}}$ Volterra functional. The properties of $\tilde{\mathcal{G}}$ are the same as for $\mathcal{G}$ and $\mathcal{F}$ in the previous models. $\mathcal{G}$ represents elasto-plasticity with strain-hardening.

4th model - Serial combination of elastic and plastic elements in parallel: $\sum_{\varrho\in\mathcal{T}} (\mathrm{E}_\varrho|\mathrm{P}_\varrho)$, the measure space $(\mathcal{T}, \mathcal{A}, \mu)$ being as in section 1.

As we saw, $\mathrm{E}_\varrho|\mathrm{P}_\varrho$ corresponds to the constitutive relation (cf. the $2nd$ model of this section)

$$\sigma \in \beta_\varrho(\varepsilon) + (\partial I_{K_\varrho})^{-1}(\dot{\varepsilon}) \qquad \mu_\varrho\text{ - a.e. in } \mathcal{T},$$

or also, if β_ϱ is injective,

$$\varepsilon = \mathcal{F}_\varrho(\sigma, \varepsilon_0)\,.$$

For the assembled model we get

$$(2.16) \qquad \begin{cases} \tilde{\sigma} \in \beta_\varrho(\varepsilon_\varrho) + (\partial I_{K_\varrho})^{-1}(\dot{\varepsilon}_\varrho) & \mu_\varrho\text{ - a.e. in } \mathcal{T}, \\ \tilde{\varepsilon} = \int_{\mathcal{T}} \varepsilon_\varrho \, d\mu_\varrho, \end{cases}$$

or also, if β_ϱ is injective μ_ϱ - a.e. in $\mathfrak{T}$,

$$\tilde{\varepsilon} = \int_{\mathfrak{T}} \mathcal{F}_\varrho(\tilde{\sigma}, \varepsilon_\varrho^0)\, d\mu_\varrho := \tilde{\mathcal{F}}(\tilde{\sigma}, \{\varepsilon_\varrho^0\}_{\varrho \in \mathfrak{T}}) \,, \tag{2.17}$$

with $\tilde{\mathcal{F}}$ Volterra functional fulfilling the same properties as in the previous models.

Let us consider the case in which the elastic elements E_ϱ's are linear, i.e. characterized by a law of the form

$$\varepsilon = \lambda_\varrho^{(1)} \sigma \qquad (\lambda_\varrho^{(1)}\text{: constant} > 0)\,. \tag{2.18}$$

Any non-empty, closed, convex set $K_\varrho \subset \mathbf{R}$ is of the form $K_\varrho = [\lambda_\varrho^{(2)}, \lambda_\varrho^{(3)}]$, with $\lambda_\varrho^{(2)}, \lambda_\varrho^{(3)} \in \mathbf{R}$ and $\lambda_\varrho^{(2)} \leq \lambda_\varrho^{(3)}$; if we require also $K_\varrho \ni 0$, then $\lambda_\varrho^{(2)} \leq 0 \leq \lambda_\varrho^{(3)}$. Consequently E_ϱ-P_ϱ is characterized by the triplet $\lambda_\varrho := (\lambda_\varrho^{(1)}, \lambda_\varrho^{(2)}, \lambda_\varrho^{(3)})$. Thus in order to describe the assembled model $\prod_{\varrho \in \mathfrak{T}} (\mathrm{E}_\varrho\text{-}\mathrm{P}_\varrho)$ one can take

$$\mathfrak{T} := \{\lambda_\varrho := (\lambda_\varrho^{(1)}, \lambda_\varrho^{(2)}, \lambda_\varrho^{(3)}) \in \mathbf{R}_\varrho^3 \colon \lambda_\varrho^{(1)} > 0,\ \lambda_\varrho^{(2)} \leq \lambda_\varrho^{(3)}\} \,, \tag{2.19}$$

or, if requiring also $K_\varrho \ni 0$,

$$\mathfrak{T}^* := \{\lambda_\varrho \in P \colon \lambda_\varrho^{(2)} \leq 0 \leq \lambda_\varrho^{(3)}\} \,; \tag{2.20}$$

then one introduces a σ-algebra $\mathcal{A}$ over $\mathfrak{T}$ (or $\mathfrak{T}^*$) and a positive measure μ, which determines the functional $\tilde{\mathfrak{G}}$ by (2.15), i.e. characterizes the behaviour of the material. If the elastic elements E_ϱ's are non-linear, then more than three (possibly infinite) parameters are required for describing each element E_ϱ-P_ϱ. A quite similar representation can be introduced for the memory functional $\tilde{\mathcal{F}}$ of the fourth model above.

Such a « spectral resolution » of the hysteresis functionals $\tilde{\mathfrak{G}}$ and $\tilde{\mathcal{F}}$ is reminiscent of the Preisach model used for one-dimensional ferromagnetism (see [8, 13] and also the 6th model below); the presence in $\mathfrak{T}$ of convex sets K_ϱ's with $K_\varrho \not\ni 0$ is quite analogous to the presence in the ferromagnetic model of dipoles for which the value $H = 0$ is not in the critical range of bi-stability.

3. Other rate-independent models.

We go on with the description of examples of rate-independent rheological models. Here, besides plastic and non-degenerate elastic elements, we shall consider Hencky's and locking elements, defined in examples (ii) and (iii) of section 1 and here denoted by H and L, respectively. We shall deal with the multivariate case.

5th model - Plastic and locking elements in parallel: P|L.
This corresponds to the constitutive relation

$$\sigma \in \partial I_{K^1}(\varepsilon) + (\partial I_{K^2})^{-1}(\dot{\varepsilon}) \tag{3.1}$$

with K^1, K^2 non-empty, closed, convex subsets of $\mathbf{R}^9_s$; here we require also that $K^1 \ni 0$, $K^2 \ni 0$ (see fig. 3). We set

$$\beta(\sigma) := \text{projection of } \sigma \text{ onto } K^2, \tag{3.2}$$

$$\alpha(\sigma) := \sigma - \beta(\sigma), \qquad \forall \sigma \in \mathbf{R}^9_s; \tag{3.3}$$

α and β are both cyclically maximal monotone graphs.

Both the applications $\sigma \mapsto \varepsilon$ and $\varepsilon \mapsto \sigma$ transform continuous tensorial functions into discontinuous ones; it seems more convenient

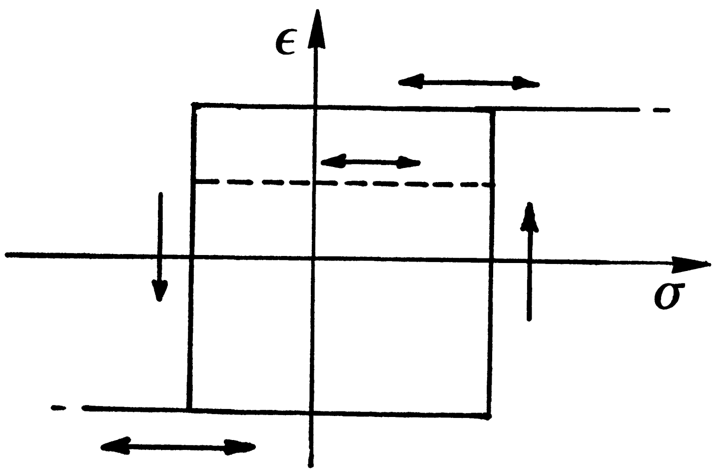

Figure 3

to consider the operator $\sigma \mapsto \varepsilon$: $C^0([0,T])^9_s \to L^\infty(0,T)^9_s$. This functional is discontinuous w.r.t. the strong topology of $C^0([0,T])^9_s$ and the weak star topology of $L^\infty(0,T)^9_s$; however, as discussed for the scalar case in [14], its closure corresponds to the following differential formulation:

$$
(3.4) \qquad \begin{cases} \varepsilon \in (\partial I_{K^1})^{-1}\big(\alpha(\sigma)\big)\,, \\ \dot\varepsilon \in \partial I_{K^2}\big(\beta(\sigma)\big)\,. \end{cases}
$$

In the scalar case, if σ and ε are replaced by H and M (respect.), the behaviour of this element is the same as that of the *relay* circuit used in electromagnetism.

6th model - Serial combination of plastic and locking elements in parallel: $\sum_{\varrho\in\mathfrak{T}} (\mathrm{P}_\varrho|\mathrm{L}_\varrho)$.

This corresponds to the rheological formulae

$$
(3.5) \qquad \begin{cases} \tilde\sigma \in \partial I_{K^1_\varrho}(\varepsilon_\varrho) + (\partial I_{K^2_\varrho})^{-1}(\dot\varepsilon_\varrho) & \mu_\varrho\text{ - a.e. in } \mathfrak{T}, \\ \tilde\varepsilon = \int_{\mathfrak{T}} \varepsilon_\varrho\, d\mu_\varrho, \end{cases}
$$

with K^1_ϱ, K^2_ϱ as in the previous model, μ_ϱ - a.e. in $\mathfrak{T}$. The state is characterized by the variables $\tilde\sigma$ and $\{\varepsilon_\varrho\}_{\varrho\in\mathfrak{T}}$.

In general the functional Φ: $C^0([0,T])^9_s \to L^\infty(0,T)^9_s$: $\tilde\sigma \mapsto \tilde\varepsilon$ is not continuous w.r.t. the strong topology of $C^0([0,T])^9_s$ and the weak star topology of $L^\infty(0,T)^9_s$: however, if the measure μ fulfils suitable non-singularity properties, Φ is continuous and can be represented in the form

$$
(3.6) \qquad \tilde\varepsilon(t) = \big[\tilde{\mathcal{F}}(\tilde\sigma, \{\varepsilon^0_\varrho\}_{\varrho\in\mathfrak{T}})\big](t)\,.
$$

$\tilde{\mathcal{F}}$ being a Volterra functional (see section 1 of [13]). In the general case, the closure of the functional Φ corresponds to the system

$$
(3.7) \qquad \begin{cases} \varepsilon_\varrho \in (\partial I_{K^1_\varrho})^{-1}\big(\alpha_\varrho(\tilde\sigma)\big) & \mu_\varrho\text{ - a.e. in } \mathfrak{T}, \\ \dot\varepsilon_\varrho \in \partial I_{K^2_\varrho}\big(\beta_\varrho(\tilde\sigma)\big) & \mu_\varrho\text{ - a.e. in } \mathfrak{T}, \\ \tilde\varepsilon = \int_{\mathfrak{T}} \varepsilon_\varrho\, d\mu_\varrho\,. \end{cases}
$$

with α_ϱ and β_ϱ defined as in (3.2), (3.3).

In the one dimensional case, if $\tilde{\sigma}$ and $\tilde{\varepsilon}$ are replaced by H and M (respect.), the classical Preisach model for ferromagnetism is obtained. So far this model has been extensively studied just in the scalar case (see [8, 13], e.g.); vectorial generalizations have been proposed in [2]; the above tensorial extension is to be compared with the second model introduced in [2].

7th model - Fracture element.

This type of rheological element is aimed to represent *local fracture* arising in a material when and where the density of elastic energy $E(\varepsilon)$ exceeds a critical threshold Ξ. Set $K := \{\varepsilon \in \mathbb{R}^9_s : E(\varepsilon) \leq \Xi\}$; this is a closed and convex set, due to the continuity and the convexity of the energy functional. We introduce the following rheological equation of state:

$$\sigma(t) = \begin{cases} a(\varepsilon(t)) & \text{if } \varepsilon(\tau) \in K \text{ for } 0 \leq \tau \leq t, \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

where $a := dE/d\varepsilon$; for the sake of simplicity, a is assumed single-valued (see fig. 4).

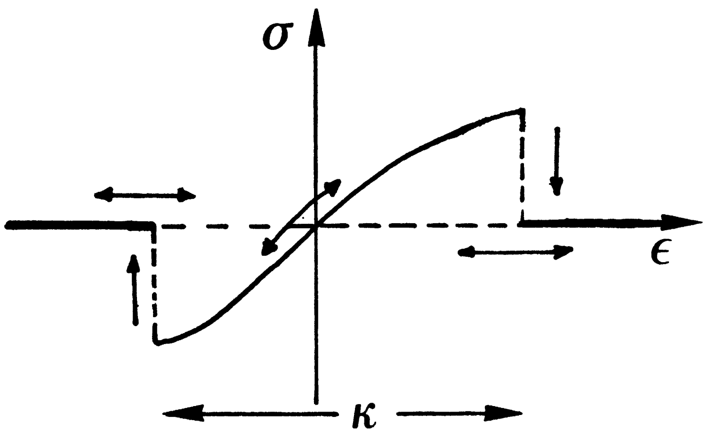

Figure 4

This model cannot be reduced to any combination of elastic and viscous elements, therefore it can be regarded as a new basic rheological element; its main feature is irreversibility. At first we « normalize » (3.8) by setting

$$\sigma = a(\varepsilon)\,\mathcal{H}(\delta(\varepsilon, K)) \tag{3.9}$$

where

$$[\mathcal{H}(\xi)]\,(t) = \begin{cases} 1 & \text{if } \xi(\tau) \geq 0 \text{ for } 0 \leq \tau \leq t, \\ 0 & \text{otherwise,} \end{cases} \tag{3.10}$$

$$\delta(\varepsilon, K) := \begin{cases} \text{distance}\,(\varepsilon, \mathbf{R}^9_s \setminus K) & \text{if } \varepsilon \in K, \\ -\,\text{distance}\,(\varepsilon, K) & \text{if } \varepsilon \in \mathbf{R}^9_s \setminus K. \end{cases}$$

The functional $\mathcal{H}$ can be compared with the univalued Heaviside function H ($H(\xi) = 0$ if $\xi < 0$, $H(\xi) = 1$ if $\xi \geq 0$) and with the functional $\sigma \mapsto \varepsilon$ of the 5th model above. $\mathcal{H}$ is discontinuous from $C^0([0, T])$ endowed with the strong topology into $L^\infty(0, T)$ endowed with the weak star topology; hence, especially in connection with partial differential equations, it is convenient to consider its closure $\overline{\mathcal{H}}$; for the same reason one introduces the maximal monotone graph $\overline{H}$ obtained as « closure » of the Heasivide function H.

A second argument has to be introduced in $\overline{\mathcal{H}}$, corresponding to the initial value of $\overline{\mathcal{H}}(\xi)$. Thus

$$\overline{\mathcal{H}} : D := \{(\xi, \eta^0) \in C^0([0, T]) \times [0, 1] \mid \eta^0 = 0 \text{ if } \xi(0) < 0\} \to \to \mathcal{P}(L^\infty(0, T)), \tag{3.11}$$

$$\forall (\xi, \eta^0) \in D, \qquad [\overline{\mathcal{H}}(\xi, \eta^0)]\,(0) = \eta^0 \tag{3.12}$$

and for any $t \in \,]0, T]$

$$\begin{cases} [\overline{\mathcal{H}}(\xi, \eta_0)]\,(t) \in [0, 1], \\ \text{if } \xi(t) \neq 0 \quad \text{then } \overline{\mathcal{H}}(\xi, \eta_0) \text{ is constant in a neighborhood of } t, \\ \text{if } \xi(t) = 0 \quad \text{then } \overline{\mathcal{H}}(\xi, \eta_0) \text{ is non-increasing at time } t. \end{cases} \tag{3.13}$$

As it is easy to check, this corresponds to the 5th model above with $K^1 = [0, 1]$, $K^2 = \mathbf{R}^+$, $\xi = \sigma$, $\eta = \varepsilon$. By (3.4) it is possible to express $\overline{\mathcal{H}}$ also in differential form:

$$\eta \in \overline{\mathcal{H}}(\xi, \eta_0) \quad \text{iff } \eta(0) = \eta^0 \text{ and } \begin{cases} \eta \in \overline{H}(-\xi^-), \\ \dot{\eta} \in \partial I_{\mathbf{R}^+}(\xi^+), \end{cases}$$

where $\bar{H}$ denotes the Heaviside graph ($\bar{H}(\xi) = \{0\}$ if $\xi < 0$, $\bar{H}(0) = [0, 1]$, $\bar{H}(\xi) = \{1\}$ if $\xi > 0$). Therefore $\sigma \in a(\varepsilon)\, \bar{\mathcal{K}}(\delta(\varepsilon, K), \sigma^0)$ corresponds to

$$\text{(3.14)} \qquad \begin{cases} \sigma \in a(\varepsilon)\bar{H}(\text{distance}\,(\varepsilon, K)), \\ \dot{\sigma} \in a(\varepsilon)\, \partial I_{\mathbf{R}^+}(\text{distance}\,(\varepsilon, \mathbf{R}^9_s \setminus K)). \end{cases}$$

A characteristic feature of this fracture element, or of its normalized formulation $\bar{\mathcal{K}}$, is that once the input ε has gone out the critical set K, then the output σ attains irreversibly a fixed state ($\sigma = 0$ in our case). This property of ***absolute irreversibility*** can make the above model useful for many applications: it can represent explosion in chemistry, death in biology, bankrupcy in economics and other calamities on other fields (it seems that happy events are less irreversible than bad ones, unfortunately). The range of possible applications of the above element are increased if one combines it with other models; this possibility is not restricted to continuum mechanics, since combinations in series and in parallel are used also in other fields, such as electromagnetism.

8th model - Parallel combination of several fracture elements: $\prod_{\varrho \in \mathcal{T}} F_\varrho$.

Let F_ϱ correspond to the formula

$$\sigma = a_\varrho(\varepsilon)\, \mathcal{K}(\delta(\varepsilon, K_\varrho), \sigma^0)$$

with a_ϱ and K_ϱ as a and K in the previous model, $\forall \varrho \in \mathcal{T}$. For the assembled model we get

$$\text{(3.15)} \qquad \begin{cases} \sigma_\varrho = a_\varrho(\tilde{\varepsilon})\mathcal{K}(\delta(\tilde{\varepsilon}, K_\varrho), \sigma^0_\varrho) \qquad \mu_\varrho\text{ - a.e. in } \mathcal{T}, \\ \tilde{\sigma} \;= \displaystyle\int_{\mathcal{T}} \sigma_\varrho \, d\mu_\varrho\,; \end{cases}$$

here the state variables are $\tilde{\varepsilon}$ and $\{\sigma_\varrho\}_{\varrho \in \mathcal{T}}$.

In general the functional $\Phi\colon C^0([0, T]) \to L^\infty(0, T)\colon \tilde{\varepsilon} \mapsto \tilde{\sigma}$ is not continuous w.r.t. the strong topology of $C^0([0, T])$ and the weak star topology of $L^\infty(0, T)$; its closure corresponds to the system

$$
(3.16)\qquad \left|\begin{array}{ll}
\sigma_\varrho \in a_\varrho(\tilde{\varepsilon})\cdot \bar{H}(-\operatorname{dist.}(\tilde{\varepsilon}, K_\varrho)) & \mu_\varrho\text{ - a.e. in } \mathfrak{T},\\
\dot{\sigma}_\varrho \in a_\varrho(\tilde{\varepsilon})\cdot \partial I_{\mathbf{R}^+}(\operatorname{dist.}(\tilde{\varepsilon}, \mathbf{R}^9_s \setminus K)) & \mu_\varrho\text{ - a.e. in } \mathfrak{T},\\
\tilde{\sigma} = \int_{\mathfrak{T}} \sigma_\varrho \, d\mu_\varrho ; &
\end{array}\right.
$$

the initial values $\sigma_\varrho = \sigma^0_\varrho$ μ_ϱ - a.e. in $\mathfrak{T}$ are to be specified.

Similarly to what we said about the Preisach model, if the measure μ has suitable non-singularity properties, Φ is continuous from $C^0([0, T])$ endowed with the strong topology into itself, and can be represented in the form

$$
\tilde{\sigma}(t) = [\tilde{\mathfrak{G}}(\tilde{\varepsilon}, \{\sigma^0_\varrho\}_{\varrho\in\mathfrak{T}})]\,(t)\,,
$$

with $\tilde{\mathfrak{G}}$ Volterra functional (cf. also(3.6)).

Other combinations of fracture elements F_ϱ with other basic elements can be considered, for instance the model $\prod_{\varrho\in\mathfrak{T}}(\mathrm{F}_\varrho|\mathrm{E}_\varrho)$ can represent the «permanent set» as described in [5], p. 491.

4. Models of the type $\prod_{\varrho\in\mathfrak{T}}(E_\varrho - V_\varrho)$.

We shall study the evolution of a spatially distributed system, under the assumption of *infinitesimal deformations*. The *dynamical equations* are

$$
(4.1)\qquad \frac{\partial^2 u_i}{\partial t^2} = \sum_{l=1}^{3} \frac{\partial \sigma_{il}}{\partial x_l} + h_i \quad \text{in } Q \qquad (i = 1, 2, 3)\,,
$$

where u denotes the displacement, h the external load and $Q := \Omega \times \times]0, T[$, with Ω bounded domain of $\mathbf{R}^3$, $T > 0$; the density is assumed to be normalized.

For the sake of simplicity, we shall consider homogeneous boundary conditions

$$
(4.2)\qquad \sum_{i=1}^{3} \sigma_{ij}\cdot \nu_j = 0 \quad \text{on } \partial\Omega \times]0, T[\qquad (i = 1, 2, 3)\,,
$$

where ν denotes the outward normal unitary vector; however our developments can be easily extended to other boundary conditions. We set

$$(4.3)\qquad \begin{cases} \varepsilon_{ij} := \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right), & (A\sigma)_{ij} := -\dfrac{1}{2}\left(\dfrac{\partial^2 \sigma_{il}}{\partial x_l \partial x_j} + \dfrac{\partial^2 \sigma_{jl}}{\partial x_l \partial x_i}\right) \\ g_{ij} := \dfrac{1}{2}\left(\dfrac{\partial h_i}{\partial x_j} + \dfrac{\partial h_j}{\partial x_i}\right) & (i, j = 1, 2, 3); \end{cases}$$

deriving (4.1) w.r.t. x we get

$$(4.4)\qquad \frac{\partial^2 \varepsilon}{\partial t^2} + A\sigma = g \quad \text{in } Q\,.$$

We remind that σ is symmetric.

We consider the parallel combination of several (possibly infinite) rheological models, each one consisting of a linear elastic element E_ϱ in series with a viscous element V_ϱ. The assembled model corresponds to the formula $\prod_{\varrho\in\mathcal{I}} (E_\varrho - V_\varrho)$, where the measure space $(\mathcal{I}, \mathcal{A}, \mu)$ is as in section 1. We have the following constitute relations:

$$\varepsilon = \lambda_\varrho \colon \sigma \quad \text{for } E_\varrho\,,$$

with λ_ϱ positive definite, symmetric 3^4-tensor in general,

$$\dot\varepsilon \in \partial\Phi_\varrho(\sigma) \quad \text{for } V_\varrho\,,$$

with Φ_ϱ proper, lower semi-continuous, convex function from the set of symmetric 3^2-tensors into $\mathbf{R} \cap \{+\infty\}$. Applying the rules ($a$), ($b$) of section 1 for combinations in series and in parallel, we get

$$\dot\varepsilon \in \lambda_\varrho \colon \dot\sigma + \partial\Phi_\varrho(\sigma) \quad \text{for } E_\varrho - V_\varrho$$

and finally for $\prod_{\varrho\in\mathcal{I}} (E_\varrho - V_\varrho)$

$$(4.5)\qquad \begin{cases} \dot{\tilde\varepsilon} \in \lambda_\varrho \colon \dot\sigma_\varrho + \partial\Phi_\varrho(\sigma_\varrho) & \mu_\varrho\text{ - a.e. in } \mathcal{I}, \\ \tilde\sigma = \displaystyle\int_{\mathcal{I}} \sigma_\varrho \, d\mu_\varrho\,; \end{cases}$$

here σ_ϱ denotes the stress in $E_\varrho - V_\varrho$, $\tilde{\varepsilon}$ and $\tilde{\sigma}$ the strain and the stress of the composed model. The state of the global model is characterized by $\tilde{\varepsilon}$ and by the *internal variables* $\{\sigma_\varrho\}_{\varrho\in\mathfrak{I}}$.

We couple the constitute relations (4.5) with the dynamical equation (1.15) written for $\tilde{\varepsilon}$ and $\tilde{\sigma}$ and integrated in time; setting $G(x,t) := \int_0^t g(x,\tau)\,d\tau + (\partial\tilde{\varepsilon}/\partial t)(x,0)$, we get

$$\text{(4.6)} \qquad \lambda_\varrho : \frac{\partial \sigma_\varrho}{\partial t} + A\int_0^t \tilde{\sigma}(x,\tau)\,d\tau + \partial\Phi_\varrho(\sigma_\varrho) \ni G \quad \text{in } Q,\ \mu_\varrho\text{ - a.e. in } \mathfrak{I}.$$

We shall use the notations of [3]. Moreover we denote by $L^2(\Omega)^9_s$ the space of (classes of) square integrable functions from Ω into the set $\mathbf{R}^9_s$ of symmetric 3^2-tensors, here identified with the set of symmetric 3^2-matrices; we also set

$$W := \left\{ v \in L^2(\Omega)^9_s \,\middle|\, \sum_{j=1}^3 \frac{\partial}{\partial x_j} v_{ij} \in L^2(\Omega),\ \sum_{j=1}^3 v_{ij}\cdot\nu_j = 0 \quad \text{on} \quad \partial\Omega\,(i=1,2,3) \right\}$$

and

$$a(u,v) := {}_{W'}\langle Au, v\rangle_W = \sum_{j,l=1}^3 \int_\Omega \frac{\partial}{\partial x_j} u_{ij} \cdot \frac{\partial}{\partial x_l} v_{il}\,dx, \qquad \forall u,v\in W.$$

We assume that

$$\text{(4.7)} \qquad \sigma^0 \in L^2(\Omega\times\mathfrak{I})^9_s; \qquad \sigma^0_\varrho \in \mathrm{Dom}(\Phi_\varrho)\ \mu_\varrho\text{ - a.e. in } \mathfrak{I}; \qquad G \in L^2(Q)^9_s;$$

$$\text{(4.8)} \qquad \forall\xi\in\mathbf{R}^9_s, \quad \underline{\lambda}|\xi|^2 \le \lambda_\varrho : \xi : \xi \le \bar{\lambda}|\xi|^2, \quad \mu_\varrho\text{ - a.e. in } \mathfrak{I}$$

$$(\underline{\lambda},\ \bar{\lambda}\text{: positive constants}).$$

We introduce a variational problem:

(P1) Find $\sigma \in H^1(0,T; L^2(\Omega\times\mathfrak{I})^9_s)$ such that

$$\tilde{\sigma} := \int_{\mathfrak{I}} \sigma_\varrho\,d\mu_\varrho \in L^2(0,T;W)$$

and

$$(4.9)\qquad \int_{\Omega\times\mathcal{T}} \lambda_\varrho : \frac{\partial\sigma_\varrho}{\partial t} : (v_\varrho - \sigma_\varrho)\,dx\,d\mu_\varrho + a\left(\int_0^t \tilde{\sigma}(x,\tau)\,d\tau, \tilde{v}-\tilde{\sigma}\right) + \int_{\Omega\times\mathcal{T}} [\Phi_\varrho(v_\varrho) - \Phi_\varrho(\sigma_\varrho)]\,dx\,d\mu_\varrho \geq \int_\Omega G : (\tilde{v}-\tilde{\sigma})\,dx \quad \text{a.e. in }]0,T[,$$

$$\forall v \in L^2(\Omega\times\mathcal{T})^9_s \text{ such that } \tilde{v} := \int_{\mathcal{T}} v_\varrho\,d\mu_\varrho \in W,$$

$$(4.10)\qquad \sigma_\varrho(0) = \sigma^0_\varrho \quad \text{a.e. in } \Omega\times\mathcal{T}.$$

THEOREM 1. Assume that (4.7), (4.8) hold and moreover that

$$(4.11)\qquad G \in W^{1,1}(0,T; L^2(\Omega)^9_s).$$

Then problem (P1) has one and only one solution. Moreover this has the further regularity

$$(4.12)\qquad \sigma \in W^{1,\infty}(0,T; L^2(\Omega\times\mathcal{T})^9_s).$$

PROOF. It is sufficient to apply the theorem III.1 of [1], taking $\mathcal{H} = L^2(\Omega\times\mathcal{T})^9_s$, $\mathcal{V} = \left\{v\in\mathcal{H}: \int_{\mathcal{T}} v_\varrho d\mu_\varrho \in W\right\}$ (Hilbert space endowed with the graph norm) and setting $u(x,\varrho,t) := \int_0^t \sigma_\varrho(x,\tau)\,d\tau$. ■

Now we consider the particular case in which the generic viscous elements V_ϱ's are replaced by *rigid perfectly plastic* elements P_ϱ's; then $\prod_{\varrho\in\mathcal{T}} (E_\varrho - P_\varrho)$ corresponds to a classical model of elasto-plasticity with strain hardening. Here of course theorem 1 still holds; we shall take into account a non-standard quasi-stationary approximation.

First we rewrite (4.4) including the density constant $\eta > 0$:

$$(4.13)\qquad \eta\frac{\partial^2\tilde{\varepsilon}}{\partial t^2} + A\tilde{\sigma} = g \quad \text{in } Q$$

or equivalently

$$(4.14)\qquad \eta\frac{\partial\tilde{\varepsilon}}{\partial t} + A\int_0^t \tilde{\sigma} = G_\eta := \int_0^t g(x,\tau)\,d\tau + \eta\frac{\partial\tilde{\varepsilon}}{\partial t}(x,0) \quad \text{in } Q;$$

then by (4.1) we get

$$(4.15)\qquad \eta\lambda_\varrho : \frac{\partial\sigma_\varrho}{\partial t} + \eta\partial\Phi_\varrho(\sigma_\varrho) + A\int_0^t \tilde{\sigma}(x,\tau)\,d\tau \ni G_\eta \quad \text{in } Q,\ \mu_\varrho\text{ - a.e. in } \mathfrak{T}.$$

Remark that here $\eta\partial\Phi_\varrho(\sigma_\varrho)$ can be replaced by $\partial\Phi_\varrho(\sigma_\varrho)$: the corresponding variational inequality is unchanged. For studying the quasi-stationary approximation we take $\eta \to 0$ (along a sequence); thus we get

$$(4.16)\qquad \partial\Phi_\varrho(\sigma_\varrho) + A\int_0^t \tilde{\sigma}(x,\tau)\,d\tau \ni G_0 := \int_0^t g(x,\tau)\,d\tau \text{ in } Q,\quad \mu_\varrho \text{ a.e. in } \mathfrak{T}.$$

We assume (4.8), $G_0 \in L^2(Q)^9_s$ and we introduce a variational problem:

(P2) Find $\sigma \in L^2(Q\times\mathfrak{T})^9_s$ such that $\tilde{\sigma} := \int_{\mathfrak{T}} \sigma_\varrho\, d\mu_\varrho \in L^2(0, T; W)$ and

$$(4.17)\qquad a\left(\int_0^t \tilde{\sigma}(x,\tau)\,d\tau, \tilde{v} - \tilde{\sigma}\right) + \int_{\Omega\in\mathfrak{T}} [\Phi_\varrho(v_\varrho) - \Phi_\varrho(\sigma_\varrho)]\,dx\,d\mu_\varrho \geqq \int_\Omega G_0 : (\tilde{v} - \tilde{\sigma})\,d,$$

a.e. in $]0, T[$

$$\forall v \in L^2(\Omega\times\mathfrak{T})^9_s \text{ such that } \tilde{v} := \int_{\mathfrak{T}} v_\varrho\, d\mu_\varrho \in W.$$

THEOREM 2. Assume that

$$4.18)\qquad \forall\eta > 0, \quad G_\eta \in W^{1,1}(0, T; L^2(\Omega)^9_s);$$

$$G_\eta \to G_0 \text{ weakly in } L^2(Q\times\mathfrak{T})^9_s,$$

$$(4.19)\qquad \exists K \text{ bounded } \subset \mathbf{R}^9_s\colon \operatorname{Dom}(\Phi_\varrho) \subset K, \quad \mu_\varrho\text{ - a.e. in } \mathfrak{T}.$$

$\forall\eta > 0$ let σ_η denote the solution (existing and unique by theorem 1) of problem $(\mathrm{P1})_\eta$, obtained from (P1) by including the factor η in the first integral in (4.9), replacing G by G_η and taking any choice of a compatible $\sigma^0 \in L^2(\Omega\times\mathfrak{T})^9_s$.

Then there exists at least one solution σ of problem (P2) such that, possibly extracting a subsequence,

$$\sigma_\eta \to \sigma \text{ weakly star in } L^\infty(Q\times\mathfrak{S})^9_s, \tag{4.20}$$

$$\int_0^t \tilde\sigma_\eta(x,\tau)\,d\tau \to \int_0^t \tilde\sigma(x,\tau)\,d\tau \text{ weakly star in } L^\infty(0,T;W). \tag{4.21}$$

Moreover $\tilde\sigma$ is uniquely determined.

PROOF. It is straightforward to deduce a priori estimates which yield (4.20) and (4.21). Moreover note that by (4.20) the first integral in $(4.9)_\eta$ (the modified equation (4.9) of $(\text{P2})_\eta$) vanishes as $\eta \to 0$ and that the other terms in $(4.9)_\eta$ are lower semi-continuous; thus taking $\eta \to 0$ we get (4.17).

Now let σ_1, σ_2 be two solutions of (P2); we easily get

$$\int_\Omega \left|\int_0^t (\tilde\sigma_1 - \tilde\sigma_2)(x,\tau)\,d\tau\right|^2 dx = 0\,, \qquad \text{a.e. in }]0,T[\,,$$

whence $\tilde\sigma_1 = \tilde\sigma_2$ a.e. in Q. ■

Note that a priori σ_ϱ is not uniquely determined μ_ϱ - a.e. in $\mathfrak{S}$.

REMARK. Usually in the quasi-static approximation of dynamical problems (4.16) is replaced by the static equation

$$\nabla\cdot\tilde\sigma = h \quad \text{in } Q\,,$$

which is of course equivalent to

$$A\int_0^t \tilde\sigma(x,\tau)\,d\tau = G_0 \quad \text{in } Q\,.$$

This last corresponds to neglecting the inertial term $\eta(\partial^2 u/\partial t^2)$ in both the elastic and plastic phases; but a priori $\partial^2 u/\partial t^2$ can be large in the plastic zone. On the other hand (4.16) corresponds to neglecting the term $\eta(\partial\varepsilon_\varrho/\partial t)$ just in the elastic phase and not in the plastic one, for each element $E_\varrho - P_\varrho$. ■

5. Models of the type $\sum_{\varrho\in\mathcal{T}}(E_\varrho|V_\varrho)$.

We consider the serial arrangement of several rheological models, each one consisting of an elastic element in parallel with a viscous one. The assembled model corresponds to the formula $\sum_{\varrho\in\mathcal{T}}(\mathrm{E}_\varrho|\mathrm{V}_\varrho)$, where the measure space $(\mathcal{T}, \mathcal{A}, \mu)$ is as in section 1.

Using the rules (a) and (b) of section 1, we get the following constitutive laws

$$(5.1)\qquad \begin{cases} \tilde{\sigma} \in \alpha_\varrho(\varepsilon_\varrho) + \partial\Psi_\varrho(\dot{\varepsilon}_\varrho) & \mu_\varrho\text{ - a.e. in } \mathcal{T} \\ \tilde{\varepsilon} = \int_{\mathcal{T}} \varepsilon_\varrho\, d\mu_\varrho; \end{cases}$$

here Ψ_ϱ is a proper, convex, lower semi-continuous function, α_ϱ a maximal monotone graph; both are defined in $\mathbf{R}^9_s$. ε_ϱ denotes the strain of $\mathrm{E}_\varrho|\mathrm{V}_\varrho$ μ_ϱ - a.e. in $\mathcal{T}$, $\tilde{\varepsilon}$ and $\tilde{\sigma}$ the strain and the stress of the composed model; the state of this last is characterized by $\tilde{\sigma}$ and by the *internal variables* $\{\varepsilon_\varrho\}_{\varrho\in\mathcal{T}}$.

Set $N := \left\{v \in W \,\middle|\, \sum_{j=1}^{3} (\partial/\partial x_j)\, v_{ij} = 0 (i = 1, 2, 3)\right\}$ and let $\hat{W}$ denote the quotient Hilbert space W/N. A induces a bijective operator $\hat{W} \to \hat{W}'$; we denote its inverse by B and the associated bilinear form by b. Note that by (1.14) $\tilde{\varepsilon} \in N^\perp$, i.e. $\int_\Omega \tilde{\varepsilon}: v\,dx = 0\ \forall v \in N$. We set $Z := \left\{v \in L^2(\Omega\times P)^9_s | \tilde{v} := \int_{\mathcal{T}} v_\varrho\, d\mu_\varrho \in N^\perp\right\}$.

Recalling the assumption of infinitesimal deformations, we couple the constitutive law (5.1) with the dynamical equation (4.4);

$$(5.2)\qquad B\frac{\partial^2 \tilde{\varepsilon}}{\partial t^2} + \partial\Psi_\varrho\left(\frac{\partial \varepsilon_\varrho}{\partial t}\right) + \alpha_\varrho(\varepsilon_\varrho) + N \ni Bg \qquad \text{a.e. in } Q\times\mathcal{T}. \quad (*)$$

This system of possibly infinite inequalities has a hyperbolic character, as B is the inverse of an elliptic second order operator; it is non-linear and its analysis seems complicate enough, therefore we shall deal just with a couple of simpler particular cases.

(*) (5.2) is obtained by applying the integration operator B to (4.4) and using (5.1); instead one could equivalent couple (5.1) with (4.1).

1st case - We assume that the elastic elements are linear, i.e. E_ϱ is characterized by

$$\sigma = \lambda_\varrho : \varepsilon , \qquad \mu_\varrho \text{ - a.e. in } \mathfrak{I} ,$$

with λ_ϱ positive definite, symmetric 3--tensor.

We assume that (4.8) holds and that

$$\text{(5.3)} \qquad \varepsilon^0 \in Z ; \qquad \tilde{\varepsilon}^1 \in W' ; \qquad \varphi : = Bg \in L^2(Q)^9_s .$$

We introduce a variational inequality:

(P3) Find $\varepsilon \in H^1(0, T; Z)$ such that $\tilde{\varepsilon} : = \int_{\mathfrak{I}} \varepsilon_\varrho \, d\mu_\varrho \in H^2(0, T; \hat{W}')$ and

$$\text{(5.4)} \qquad b\left(\frac{\partial^2 \tilde{\varepsilon}}{\partial t^2}, \; \tilde{v} - \frac{\partial \tilde{\varepsilon}}{\partial t}\right) + \int_{\Omega \times \mathfrak{I}} \lambda_\varrho : \varepsilon_\varrho : \left(v_\varrho - \frac{\partial \varepsilon_\varrho}{\partial t}\right) dx \, d\mu_\varrho +$$

$$+ \int_{\Omega \times \mathfrak{I}} \left[\Psi_\varrho(v_\varrho) - \Psi_\varrho\left(\frac{\partial \varepsilon_\varrho}{\partial t}\right)\right] dx \, d\mu_\varrho \geq \int_\Omega \varphi : \left(\tilde{v} - \frac{\partial \tilde{\varepsilon}}{\partial t}\right) dx, \qquad \text{a.e. in }]0, T[,$$

$$\forall v \in Z \left(\tilde{v} : = \int_{\mathfrak{I}} v_\varrho \, d\mu_\varrho \qquad \text{a.e. in } \Omega \right),$$

$$\text{(5.5)} \qquad \left.\frac{\partial \tilde{\varepsilon}}{\partial t}\right|_{t=0} = \tilde{\varepsilon}^1 \quad \text{in } W'$$

$$\text{(5.6)} \qquad \varepsilon_\varrho(x, 0) = \varepsilon^0_\varrho(x) \quad \text{a.e. in } \Omega \times \mathfrak{I} .$$

THEOREM 3. Assume that (4.8) and (5.3) hold. Then problem (P3) has one and only one solution.

PROOF. We introduce a family of regularized problems, corresponding to replacing (5.2) by

$$\eta B \frac{\partial^2 \varepsilon^\eta_\varrho}{\partial t^2} + B \frac{\partial^2 \tilde{\varepsilon}^\eta}{\partial t^2} + \partial \Psi_\varrho\left(\frac{\partial \varepsilon^\eta_\varrho}{\partial t}\right) + \lambda_\varrho : \varepsilon^\eta_\varrho + N \ni Bg \qquad \text{a.e. in } Q \times \mathfrak{I}$$

with η: constant > 0. We still apply theorem III.1 of [1], now with the choice $\mathcal{H} = L^2(\mathfrak{I}; \hat{W}')$ and $\mathcal{V} = L^2(\Omega \times \mathfrak{I})$; thus for any η we get the existence and uniqueness of the solution ε^η and moreover

$$\|\varepsilon^\eta\|_{H^1(0,T;Z)} \leq C \text{ (constant independent of } \eta) ,$$

$$\|\tilde{\varepsilon}^\eta\|_{W^{1,\infty}(0,T;\hat{W}')} \leq C .$$

Therefore ε^η (or a suitable subsequence) converges to some ε; using semi-continuity properties we get (5.4).

It is straightforward to check that the solution of (P3) is unique. ■

2nd case - The viscous elements are linear, i.e. μ_ϱ - a.e. in $\mathfrak{T}V_\varrho$ is characterized by

$$\sigma = \lambda_\varrho : \dot{\varepsilon} ,$$

with λ_ϱ positive definite, symmetric 3^4-tensor. We assume (4.8), (5.3), and that

(5.7) $\forall v \in \mathbb{R}^9_s, \ |\alpha_\varrho(v)| \leq L|v| + M , \quad \mu_\varrho$ - a.e. in $\mathfrak{T}(L, M \in \mathbb{R}^+)$.

We introduce a variational problem

(P4) Find $\varepsilon \in H^1(0, T; Z)$ such that, setting

$$\tilde{\varepsilon} := \int_{\mathfrak{T}} \varepsilon_\varrho \, d\mu_\varrho \quad \text{a.e. in } Q ,$$

(5.8) $$B \frac{\partial^2 \tilde{\varepsilon}}{\partial t^2} + \lambda_\varrho : \frac{\partial \varepsilon_\varrho}{\partial t} + \alpha_\varrho(\varepsilon_\varrho) + N \in \varphi \quad \text{a.e. in } Q \times \mathfrak{T}$$

(5.9) $$\left.\frac{\partial \tilde{\varepsilon}}{\partial t}\right|_{t=0} = \tilde{\varepsilon}^1 \quad \text{in } W'$$

(5.10) $$\varepsilon_\varrho(x, 0) = \varepsilon^0_\varrho(x) \quad \text{a.e. in } \Omega \times \mathfrak{T} .$$

Remark. (5.8) yields $B(\partial^2 \tilde{\varepsilon}/\partial t^2) \in L^2(Q)^9_s$; this gives a meaning to (5.9). ■

THEOREM 4. Assume that (4.8), (5.3) hold and that moreover

$$(5.11)\qquad \begin{cases} \exists \Lambda \in \mathbb{R}^+ : \forall \varepsilon^1, \varepsilon^2 \in L^2(\mathcal{T})^9_s , \\ \|\alpha_\varrho(\varepsilon^1_\varrho) - \alpha_\varrho(\varepsilon^2_\varrho)\|_{L^2(\mathcal{T})^9_s} \le \Lambda \|\varepsilon^1_\varrho - \varepsilon^2_\varrho\|_{L^2(\mathcal{T})^9_s} . \end{cases}$$

Then problem (P4) has one and only one solution.

PROOF. The result is a particular case of theorem 5 given below. ■

Generalization of the 2nd case. We consider linear viscous elements and we replace the elastic elements E_ϱ's by elements M_ϱ's with *memory*, characterized by constitutive relations of the form

$$(5.12)\qquad \sigma(t) = [\mathcal{G}_\varrho(\varepsilon(\cdot), \sigma^0)](t) \qquad \mu_\varrho\text{ - a.e. in } \mathcal{T} ,$$

where the argument $\varepsilon(\cdot)$ denotes the function $t \mapsto \varepsilon(t)$ (not just the tensor $\varepsilon(t)$), $\sigma^0 \in \mathbb{R}^9_s$ and $\mathcal{G}_\varrho$ is a Volterra (i.e. causal) functional. Henceforth we shall write just ε, instead of $\varepsilon(\cdot)$. Examples of such functionals were provided in section 2.

Analogously to (5.1), $\sum_{\varrho\in\mathcal{T}} (\mathrm{M}_\varrho | V_\varrho)$ corresponds to the constitutive relations

$$(5.13)\qquad \begin{cases} \tilde\sigma = \lambda_\varrho : \dot\varepsilon_\varrho + \mathcal{G}_\varrho(\varepsilon_\varrho, \tilde\sigma^0) \qquad \mu_\varrho\text{ - a.e. in } \mathcal{T}, \\ \tilde\varepsilon = \int_{\mathcal{T}} \varepsilon_\varrho \, d\mu_\varrho . \end{cases}$$

We detail the assumptions on the $\mathcal{G}_\varrho$'s

$$(5.14)\qquad \begin{cases} \forall \varepsilon \in C^0([0,T])^9_s, \quad \forall \sigma^0 \in \mathbb{R}^9_s, \quad \forall t \in [0,T], \\ |[\mathcal{G}_\varrho(\varepsilon, \sigma^0)](t)| \le L|\varepsilon(t)| + M, \quad \mu_\varrho\text{ - a.e. in } \mathcal{T}(L, M \in \mathbb{R}^+) \end{cases}$$

$$(5.15)\qquad \begin{cases} \forall \varepsilon \in C^0([0,T])^9_s, \quad \forall \sigma^0 \in \mathbb{R}^9_s, \\ \mathcal{G}_\varrho(\varepsilon, \sigma^0) \in C^0([0,T])^9_s \quad \text{and } [\mathcal{G}_\varrho(\varepsilon, \sigma^0)](0) = \sigma^0, \end{cases}$$

$$(5.16)\qquad \begin{cases} \forall t \in]0,T], \ \forall \varepsilon_1, \varepsilon_2 \in C^0([0,T])^9_s, \ \forall \sigma^0 \in \mathbb{R}^9_s, \text{ if } \varepsilon_1 = \varepsilon_2 \text{ in } [0,t] \\ \text{then } [\mathcal{G}_\varrho(\varepsilon_1, \sigma^0)](t) = [\mathcal{G}_\varrho(\varepsilon_2, \sigma^0)](t) \text{ (Causality).} \end{cases}$$

$$(5.17)\qquad \begin{cases} \forall \varepsilon \in C^0([0,T])^9_s, \ \forall \sigma^0 \in \mathbb{R}^9_s, \ \forall \bar t, t \in [0,T] \ (\bar t < t), \\ \qquad\qquad \text{setting } \sigma^{\bar t} := [\mathcal{G}_\varrho(\varepsilon, \sigma^0)](\bar t), \\ [\mathcal{G}_\varrho(\varepsilon, \sigma^0)](t) = [\mathcal{G}_\varrho(\varepsilon(\bar t + \cdot), \sigma^{\bar t})](t - \bar t) \text{ (Semigroup property).} \end{cases}$$

We also assume that (5.3) holds and that

$$\tilde{\sigma}^0 \in L^2(Q)^9_s . \tag{5.18}$$

We introduce a further variational problem:

(P5) Find $\varepsilon \in H^1(0, T; Z)$ such that, setting

$$\tilde{\varepsilon} := \int_{\mathfrak{T}} \varepsilon_\varrho \, d\mu_\varrho \quad \text{a.e. in } Q ,$$

$$B \frac{\partial^2 \tilde{\varepsilon}}{\partial t^2} + \lambda_\varrho : \frac{\partial \varepsilon_\varrho}{\partial t} + \mathcal{G}_\varrho(\varepsilon_\varrho, \tilde{\sigma}^0) + N \ni \varphi \quad \text{a.e. in } Q \times \mathfrak{T}, \tag{5.19}$$

$$\left.\frac{\partial \tilde{\varepsilon}}{\partial t}\right|_{t=0} = \tilde{\varepsilon}^1 \quad \text{in } \hat{W}' \tag{5.20}$$

$$\varepsilon_\varrho(x, 0) = \varepsilon^0_\varrho(x) \quad \text{a.e. in } \Omega \times \mathfrak{T} . \tag{5.21}$$

See the remark following (P4).

THEOREM 5. Assume that (4.8), (5.3), (5.14), ..., (5.18) hold and moreover that

$$\begin{cases} \text{there exists } \eta : \mathbf{R}^+ \to \mathbf{R}^+ \text{ such that } \eta(\xi) \to 0 \text{ as } \xi \to 0 \text{ and} \\ \forall t', t'' \in [0, T]\ (t' < t''),\ \forall v^1, v^2 \in H^1(0, T; L^2(\mathfrak{T})^9_s),\ \forall \sigma^0 \in \mathbf{R}^9_s \\ \|\mathcal{G}_\varrho(v^1_\varrho, \sigma^0) - \mathcal{G}_\varrho(v^2_\varrho, \sigma^0)\|_{L^2(t',t'';L^2(\mathfrak{T})^9_s)} \leq \\ \qquad \leq \eta(t'' - t') \cdot \|v^1_\varrho - v^2_\varrho\|_{H^1(t',t'';L^2(\mathfrak{T})^9_s)} . \end{cases} \tag{5.22}$$

Then problem (P5) has one and only one solution.

PROOF. Set $X := H^1(0, T, Z)$. For any $v \in X$, let $w \in X$ be the solution (existing and unique) of the linear problem

$$B \frac{\partial^2 \tilde{w}}{\partial t^2} + \lambda_\varrho : \frac{\partial w_\varrho}{\partial t} + \mathcal{G}_\varrho(v_\varrho, \tilde{\sigma}^0) + N \ni \varphi \quad \text{a.e. in } Q \times \mathfrak{T} \tag{5.23}$$

$$\left.\frac{\partial \tilde{w}}{\partial t}\right|_{t=0} = \tilde{\varepsilon}^1 \quad \text{in } W', \tag{5.24}$$

$$w_\varrho(x, 0) = \varepsilon^0_\varrho(x) \quad \text{a.e in } \Omega \times \mathfrak{T} ; \tag{5.25}$$

here $\tilde{w} := \int_{\mathfrak{T}} w_\varrho \, d\mu_\varrho$. Set $w = \Lambda(v)$. We shall show that $\Lambda: X \to X$ is a contraction for T small enough. Let $v^1, v^2 \in X$; set $w^i := \Lambda(v^i)$ $(i = 1, 2)$, $\bar{v} := v^1 - v^2$, $\bar{w} := w^1 - w^2$, $\bar{\tilde{w}} := \tilde{w}_1 - \tilde{w}_2$. Taking the difference between (5.23) written for $i = 1, 2$, multiplying by $\partial\bar{w}\text{-}/\partial t$ and integrating w.r.t. x, t, ϱ we get

$$\frac{1}{2}\left\| \frac{\partial \bar{\tilde{w}}}{\partial t}(T) \right\|^2_{H^{-1}(\Omega)^9_s} + \left\| \lambda_\varrho^{\frac{1}{2}} : \frac{\partial \bar{w}_\varrho}{\partial t} \right\|^2_{L^2(Q\times\mathfrak{T})^9_s} \leq$$

$$\leq \left| \int_{Q\times\mathfrak{T}} \{[\mathcal{G}_\varrho(v^1_\varrho, \tilde{\sigma}^0)](t) - [\mathcal{G}_\varrho(v^2_\varrho, \tilde{\sigma}^0)](t)\} : \frac{\partial \bar{w}_\varrho}{\partial t}(t)\, dx\, dt\, d\mu_\varrho \right| \leq$$

$$\leq \|\mathcal{G}_\varrho(v^1_\varrho, \tilde{\sigma}^0) - \mathcal{G}_\varrho(v^2_\varrho, \tilde{\sigma}^0)\|_{L^2(Q\times\mathfrak{T})^9_s} \cdot \left\| \frac{\partial \bar{w}_\varrho}{\partial t} \right\|_{L^2(Q\times\mathfrak{T})^9_s} \leq (\text{by } (3.22)) \leq$$

$$\leq \eta(T)\, \|\bar{v}\|_X \cdot \|\bar{w}\|_X \leq \eta(T)\, \underline{\lambda}^{-1} \|\lambda^{\frac{1}{2}} : \bar{v}\|_X \cdot \|\lambda^{\frac{1}{2}} : \bar{w}\|_X .$$

If T is such that $\eta(T)\,\underline{\lambda}^{-1} < 1$, then Λ is a contraction in X and therefore it has one and only one fixed point, which solves (P5) in $[0, T]$. Repeating this procedure step by step in time we get the existence and uniqueness of the solution of (P5) for any $T > 0$. ■

REMARK. The results of this section can be extended to the quasi-static case, corresponding to cancelling the term $B(\partial^2\tilde{\varepsilon}/\partial t^2)$ in (5.2); consequently in (P3) $\tilde{\varepsilon}$ loses the regularity $H^2(0, T; \hat{W}')$ and the initial condition (3.5) has to be removed; similar modifications have to be introduced in (P4) and (P5). ■

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