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On the Homology Groups of q-Complete Spaces.

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Sunto - Sia X uno spazio complesso q-completo n-dimensionale; allora $H_k(X, \mathbb{Z}) = 0$ per ogni $k > n + q$. Sia poi (X, Y) una q-coppia di Runge di spazi q-completi e Y privo di singolarità; allora $H_k(X \bmod Y, \mathbb{Z}) = 0$ per ogni $k > n + q$.

It is known (Sorani [8]) that if X is a q-complete manifold then $H_k(X, \mathbb{Z}) = 0$ for $k > n + q$ and $H_{n+q}(X, \mathbb{Z})$ is a free group. The proof of this theorem comes from ideas of Serre, Thom and Andreotti-Frankel; but it does seem to be easily generalizable to the singular case. In this paper we prove that if X is a q-complete n-dimensional complex space then $H_k(X, \mathbb{Z}) = 0$ if $k > n + q$. We don't know if $H_{n+q}(X, \mathbb{Z})$ is torsion free or free. We use a lemma (furuncle-lemma) of Andreotti-Grauert and a theorem of Coen which extends the results of Sorani to the case of an open subset of a Stein space. Moreover we apply our theorem to obtain a vanishing theorem for the relative homology of q-Runge pairs.

§ 0. We consider throughout this paper analytic complex spaces countable at the infinity. A complex space X is said to be *q-complete* when there exists a C^∞-function $h: X \to \mathbb{R}$ such that $X(c) = \{x \in X | h(x) < c\}$ is relatively compact in X for every $c \in \mathbb{R}$, and every $x \in X$ has a neighborhood V with the following property: there exist an

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isomorphism χ of V onto an analytic subset A of an open subset U of $\mathbb{C}^n$ and a C^∞-function $\varphi: U \to \mathbb{R}$ such that $h = \varphi \circ \chi$ and the Levi form

$$\mathcal{L}(\varphi, y)(u) = \sum_{i,j=1} \left(\frac{\partial^2 \varphi}{\partial z_i \, \partial \bar{z}_j} \right) u_i \bar{u}_j$$

has at least $n - q$ positive eigenvalues at every point $y \in U$; the function h is said to be *strongly q-plurisubharmonic.*

If X is a complex space and Y an open subset of X, the pair (X, Y) is said to be a *q-Runge pair* if the natural homomorphism

$$\varrho_Y^X: H^q(X, \Omega_X^p) \to H^q(Y, \Omega_Y^p)$$

has dense image for every $p = 0, 1, \ldots, n$, where Ω_X^p is the sheaf of holomorphic p-forms (see for instance [5]).

We recall the following theorem that we will use in the proof of our result:

THEOREM 0.1 (Coen, [4]). *Let X be a q-complete open subspace of a Stein space S; let* $\dim X = n$. *Then*

$$H_k(X, \mathbb{Z}) = 0 \quad \textit{if } k > n + q \textit{ and}$$

$$H_{n+q}(X, \mathbb{Z}) \quad \textit{is torsion free.}$$

A similar theorem was known for manifolds:

THEOREM 0.2 (Sorani, [8]). *Let X be a q-complete manifold, and let* $\dim X = n$. *Then*

$$H_k(X, \mathbb{Z}) \quad \textit{if } k > n + q \textit{ and}$$

$$H_{n+q}(X, \mathbb{Z}) \quad \textit{is free.}$$

By means of the results of Ferrari ([5] and [6]) and Le Potier [7] we know something else about these groups:

THEOREM 0.3. *Let X be a q-complete complex space, and let* $n = \dim X$. *Then $H_k(X, \mathbb{C}) = 0$ and $H_k(X, \mathbb{Z})$ is a torsion group for each $k > n + q$.*

§ 1. In order to prove the theorem we need the following

LEMMA 1.1 (Benedetti, [2]). *Let X be a reduced q-complete complex space. Then the function h defining the q-completeness of X can be chosen such that the set* $\{$local minima of h in $X\}$ *is discrete in X.*

The proof of our theorem requires, besides this result, the Mayer-Vietoris sequence and the furuncle-lemma ([1], p. 237).

THEOREM 1.2. *Let X be a q-complete complex space, and let* $\dim X = n$. *Then $H_k(X, \mathbb{Z}) = 0$ if $k > n + q$.*

PROOF. Without loss of generality we can suppose X reduced. Let h be a non-negative function chosen as in 1.1. For every $t \in \mathbb{R}$ we put $X(t) = \{x \in X | h(x) < t\}$ and $B(t) = \{x \in X | h(x) = t\}$. Every open set $X(t)$ is a q-complete space. Let $t_0 = \min_X h(x)$; it follows that $B(t_0)$ is finite and then, thanks to the property of h, it is possible to find $d \in \mathbb{R}$, $d > t_0$, such that $X(d)$ is contained in an open Stein set. Therefore (theorem 0.1.) $H_k(X(t), \mathbb{Z}) = 0$ if $k > n + q$ and $t < d$. Then let us consider the set

$$A = \{t \in \mathbb{R} | \forall r < t \text{ and } \forall k > n + q \;\; H_k(X(r), \mathbb{Z}) = 0\} \neq \emptyset .$$

We will see that $A = [t_0, +\infty[$ by means of the furuncle-lemma. Let $t \in A$; we claim that there exists $\varepsilon > 0$ such that $t + \varepsilon \in A$.

We cover $\partial X(t)$ with a finite family $\{U_i\}_{1 \leqslant i \leqslant p}$ of open relatively compact Stein sets for which there exist closed embeddings $\psi_i \colon U_i \to V_i$, with V_i open subset of $\mathbb{C}^{n_i}$, and non-negative strongly q-plurisubharmonic functions $h_i \colon V_i \to \mathbb{R}$ such that $h_i \circ \psi_i = h$. Then we consider a family $\{W_i\}$ of open sets covering $\partial X(t)$ and such that $W_i \subset\subset U_i$ for every i, and a family $\{\varrho_i\}$ of C^∞-functions, non-negative, such that ϱ_i has compact support in U_i and $\varrho_i(x) > 0$ for every $x \in W_i$.

It is possible to choose p constants $c_i > 0$, $1 \leqslant i \leqslant p$, such that the functions $f_i = h - \sum_{k=1}^{i} c_k \varrho_k$ are strongly q-plurisubharmonic ones and the sets $C_i = \{x \in X | f_i(x) < t\}$ q-complete.

Since $B(t) \setminus \partial X(t)$ is a finite set, by lessening if necessary the constants c_k we can suppose that no point $x \in B(t) \setminus \partial X(t)$ is in C_p; then there exist an open Stein set $V \subset X$ and an $\varepsilon > 0$ such that $V \cap C_p = \emptyset$ and $X(t + \varepsilon) \subset C_p \cup V$. Moreover, from the construction we see that, if we put $C_0 = X(t)$, $C_i \setminus C_{i-1} \subset\subset U_i$ for $1 \leqslant i \leqslant p$.

Let now $t' \leqslant t + \varepsilon$. For every $i = 0, 1, \ldots, p$ $C_i \cap X(t')$ is q-complete too. Indeed, f_i is constructed from h through small perturbations, and therefore the Levi forms of h and of f_i in a point x are positive definite on the same q-codimensional subspace. Then the following function determines the q-completeness of $X(r') \cap C_i$.

$$g(x) = \frac{1}{t - f_i(x)} + \frac{1}{t' - h(x)} \, .$$

Now, put $Y_i = X(t') \cap C_i$; in particular Y_0 is $X(t)$. We show by induction that $H_k(Y_i, \mathbb{Z}) = 0$ for $k > n + q$ for every i. It is true (by assumption) for $i = 0$. Let now $i \geqslant 1$ and let us consider the Mayer-Vietoris sequence of the pair $(Y_{i-1}, Y_i \cap U_i)$:

$$H_k(Y_{i-1} \cap U_i, \mathbb{Z}) \to H_k(Y_{i-1}, \mathbb{Z}) \oplus H_k(Y_i \cap U_i, \mathbb{Z}) \to$$

$$\to H_k(Y_i, \mathbb{Z}) \to H_{k-1}(Y_{i-1} \cap U, \mathbb{Z})$$

Y_{i-1} and Y_i are q-complete and therefore $Y_{i-1} \cap U_i$ and $Y_i \cap U_i$ are q-complete open subsets of the Stein space U_i. Applying 0.1. and the induction we find $H_k(Y_i, \mathbb{Z}) = 0$ if $k > n + q + 1$ and

$$0 \to H_{n+q+1}(Y_i, \mathbb{Z}) \to H_{n+q}(Y_{i-1} \cap U_i, \mathbb{Z}) \qquad \text{if } k = n + q + 1.$$

Thanks to 0.3 $H_{n+q+1}(Y_i, \mathbb{Z})$ is a torsion group; on the other hand $H_{n+q}(Y_{i-1} \cap U_i, \mathbb{Z})$ is torsion free; therefore $H_{n+q+1}(Y_i, \mathbb{Z}) = 0$. Then in particular $H_k(X(t') \cap C_p, \mathbb{Z}) = 0$ if $k > n + q$; since finally $X(t') = (X(t') \cap C_p) \cup (X(t') \cap V)$, and this union is disjoint, also $H_k(X(t'), \mathbb{Z}) = 0$ for each $k > n + q$.

Therefore A is open. If we suppose $s = \sup A < +\infty$, we can find a sequence of points of A $t_n \to s$. But then

$$H_k(X(s), \mathbb{Z}) = \lim H_k(X(t_n), \mathbb{Z}) = 0$$

and this is a contradiction, since $s \notin A$. Then $\sup A = +\infty$. In particular $m \in A$ for every $m \in \mathbb{N}$, and then

$$H_k(X, \mathbb{Z}) = \lim H_k(X(m), \mathbb{Z}) = 0 \qquad \text{for each } k > n + q.$$

REMARK. This theorem allows us to remove the assumption of a Stein environment in several results; for instance, in the corollaries 2.1 and 2.4 of [4].

§ 2. We recall the following proposition:

PROPOSITION 2.1 (Le Potier [7]). *Let X be a complex space, and let $n = \dim X$. Then there exists a canonical homomorphism*

$$\theta^{n,q}: H^q(X, \Omega^n_X) \to H^{n+q}(X, \mathbb{C})\,;$$

moreover, it is surjective if X is q-complete.

If X is a complex manifold $H^{n+q}(X, \mathbb{C})$ has a natural topology, thanks to De Rham's theorem; moreover we have the following

LEMMA 2.2 (see Le Potier [7], Remarque III, 6). *Let X be a complex manifold. Then $\theta^{n,q}$ is continuous with respect to the natural topologies.*

PROOF. We can factorize the map $\theta^{n,q}$, with $q > 0$ (the case $q = 0$ is similar), in the following way:

$$H^q(X, \Omega^n_X) \xrightarrow{g} \frac{\operatorname{Ker}\big(\Gamma(X, \mathcal{A}^{n,q}) \to \Gamma(X, \mathcal{A}^{n,q+1})\big)}{\operatorname{Im}\big(\Gamma(X, \mathcal{A}^{n,q-1}) \to \Gamma(X, \mathcal{A}^{n,q})\big)} \xrightarrow{h}$$

$$\xrightarrow{h} \frac{\operatorname{Ker}\big(\Gamma(X, \mathcal{E}^{n+q}) \to \Gamma(X, \mathcal{E}^{n+q+1})\big)}{\operatorname{Im}\big(\Gamma(X, \mathcal{E}^{n+q-1}) \to \Gamma(X, \mathcal{E}^{n+q})\big)} \xrightarrow{k} H^{n+q}(X, \mathbb{C})$$

where $\mathcal{A}^{n,q}$ is the sheaf of C^∞-differential forms of type (n, q) and $\mathcal{E}^k$ is the sheaf of C^∞-differential forms of type k. The map g is continuous (with respect to the Fréchet topologies on the modules of sections), since $\mathcal{A}^{n,q}$ is a fine resolution of Fréchet sheaves of Ω^n (by means of the results of [3]); h is continuous since it comes from the natural inclusion of (n, q)-forms into $(n + q)$-forms; k is continuous by definition.

THEOREM 2.3. *Let (X, Y) be a q-Runge pair of q-complete spaces; let Y be free of singularities. Then $H_k(X \bmod Y, \mathbb{Z}) = 0$ for each $k > n + q$.*

PROOF. If $k > n + q + 1$ the theorem follows from the relative homology sequence of the pair (X, Y) and from theorem 1.2.

Let now $k = n + q + 1$. We begin proving that $H_{n+q+1}(X \bmod Y, \mathbf{C}) = 0$. In the following commutative diagram

$$\begin{array}{ccc} H^q(X, \Omega^n_Y) & \xrightarrow{\theta^{n,q}_X} & H^{n+q}(X, \mathbf{C}) \\ \downarrow{\scriptstyle \varrho^X_Y} & & \downarrow{\scriptstyle \nu^X_Y} \\ H^q(Y, \Omega^n_X) & \xrightarrow{\theta^{n,q}_Y} & H^{n+q}(Y, \mathbf{C}) \end{array}$$

$\Theta^{n,q}_Y$ is continuous and surjective (applying lemmas 2.1 and 2.2) and ϱ^X_Y has dense image by hypotheis; thus ν^X_Y has dense image too. Moreover, the natural algebraic pairing $\langle H^{n+q}(X, \mathbf{C}), H_{n+q}(X, \mathbf{C})\rangle$ is also topological (see Sorani [9]); then the natural homomorphism

$$H_{n+q}(Y, \mathbf{C}) \xrightarrow{j} H_{n+q}(X. \mathbf{C})$$

is injective. Thus considering the exact sequence

$$0 \to H_{n+q+1}(X \bmod Y, \mathbf{C}) \to H_{n+q}(Y, \mathbf{C}) \xrightarrow{j} H_{n+q}(X, \mathbf{C})$$

we find $H_{n+q+1}(X \bmod Y, \mathbf{C}) = 0$; then $H_{n+q+1}(X \bmod Y, \mathbb{Z})$ is a torsion group. But in the natural relative exact sequence

$$H_{n+q+1}(X, \mathbb{Z}) \to H_{n+q+1}(X \bmod Y, \mathbb{Z}) \xrightarrow{f} H_{n+q}(Y, \mathbb{Z})$$

f is injective, applying theorem 1.2; moreover $H_{n+q}(Y, \mathbb{Z})$ is a torsion free group (proposition 0.2). Therefore $H_{n+q+1}(X \bmod Y, \mathbb{Z}) = 0$.

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