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Axiomatic Foundations of the Kinematics Common to Classical Physics and Special Relativity.

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SOMMARIO - Si costruisce una teoria comune alla cinematica classica e a quella relativistica, usando quattro nozioni primitive, riducibili alla relazione di precedenza $\prec$ tra punti eventi, e quella $\mathcal{O}$ di essere un punto evento $\mathcal{E}$ occupato da un punto materiale M. Si definiscono fisicamente, mediante condizioni di necessità e possibilità fisica, i riferimenti inerziali; e si caratterizzano le trasformazioni matematiche tra questi. Si definiscono gli spazi inerziali, che poi risultano essere ∞^3. Risulta pure che vale o il caso classico o quello relativistico. Si caratterizzano questi casi in vari modi, ciascuni dei quali può essere usato come postulato addizionale per arrivare ad una teoria classica $\mathcal{T}_c$ o relativistica $\mathcal{T}_r$.

Una differenza sussistente tra concetti fondamentali corrispondenti di due teorie incompatibili $\mathcal{T}_1$ e $\mathcal{T}_2$, usualmente trattati come primitivi, viene ridotta, nel caso presente, a una differenza tra nozioni definite nelle teorie $\mathcal{T}_c$ e $\mathcal{T}_r$ (sostituite a $\mathcal{T}_1$ e $\mathcal{T}_2$); e questa differenza è ora dovuta unicamente a differenze tra postulati, in quanto $\mathcal{T}_c$ e $\mathcal{T}_r$ hanno le stesse nozioni primitive. Tale riduzione ha interesse, per es., in quanto in certi casi alcuni autori ritengono problematico un confronto tra $\mathcal{T}_1$ e $\mathcal{T}_2$.

1. Introduction (**).

We present a system of purely kinematic axioms complying with both classical physics and special relativity; and we derive the main

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consequences of it, in particular that either the classical case or the relativistic one holds [Theor. 4.2].

The spatial metric or the space-time one is not assumed as a primitive notion. These metrics can be defined on the basis of some properties for the motions possible for certain isolated particle systems. Incidentally the present treatment is based on a modal logic—see e.g. [1], [4], and [2] or [3].

More in detail, we use only four primitive notions: *event point*, *particle*, the relation of *being in the past of ...*, and the one of *being* (*an event point*) *occupied by* (*a particle*). We define inertial (affine) frames according to the criteria used, e.g., in [1] to define mass within a theory of classical particle mechanics of the Mach-Painlevé type; more precisely this occurs in that this definition is based on a condition of (causal) implication and one of (causal) possibility—see (*a*) and (*b*) in Def. 2.3.

On the basis of suitable postulates, Posts 2.1-5, we define inertial spaces and inertial instants [N. 5]. Furthermore we characterize the classical and relativistic cases in various ways [NN. 4, 6, 8]. Any among these characterizations can be used as an additional axiom to turn our theory into an exclusively classical one, $\mathcal{T}_c$, or into an exclusively relativistic theory, $\mathcal{T}_r$.

In the relativistic case an (oriented) time metric and a spatial metric can be determined on every inertial affine space, so that it is turned into a Euclidean space (Galilean space), on the basis of our purely kinematic postulates [N. 7]. The analogue for classical physics cannot be done; one has to use, e.g., forces at a distance—as in [1]—or contact forces.

In both the classical case and the relativistic one the inertial spaces can be proved to be ∞^3, [Theor. 8.2], and the inertial (affine) frames are proved to be those related to any of them by certain transformations (generalized Galilei or Lorentz transformations)—see Theor. 8.2.

* * *

It is often remarked that special relativity, SR, is based on kinematic notions different from their correspondents in classical physics, CP. In the present paper *the same primitive notions are used for* $\mathcal{T}_r$ *and* $\mathcal{T}_c$, and within our frame work the difference above concerns some (corresponding) defined notions belonging to theories $\mathcal{T}_r$ and $\mathcal{T}_c$; and *it arises only because of the additional postulates*, which may be the following two for $\mathcal{T}_r$ and $\mathcal{T}_c$ respectively: *the set of possible speeds* *for isolated particles, with respect to an inertial frame* (*or space*), *is bounded* [*unbounded*].

Our reduction of the difference between corresponding basic primitive notions of two previous theories $\mathcal{T}_1$ (e.g. CP) and $\mathcal{T}_2$ (e.g. SR), to a difference in defined notions due only to a difference between the axioms of two new theories $\mathcal{T}_1'$ and $\mathcal{T}_2'$ having the same primitive notions, may be useful to compare $\mathcal{T}_1$ with $\mathcal{T}_2$. The interest of this is supported by the fact that for some choices of $\mathcal{T}_1$ and $\mathcal{T}_2$ (in the above situation), e.g., Feyerabend finds the comparison of $\mathcal{T}_1$ and $\mathcal{T}_2$ difficult, or even impossible—cf. [5] (1).

Let us add that, if $\mathcal{T}_1$ and $\mathcal{T}_2$ have the same extralogical primitive terms, but are logically incompatible, then according to extensional logic the notions designated by (all) primitive or defined terms cannot coincide. However, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are physical theories, their interpretations (given intuitively) are modal. Thus $\mathcal{T}_1$ and $\mathcal{T}_2$ can express the same conceivable phenomena. Their postulates tell us which of them can really take place. Hence the extensions of a term t of $\mathcal{T}_1$ (and $\mathcal{T}_2$), admitted in $\mathcal{T}_i$, depends on $\mathcal{T}_i$'s postulates; and consequently so does t's intension $\tilde{t}_{\mathcal{T}_i}$ in $\mathcal{T}_i$ $(i = 1, 2)$. Let us remark that both $\tilde{t}_{\mathcal{T}_1}$ and $\tilde{t}_{\mathcal{T}_2}$ are subintensions induced by a wider intension $\tilde{t}$ conceivable on the basis of the intuitive characterizations of the primitive notions of $\mathcal{T}_1$ (and $\mathcal{T}_2$). In this sense do the terms of $\mathcal{T}_1$ and $\mathcal{T}_2$ have the same meanings.

The afore-mentioned situation occurs in common practice, e.g. when $\mathcal{T}_1$ and $\mathcal{T}_2$ are (incompatible) theories concerning the constitutive equations of a given real body (on which experiments can be made).

The theories $\mathcal{T}_c$ and $\mathcal{T}_r$, used to characterize the difference between

(1) As it appears from G. Giorello's prefazione in [5] (*b*), p. 2, according to a thesis of Feyerabend (in its most radical version) *the meanings of the extralogical terms of a theory change with the whole context of the theory*; and Feyerabend believes that by this change, the presupposition that two theories $\mathcal{T}_1$ and $\mathcal{T}_2$ (on the same field) have any expression in common, can be denied, so that $\mathcal{T}_1$ and $\mathcal{T}_2$ cannot be compared (on the basis of their empirical content in Paper's sense).

Let us add that, if the primitive notions of a physical theory lack a satisfactory intuitive characterization, then a good help can be afforded by its applications to real cases, made by the author. Furthermore Feyerabend's thesis above practically implies that we don't know (precisely) the notions used in any physical theory (in that we usually never are at the end of the context of any theory).

CP and SR, are substantially in the situation of $\mathcal{T}_1$ and $\mathcal{T}_2$ in example above. In particular they have the same notions in the afore-mentioned sense; and thus they are comparable.

2. Primitive notions and postulates for a theory $\mathcal{T}$ belonging to both classical physics and special relativity.

As primitive notions of our theory $\mathcal{T}$ on space time, made from the points of view of both classical physics and special relativity, we assume: *event point*, EP (identified with the class EP of event points), *mass point*, MP (or particle), the relation $\prec$ of *being an event point* (*strictly*) *in the past of* another one, and the relation $\mathcal{O}$ between $\mathcal{E}$ and M: $\mathcal{E}$ *is an* EP *occupied by the mass point* M (2). The set W_M of these event points is called the world line of M.

POST. 2.1. $M \in \mathrm{MP}$ *implies* $W_M \subset \mathrm{EP}$.

POST. 2.2. *The relation* $\prec$ *is an unbounded partial order in* EP (3).

The predicates (or classes) MP and EP, and the relation $\prec$ are usually regarded as independent of phenomena. This independence has some logical consequences—see Post. 2.3 (*a*) below—that, usually, are not stated explicitly. The same can be said of the fact that the relation $\mathcal{O}$ and the class W_M are usually used extensionally, i.e. as coinciding with their extensionalizations (4):

(2) Among our primitive notions, EP and MP could be defined in terms of the other two:

(α) *by* EP *we call the field of the binary relation* $\prec$, i.e. the set $\{x$: *for some* y *either* $x \prec y$ *or* $y \prec x\}$;

(β) *by* MP *we call the set of the possible first members of binary relation* $\mathcal{O}$—*i.e.* $x \in \mathrm{MP}$ *iff for some* y, $\mathcal{O}$ *can hold for* x *and* y.

(3) That $\prec$ is an unbounded (strict) partial order in EP means:

(i) $\mathcal{E} \prec \mathcal{E}'$ and $\mathcal{E}' \prec \mathcal{E}''$ imply $\mathcal{E}, \mathcal{E}' \in \mathrm{EP}$, $\mathcal{E}' \not\prec \mathcal{E}$ and $\mathcal{E} \prec \mathcal{E}''$; and

(ii) $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{EP} \Rightarrow (\exists \mathcal{E}, \mathcal{E}' \in \mathrm{EP})(\mathcal{E} \prec \mathcal{E}_r \prec \mathcal{E}'$ for $r = 1, 2)$.

(4) By definition the *extensionalization* $P^{(e)}$ of a property P holds for x iff P holds for some y which happens to equal x. We identify (also non-extensional) properties with classes. As an example, let $x \in P$ hold by definition (of P) iff x necessarily equals 1; then $y \in P^{(e)}$ iff y equals 1 (i.e. happens to equal 1 without excluding the possibility of $y \neq 1$).

$$\mathcal{O} = \mathcal{O}^{(e)}, \qquad W_M = W_M^{(e)}\text{—see [2], p. 67, or [3], ftn. 15, p. 302}.$$

For the sake of completeness we state the facts above explicitly by means of Post. 2.3 below, even if we think that our presentation would meet the usual standard of rigour (in its field) also without Post. 2.3.

POST. 2.3. (*a*) *The notions* MP, EP, *and* $\prec$ *are* (*modally*) *absolute*—see [2], pag. 67, or [3], pag. 292 (5).

(*b*) *The notion* $\mathcal{O}$ *is extensional* (*hence so is* W_M).

Obviously, if $\mathcal{W} \subset$ PE (and $\mathcal{W}$ is a mathematical subset of EP, i.e. also $\mathcal{W} \in$ Abs holds) and for some $M \in$ MP it is physically possible—briefly phys. poss., see [4]—that $\mathcal{W} = W_M$, then we say that $\mathcal{W}$ is a *phys. poss. world line of some mass point*, or briefly $\mathcal{W}$ is a PW; thus the mathematical set of these lines in PE will be denoted by PW.

POST. 2.4. $\mathcal{E} \prec \mathcal{E}'$ *or* $\mathcal{E}' \prec \mathcal{E}$, *iff* $\mathcal{E} \neq \mathcal{E}'$ *and* $\mathcal{E}$ *and* $\mathcal{E}'$ *belong to the same* PW, i.e. $(\exists \mathcal{W} \in \mathrm{PW})\ \mathcal{E}, \mathcal{E}' \in \mathcal{W}$.

By Post. 2.4 we have the following

THEOR. 2.1. *If* $\mathcal{W} \in$ PW, *the restriction* $(\prec \cap\, \mathcal{W}^2)$ *of the relation* $\prec$ *to* $\mathcal{W}$ *is a total* (*or simple*) *order* (6).

We regard the condition $W_M = \phi$, the empty set, as schematizing the case where $M(\in \mathrm{MP})$ is very far from the observer, i.e. M is at infinity. We say that M exists (or is in EP) in case $W_M \neq \phi$. Hence it is reasonable to consider the cases when in EP there is one MP, no MPs, or some but not all MPs.

DEF. 2.1. (*a*) *We say that* $(M_1, \ldots, M_m)$ *is an isolated particle system—or that* M_1 *is an isolated particle in case* $m = 1$—, *if* $W_M \neq \phi$ *only for* $M = M_i(\in \mathrm{MP})$ $(i = 1, \ldots, m)$ (*and if the electromagnetic field vanishes, in case an extension of* $\mathcal{C}$ *is referred to, that includes this field*).

(5) The property (or class) P is (*modally*) *absolute*, briefly $P \in$ Abs, if (i) it is *modally constant* i.e. $x \in P$ must hold as soon as it can hold, and (ii) P is *modally separated*, i.e. the possibily of $x, y \in P$ and $x = y$ implies that x must equal y. The definition of absolute relations is quite similar.

(6) For any set A, we can write $A^1 = A$, $A^{n+1} = A^n \times A$ where « $\times$ » is the cartesian product.

(*b*) *We denote by* PWI *the class of the phys. poss. world lines of isolated particles.*

DEF. 2.2. (*a*) *We say that* φ *is an admissible frame* (*for the space time* EP) *if* φ *is a bijection of* EP *onto* $\mathbb{R}^4$ *for which, by writing* $x^\alpha = \varphi^\alpha(\mathcal{E})$ *when* $(x^0, \ldots, x^3) = \varphi(\mathcal{E})$, *we have* (7)

(i) $\varphi^0(\mathcal{E}) \prec \varphi^0(\mathcal{E}')$ *for* $\mathcal{E} \prec \mathcal{E}'$ *and*

(ii) *every* $\mathcal{W} \in \mathrm{PWI}$ *has a representation of the form* (8)

$$x^r = f^r(x^0) \quad \text{with} \quad f^r \in C^{(2)}\big(x^\alpha = \varphi^\alpha(\mathcal{E})\big)\,, \tag{2.1}$$

i.e. $\mathcal{E} \in \mathcal{W}$ iff $(2.1)_3$ implies $(2.1)_1$.

(*b*) *If in addition* $\mathcal{W} \in PW$, *in some neighborhood of* $\mathcal{E}(\in \mathcal{W})$ $\mathcal{W}$ *has a representation of the form* (2.1), *and*

$$v^r = \frac{dx^r}{dx^0} = \frac{df^r}{dx^0}(x^0)\,, \qquad x^0 = \varphi^0(\mathcal{E})\,, \tag{2.2}$$

then $\boldsymbol{v} = (v^1, v^2, v^3)$ *will be called the* φ*-velocity of* $\mathcal{W}$ *at* $\mathcal{E}$; *and its modulus* $|\boldsymbol{v}|$ *is defined by*

$$|\boldsymbol{v}| = [(v^1)^2 + (v^2)^2 + (v^3)^2]^{\frac{1}{2}} \qquad (\geqslant 0)\,. \tag{2.3}$$

DEF. 2.3. *We say that the admissible frame* φ *for* EP *is* (*affine*) *inertial in case, for some* $\beta \geqslant 0$,

(7) Here (and in the sequel) $\mathbb{R}$ and e.g. $\mathbb{R}^4$ are used in an absolute way, as well as all mathematical notions in purely mathematic contexts. However in the assertion « the φ-velocity of the particle M, at the instant t, is an (element of) $\mathbb{R}^3$ », $\mathbb{R}^3$ is used in an extensional way. Strictly speaking we ought to use $(\mathbb{R}^3)^{(e)}$. But as is explained in, e.g., [2], pp. 86-94, or [3], p. 293, more at length, this is an instance of a widespread double use of names. In textbooks of logic other similar multiple uses are often mentioned in connection with, e.g., « or » or « if ..., then ... ».

(8) Greek [Latin] indeces are meant to run from 0 [1] to 3. Furthermore Einstein's convention is used (e.g. $a_\alpha x^\alpha = \sum_{\alpha=0}^{3} a_\alpha x^\alpha$).

(a) *if M is (any) isolated particle, then* (i) *(necessarily) its φ-velocity is constant*—i.e. *the φ-velocities of W_M at $\mathcal{E}$ and $\mathcal{E}'$ coincide whenever $\mathcal{E}$, $\mathcal{E}' \in W_M$—and* (ii) $|\boldsymbol{v}| < 1/\beta$; *furthermore*

(b) *if $\boldsymbol{x} \in \mathbb{R}^4$, $\boldsymbol{v} \in \mathbb{R}^3$, and $|\boldsymbol{v}| < 1/\beta$, then for some particle M it is phys. poss. to occupy $\mathcal{E} = \varphi^{-1}(\boldsymbol{x})$ and to have $\boldsymbol{v}$ as φ-velocity—see ftn. 7.*

The number β that fulfils (a) and (b) is unique. We shall denote it by β_φ.

POST. 2.5. *There exists an (affine) inertial frame.*

Then, by Def. 2.3, infinitely many inertial frames exist. More detailed results on this subject are afforded by Theor. 8.2.

3. The natural affine structure of space time. Linearity of the transformations between affine inertial frames.

We consider two inertial frames φ and ψ:

$$(3.1) \qquad x^\alpha = \varphi^\alpha(\mathcal{E}) \,, \qquad \xi^\alpha = \psi^\alpha(\mathcal{E}) \qquad (\mathcal{E} \in \mathrm{EP}) \,.$$

A line r in EP represented in φ by equations in the parameter λ, of the form

$$(3.2) \qquad x^\alpha = a^\alpha \lambda + b^\alpha \qquad \left(a^\alpha \text{ and } b^\alpha \text{ const.}; \ \sum_{\alpha=0}^{3} (a^\alpha)^2 > 0\right) ,$$

will be called a *(space-time) φ-straight line.* Among these are the PWIs. Equations (3.2) represent a PWI in φ iff

$$(3.3) \qquad \beta_\varphi^2 \sum_i (a^i)^2 < (a^0)^2 \text{—i.e. } \operatorname{tg} \vartheta_r < 1/\beta_\varphi\text{—} , \qquad a^0 \neq 0 ,$$

where $\vartheta_r \in [0, \pi/2]$ is the angle between the line r and the φ-axis $x^0 = $ var., or *time φ-axis*.

THEOR. 3.1. *If φ and ψ are (affine) inertial frames, the φ-straight lines are the ψ-straight lines.*

Indeed let us consider an inertial frame φ, a φ-straight line s outside PWI, hence with $\operatorname{tg} \vartheta_s \geqslant 1/\beta_\varphi$ ($\vartheta_s = \pi/2$ if $\beta_\varphi = 0$), and three points

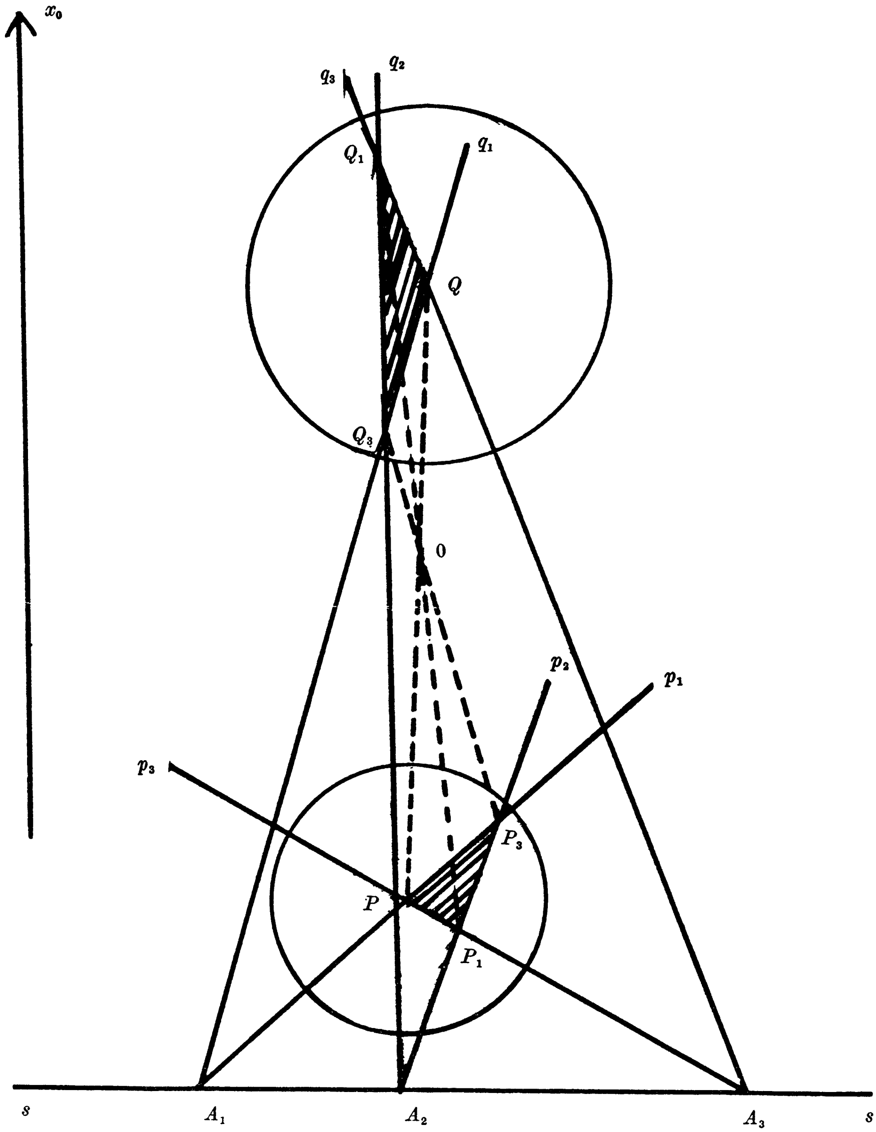

Fig. 1.

A_1 to A_3 on s, their indices being increasing towards right—see fig. 1. We can choose the points P and Q with

$$\varphi^0(A_2) < \varphi^0(P) < \varphi^0(Q)\,, \qquad \varphi^r(A_2) = \varphi^{(r)}(P) = \varphi^{(r)}(Q)\,,$$

and so far from A_2 and one another, that

(i) $(3.3)_2$ holds when r coincides with any among the φ-straight lines A_hP and A_hQ $(h = 1, 2, 3)$, and

(ii) for some (small) $\varepsilon > 0$, $(3.3)_2$ holds for any φ-straight line r in the plane (s, P) $\big(= (s, Q)\big)$, whose (cartesian) φ-distance from P and Q is $\leqslant\varepsilon$—where the φ-distance is the one that has a cartesian expression in φ-coordinates.

We can now consider a φ-straight line p_h $(= p_{h+3})$ $[q_h$ $(= q_{h+3})]$ through A_h, near A_hP $[A_hQ]$ $(h = 1, 2, 3)$ in such a way that

(iii) the intersection points

$$P_h = (p_{h+1}, p_{h+2}) \qquad [Q_h = (q_{h+1}, q_{h+2})] \tag{3.4}$$

fail to be on a same straight line, and

(iv) their φ-distances from $P[Q]$ are less than ε (9).

The triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are homological with respect to φ—i.e. in the affine space (based on the set) EP for which φ is an isomorphism with $\mathbb{R}^4$—. Hence the φ-straight lines P_hQ_h $(h = 1, 2, 3)$ pass through a same point O. Furthermore, by (ii) and (iv), $(3.3)_2$ holds if r is any of them. Hence the φ-straight line P_hQ_h is in PWI as well as the φ-straight lines $P_{h+1}P_{h+2}$ and $Q_{h+1}Q_{h+2}$ $(h = 1, 2, 3)$. Furthermore the frame ψ is assumed to be inertial. Then by Def. 2.3 the above φ-straight lines are also ψ-straight lines. Thus the above

(9) We set $p_1 = A_1P$ and $p_3 = A_3P$, hence $P_2 = P$ by (3.4). Then we choose P_3 on p_1 with $\varphi_0(P_3) > \varphi_0(P)$—hence P_3 is above P. In addition we assume that $A_2P \perp A_1A_2$ and A_3 is at the right of A_1. Then P_3 is at the right of P. Setting $p_2 = A_2P_3$, p_2 is at the right of P and intersects p_3 in a point $P_1\,(\neq P_2)$ which tends to P when $P_3 \to P$. Hence we can choose P_3 with $|P_3P| < \varepsilon$ and $|P_1P| < \varepsilon$.

Let us construct the triangle $Q_1Q_2Q_3$ in the analogous way, but interchanging the roles of A_1 and A_3, so that q_2 passes at the left of Q (and P); hence it is not parallel with p_2. Obviously p_i intersects q_i $(i = 1, 2, 3)$.

triangle are homological also with respect to ψ. Hence the intersections $A_h = (p_h, q_h)$ $(h = 1, 2, 3)$ of their homologous sides belong to a same ψ-straight line $s_\psi (= A_1 A_2)$.

By keeping A_1 and A_2 fixed and having A_3 describe $s - \{A_1, A_2\}$, one sees that $s \subseteq s_\psi$. By interchanging the roles of φ and ψ, one shows that $s_\psi \subseteq s$, so that the arbitrary φ-straight line s outside PWI is a ψ-straight line.

In addition the PWIs are both φ- and ψ-straight lines by Def. 2.3. Hence the thesis holds. q.e.d.

By Theor 3.1, any φ-straight line, where φ is an inertial frame in EP, can be said to be a *straight line of* EP. Thus EP can be regarded as a 4-dimensional affine space, in which inertial frames are special affine frames in EP (in that their time coordinates increase towards future). Therefore we have the following

THEOR. 3.2. *The transformations $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ between any two inertial frames φ and ψ are linear*:

$$x^\alpha = x^\alpha_\beta \xi^\beta + \bar{x}^\alpha , \qquad \xi^\alpha = \xi^\alpha_\beta x^\beta + \bar{\xi}^\alpha , \tag{3.5}$$

where x^α_β, $\bar{x}^\alpha$, ξ^α_β and $\bar{\xi}^\alpha$ are real constants, and

$$\|x^\alpha_\beta\| \neq 0 \neq \|\xi^\alpha_\beta\| , \qquad x^0_0 > 0 < \xi^0_0 \qquad \left(\|x^\alpha_\beta\| = \det (x^\alpha_\beta)\right) . \tag{3.6}$$

4. Proof of the disjunction: classical case or relativistic case. Velocity transformations.

We shall say that an r-subspace of EP (i.e. an r-dimensional one) is *space-like* if no PWI belongs to it.

For any inertial frame φ, every subspace of EP is filled with φ-straight lines, so that it can be regarded as a φ-subspace. By the assertion including $(3.3)_2$, we obviously have the following

THEOR. 4.1. *If φ is an inertial (affine) frame of* EP, *then (a) to (d) below hold.*

(*a*) *A 3-subspace of* EP *is space-like iff its normal forms with the time φ-axis an angle χ, for which* tg $\chi \leqslant \beta_\varphi$;

(*b*) $\beta_\varphi = 0$ *iff the space-like 3-subspaces of* EP *are orthogonal to the time φ-axis*;

(*c*) $\beta_\varphi = 0$ *iff the above subspaces are parallel with one another*;

(*d*) $\beta_\varphi = 0$ *iff for* $\mathcal{E} \in \mathrm{EP}$ *exactly one space-like* 3*-subspace of* EP *passes through* $\mathcal{E}$.

By thesis (*c*) and (*d*) of Theor 4.1 we have the theorem below, which substantially says that space-time is in harmony with either classical physics or special relativity.

THEOR. 4.2. *We have* $\beta_\varphi = 0$ *either for every inertial frame* φ *or for none.*

The disjuncts above will be called just the *classical* and *relativistic cases*.

Note that by Post. 2.5 *the alternatives asserted by* Theor 4.2 *can be meant in an exclusive way*; and that Post. 2.5 is essential to reach this goal.

Now let us consider the velocities $v^r = dx^r/dx^0$ and $w^r = d\xi^r/d\xi^0$ of any mass (or moving) point with respect to the inertial frames φ and ψ, mutually related by (3.5). Then

$$v^r = \frac{x^r_s w^s + x^r_0}{x^0_s w^s + x^0_0}, \qquad w^r = \frac{\xi^r_s v^s + \xi^r_0}{\xi^0_s v^s + \xi^0_0}. \tag{4.1}$$

By the linearity of the transformations (3.5), the 4-dimensional vector space $\mathrm{EP}' = \{\mathcal{E}' - \mathcal{E}\colon \mathcal{E}, \mathcal{E}' \in \mathrm{EP}\}$ (over the real field) naturally associated with EP, in connection with the inertial frame φ, is independent of φ. Hence a unique topology on EP′ is compatible with the continuity of the vector operations. This naturally induces a unique (*inertial*) topology on EP. Incidentally this is the φ-transform of the natural topology on $\mathbb{R}^4$ for some inertial frame φ; and it is independent of φ in harmony with the continuity of the transformations (3.5).

5. Inertial affine spaces and inertial instants.

It is not unusual to regard the PWIs as *inertial* (*geometrical*) *points*. Let Σ_φ be the set formed by those PWIs that have zero φ-velocity—see (2.2). Hence every point $\mathcal{W} \in \Sigma_\varphi$ has a unique representation in φ, of the kind (3.2), with $a^r = 0 = b^0$ and $a^0 = 1$. Hence we can regard b^1 to b^3 as the affine co-ordinates of $\mathcal{W}$ induced by φ. Thus Σ_φ receives the structure of an affine space and can be called an *inertial affine space*.

By (3.5) the φ-velocity of the point $P_\psi(\xi^r)$ of Σ_ψ, with co-ordinates ξ^r in ψ, is

$$(5.1) \qquad \tau^r_{\psi,\varphi} = \frac{dx^r}{dx^0} = \frac{x^r_0}{x^0_0};$$

thus it is a constant, independent of ξ^r and any time co-ordinate. Hence the motion of Σ_ψ with respect to φ is a uniform rectilinear translation; therefore $\{\tau^r_{\psi,\varphi}\}$ ($\in \mathbb{R}^3$) will be called the *translation φ-velocity* of ψ.

We now *assume* $\tau^r_{\psi,\varphi} = 0$, so that $x^r_0 = 0 \neq x^0_0$; hence by (3.5-6), (4.1) becomes

$$(5.2) \qquad v^r = \frac{x^s_r w^s}{x^0_s w^s + x^0_0} \quad \text{with} \quad \|x^r_s\| \neq 0 \quad (\tau^r_{\psi,\varphi} = 0).$$

Then in the *classical case* ($\beta_\psi = 0$) the relations (3.5) between φ and ψ hold with

$$(5.3) \qquad x^r_0 = x^0_r = 0 = \xi^r_0 = \xi^0_r, \qquad \|x^r_s\| = \|\xi^r_s\|^{-1} \neq 0 \qquad (\text{for } \tau^r_{\psi,\varphi} = 0).$$

Indeed, if for some index a $x^0_a \neq 0$ holded, then $(4.1)_1$ would imply $|\boldsymbol{v}| = \infty$ for $w^s = -\delta^s_a x^0_0/x^0_s$ (because $(5.2)_2$ yields $x^r_a \neq 0$ for some r). This completes the proof of $(5.3)_{1\text{-}2}$. By interchanging the roles of φ and ψ we obtain $(5.3)_{3\text{-}4}$. Since $\|x^\alpha_\beta\| = \|\xi^\alpha_\beta\|^{-1}$—see (3.5)—by $(5.3)_{1\text{-}4}$ we obtain $(5.3)_{5\text{-}6}$.

Now we consider the *relativistic case* ($\beta_\varphi > 0$ for every inertial frame φ [Theor 4.2]). Then the analytic transformation $(5.2)_1$ of w^s into v^r is a diffeomorphism (bicontinuous bijection) of the open sphere $|\boldsymbol{w}| < 1/\beta_\psi$ onto the open sphere $|\boldsymbol{v}| < 1/\beta_\varphi$. Thus it is bounded, and hence it can be extended to a diffeomorphism between the closures of these spheres. Hence $|\boldsymbol{w}| = 1/\beta_\psi$ iff $|\boldsymbol{v}| = 1/\beta_\varphi$. In addition we set $w^s = \pm\delta^s_a/\beta_\psi$ ($\beta_\psi \neq 0$). Then (since $x^0_0 \neq 0$) (5.2) yields [10]

$$(5.4) \qquad (x^0_a \pm x^0_0 \beta_\psi)^2 = \beta^2_\varphi \sum_{r=1}^{3} (x^r_a)^2, \qquad \text{so that } x^0_a = 0 \ (\text{for } \tau^r_{\psi,\varphi} = 0).$$

[10] Indeed (5.2) for $w^s = \pm\delta^s_a/\beta_\psi$ with $\beta_\psi \neq 0$ implies

$$(a) \qquad v^r = \frac{\pm x^r_a/\beta_\psi}{\pm x^r_a/\beta_\psi + x^0_0}, \qquad \text{hence } (v^r)^2 = \frac{(x^r_a)^2}{(x^0_0\beta_\psi \pm x^0_a)^2}.$$

Since $|\boldsymbol{w}| = 1/\beta_\psi$, we have $\sum_r (v^r)^2 = |\boldsymbol{v}|^2 = \beta_\varphi^{-2}$. Hence $(a)_2$ yields $(5.4)_1$.

Hence $(5.2)_1$, is a nonsingular homographic transformation, so that $x^r_s = \pm x^0_0 \alpha^r_a d^a_s$, where the matrix (α^r_a) is orthogonal, $\det(\alpha^r_a) = 1$, the matrix (d^a_s) is symmetric, and it has positive proper values. Furthermore, in our case $\beta_\psi |\boldsymbol{w}| = \beta_\varphi |\boldsymbol{v}|$, which easily implies $d^a_s = \delta^a_s \beta_\psi / \beta_\varphi$. Then

$$(5.5) \quad \beta_\varphi x^r_s = \pm \beta_\psi x^0_0 \alpha^r_s \quad \text{with} \quad \sum_i \alpha^r_i \alpha^s_i = \delta^{rs} \,(\text{for } \tau^r_{\psi,\varphi} = 0, \ i.e.\ \tau^r_{\varphi,\psi} = 0)\,.$$

Let us now note that *conditions* (5.3) *hold also in the relativistic case,* because so does $(5.4)_2$. Then, by (5.2), for $\tau^r_{\varphi,\psi} = 0$ we have $\boldsymbol{v} = 0$, iff $\boldsymbol{w} = 0$. Hence, *for* $\tau^r_{\varphi,\psi} = 0$ *the inertial spaces* Σ_φ *and* Σ_ψ *coincide as sets. They also coincide as affine spaces* because by $(5.3)_{1\text{-}2}$ the transformation $(3.5)_1$ becomes

$$(5.6) \qquad x^r = x^r_s \xi^s + \bar{x}^r\,, \qquad x^0 = x^0_0 \xi^0 + \bar{x}^0 \qquad (\text{for } \tau^r_{\varphi,\psi} = 0)\,,$$

where x^r_s, x^0_0 and $\bar{x}^\alpha$ are constants and $x^0_0 > 0 \neq \|x^r_s\|$.

If φ is an inertial frame and $x^0 \in \mathbb{R}$, we shall say that

$$(5.7) \qquad \vartheta = \mathrm{Inst}_\varphi(x^0) = \{\mathcal{E}\colon \varphi^0(\mathcal{E}) = x^0\} \qquad (\subseteq \mathrm{EP})$$

is a *φ-instant of absciss x^0*. Obviously φ-instants constitute a partition of EP. Let Inst_φ be their class. Then *we have* $\mathrm{Inst}_\varphi = \mathrm{Inst}_\psi$ (*besides* $\Sigma_\varphi = \Sigma_\psi$) *in case* $\tau^r_{\varphi,\psi} = 0$. Hence φ-instants can be called instants relative to the inertial space $\Sigma = \Sigma_\varphi = \Sigma_\psi$, or *$\Sigma$-instants.*

6. Characterizations of the classical and relativistic cases.

THEOR. 6.1. (*a*) *we have the classical case as soon as* $(5.6)_2$ *holds for a particular choice of the inertial frames* φ *and* ψ *with* $\tau^r_{\varphi,\psi} \neq 0$.

(*b*) *We have the classical case if and only if* $(5.6)_2$ *holds for arbitrary inertial affine frames* φ *and* ψ, *so that for these frames we have*

$$(6.1) \qquad x^r = x^r_s \xi^s + \bar{x}^r\,, \qquad x^0 = x^0_0 \xi^0 + \bar{x}^0; \qquad v^r = \frac{x^r_s w^s + x^r_0}{x^0_0}\,.$$

(*c*) *We have the classical case, iff the set of Σ-instants is independent of the inertial space Σ.*

Indeed assume that $(5.6)_2$ and $\beta_\psi > 0$ hold for the inertial frames φ and ψ, with $\tau^r_{\varphi,\psi} \neq 0$. This inequality implies $v^r \neq 0$ for $w^s = 0$. In addition the comparison of $(5.6)_2$ with $(3.5)_1$ yields $x^0_s = 0$, so that $(4.1)_1$ becomes $(6.1)_3$. Then we must have $x^a_0 \neq 0$ for some $a \in \{1, 2, 3\}$, so that, when the vector w^s describes (in $\mathbb{R}^3$) the open sphere of radius $1/\beta_\psi$ and center at the origin O, v^r describes an ellipsoid which cannot coincide with the sphere of center O and radius $1/\beta_\varphi$. But just this must happen for inertial frame φ, by Def. 2.3. Hence the assumption $\beta_\psi > 0$ is incompatible with $(5.6)_2$. Thus part (*a*) holds.

Now let us note that *there exist two inertial frames φ and ψ with* $\tau^r_{\varphi,\psi} \neq 0$. Indeed one of them, say φ, exists by Post. 2.5; and for $\beta_\varphi = 0$ [$\beta_\varphi > 0$] the product $\psi = T \circ \varphi$ of φ with any Galilei [Lorentz] transformation T is another inertial frame by Def. 2.3.

The italicized assertion above and thesis (*a*) imply the part « only if » of thesis (*b*). In order to prove the remaining part (« if ») of (*b*) we assume $\beta_\varphi = 0 (= \beta_\psi)$; hence every $\boldsymbol{w} \in \mathbb{R}^3$ is the ψ-velocity of some inertial point. If $x^0_a \neq 0$ holded for some $a \in \{1, 2, 3\}$, then the denominator of the fraction in (4.1) would vanish when $w^a = - x^0_0 / x^0_a$ and $w^s = 0$ for $s \neq a$. But $|\boldsymbol{v}| < \infty$; hence the numerator of the same fraction ought to vanish, which implies $w^a = - x^r_0 / x^r_a$ $(r = 1, 2, 3)$. Thus the 1^{st} and $(a+1)$-th colomns of the matrix (x^α_β) should be proportional in contrast to $(3.6)_1$. Then $(5.3)_{1\text{-}2}$ must hold also for $\tau^r_{\psi,\varphi} \neq 0$. Then the transformation $(3.5)_1$ reduces to $(6.1)_{1\text{-}2}$ and (4.1) simplifies into $(6.1)_3$. Thus thesis (*b*) is completely proved.

Obviously $(6.1)_{1\text{-}2}$ hold for all inertial frames φ and ψ, iff the Σ-instants are independent of the inertial space Σ. Hence thesis (*b*) implies thesis (*c*). q.e.d.

THEOR. 6.2. (*a*) [(*b*)] *The relativistic case holds iff, for some* [*every*] *choice of the inertial frames φ and ψ with $\tau^r_{\varphi,\psi} \neq 0$, in the corresponding transformation formula* $(3.5)_1$ *we have $x^0_s \neq 0$ (for some s).*

(*c*) [(*d*)] *The relativistic case holds iff, for some* [*every*] *couple (Σ, Σ') of distinct inertial spaces, either some Σ-instant differs from every Σ'-instant or*

($\mathcal{A}$) *every Σ-instant differs from every Σ'-instant.*

PROOF. Up to the disjunct ($\mathcal{A}$), parts (*a*) and (*c*) follow from parts (*b*) and (*c*) of Theor 6.1; furthermore by part (*a*) of Theor 6.1 one easily checks parts (*b*) and (*d*) of Theor 6.2.

That we can also insert ($\mathcal{A}$) is well known on the basis of special Lorentz transformations. q.e.d.

7. On the relativistic case.

We assume

$$\tau^r = \tau^r_{\psi,\varphi} \neq 0\,, \qquad \beta_\varphi > 0\,. \tag{7.1}$$

Furthermore, besides (3.5), we consider a (mathematical) Lorentz transformation $\mathfrak{L}_1$: $z = z(x)$, that is the product $\mathfrak{L}\alpha(= \mathfrak{L}\circ\alpha)$ of the following two:

$$y^0 = x^0\,, \qquad y^r = \alpha^r_s x^s \quad \text{with} \quad \sum_{i=1}^{3} \alpha^r_i \alpha^s_i = \delta^{rs}\,, \tag{7.2}$$

$$\left|\begin{array}{l} z^0 = \dfrac{\beta^2_\varphi \tau y^1 - y^0}{\sqrt{1-\tau^2\beta^2_\varphi}}\,, \qquad z^1 = \dfrac{y^1 - \tau y^0}{\sqrt{1-\tau^2\beta^2_\varphi}}\,, \\ \\ z^2 = y^2\,, \qquad z^3 = y^3 \quad \text{with} \quad \tau = \left[\sum_{r=1}^{3} (\tau^r)^2\right]^{\frac{1}{2}} \geqslant 0\,, \end{array}\right. \tag{7.3}$$

where the (properly) orthogonal matrix (α^r_i) is such that in the frame $\alpha\varphi(= \alpha\circ\varphi)$, or (y), the vector τ^r has the components $(\tau, 0, 0)$. According to Def. 2.3 $\alpha\varphi$ is obviously inertial, and so is also the frame $\mathfrak{L}\alpha\varphi$, or (z).

By (7.2-3) $\tau^r_{\chi,\varphi} = \tau^r_{\psi,\varphi}$, where $\chi = \mathfrak{L}_1\varphi$ is the frame (z), so that the geometrical points of Σ_ψ and Σ_χ coincide. Hence in the transformation $\chi\psi^{-1}$, i.e.

$$z^\alpha = z^\alpha_\beta \xi^\beta + \bar{z}^\alpha \qquad \text{with } z^\alpha_\beta \text{ and } \bar{z}^\alpha \text{ constants}\,, \tag{7.4}$$

between ψ and χ, ξ^1 to ξ^3 are functions of z^1 to z^3 and conversely; hence $z^r_0 = 0$, i.e. $\tau^r_{\chi,\psi} = 0$. Let z^r_β and γ^r_i be the respective analogues for the mutually joined frames χ and ψ, of the quantities x^r_β and α^r_i which occur in (5.5) and refer to the frames ψ and φ, assumed there to be mutually joined. Then the analogues of (5.5-6) hold for the transformation $\chi\psi^{-1}$, i.e. (7.4). Hence

$$\beta_\chi z^r_s = \pm \beta_\psi z^0_0 \gamma^r_s \quad \text{with} \quad \sum_i \gamma^r_i \gamma^s_i = \delta^{rs},\ \beta_\chi = \beta_\varphi\,, \tag{7.5}$$

$$z^r = z^r_s \xi^s + \bar{z}^r\,, \qquad z^0 = z^0_0 \xi^0 + \bar{z}^0 \qquad (z^0_0 > 0)\,, \tag{7.6}$$

where the (added) equality $(7.5)_3$ holds because β_φ is left unaltered by both transformations (7.2-3). By (7.5-6)

$$(7.7) \qquad \chi\psi^{-1} = Tz_0^0 \Omega^{-1}\gamma \qquad (\text{e.g. } \chi\psi^{-1} = \chi\circ\psi^{-1}),$$

where γ is a (proper) spatial rotation, Ω^{-1} is the (possibly improper) spatial homothety of parameter $\pm\beta_\psi/\beta_\chi$, z_0^0 is the space-time homothety $(z_0^0\delta_\beta^\alpha)$, and T is the space time translation of vector $-\bar{z}^\alpha$. Hence, in particular,

$$(7.8) \qquad \gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \gamma_s^r & \\ 0 & & & \end{pmatrix}, \qquad \Omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \pm\dfrac{\beta_\varphi}{\beta_\psi}\,\delta_s^r & \\ 0 & & & \end{pmatrix} \qquad (\beta_\chi = \beta_\varphi).$$

Furthermore we know that $\chi\varphi^{-1} = \mathfrak{L}\alpha$—see (7.2-3); hence

$$(7.9) \qquad \psi\varphi^{-1} = T'(z_0^0)^{-1}\Omega L \qquad \text{where} \qquad L = \gamma^{-1}\mathfrak{L}\alpha,\ T' = \gamma^{-1}\Omega T^{-1}\Omega^{-1}\gamma,$$

because by (7.7) and $(7.9)_3$ $\psi\varphi^{-1} = (\chi\psi^{-1})^{-1}\chi\varphi^{-1} = \gamma^{-1}\Omega(z_0^0)^{-1}T^{-1}\mathfrak{L}\alpha = T'\gamma^{-1}\Omega(z_0^0)^{-1}\mathfrak{L}\alpha$.

Since T' is a space time transformation (as well as T), we have proved the following

THEOR. 7.1. *In the relativistic case the transformation $\psi\varphi^{-1}$ between any two (affine) inertial frames φ and ψ is the product of a general Lorentz transformation L (relating frames connected with the same units for length and time), a spatial homothety Ω (which is improper iff the frames φ and ψ have opposite orientations), a space time homothety $(z_0^0)^{-1}$, and a space time translation T'.*

Furthermore Ω is the identity [the spatial inversion with respect to the origin] iff $\beta_\varphi = \beta_\psi$ and φ and ψ have coinciding [opposite] orientations.

Lastly $\mathfrak{L} = I$ iff $\tau^r_{\psi,\varphi} = 0$, i.e. $\Sigma_\varphi = \Sigma_\psi$ (and $\tau = 0$), cf. (7. 2-3).

Hence in the relativistic case the transformations $(\psi\varphi^{-1})$ considered above form a subgroup of the group of affine transformations, which is proper (in harmony with the role of β_φ in Def. 2.3, and with the requirement that the time coordinate must increase towards future). A proper subgroup of it is afforded by the transformations (7.9) with $\Omega = \pm I$ $[\Omega = I]$ and hence with $\beta_\varphi = \beta_\psi$; and in this case φ and ψ can be said to belong to the same *Römer class* [*oriented Römer class*]. If $\beta_\varphi = 1$, then those among the preceding transformations, for which $x_0^0 = 1$, also form a group.

DEF. 7.1. (*a*) *Let φ and ψ be inertial frames; and for all ϑ, ϑ', $\mathcal{E}_1$, and $\mathcal{E}_2$ with*

$$\vartheta \in \mathrm{Inst}_\varphi, \qquad \vartheta' \in \mathrm{Inst}_\psi; \qquad \mathcal{E}_1, \mathcal{E}_2 \in \vartheta \cap \vartheta', \tag{7.10}$$

assume that the (Euclidian) φ-distance $\delta_\varphi(\mathcal{E}_1, \mathcal{E}_2)$ assigned to $\mathcal{E}_1$ and $\mathcal{E}_2$ is λ times $\delta_\psi(\mathcal{E}_1, \mathcal{E}_2)$. Then we say that λ is the ratio of the space units of ψ and φ:

(*b*) *If in addition the ratio $\sigma = \beta_\psi/\beta_\varphi$ of the limit speeds in φ and ψ equals λ/τ, we say that τ is the ratio of the time units of ψ and φ.*

(*c*) *If $\lambda = \tau = 1$, we say that the units of φ and ψ coincide.*

By means of Theor 7.1 and the theory of (special) Lorentz transformations developed by ordinary textbooks on relativity, it is easy to prove the following

THEOR. 7.2. *If φ and ψ are inertial frames, then in the relativistic case the ratios λ and τ above*—c.f. Def. 7.1 (*a*), (*b*)—*exist. Furthermore if $\Sigma_\varphi = \Sigma_\psi$, so that* (5.5-6) *hold, then we have*

$$\tau = x_0^0, \qquad \lambda = x_0^0 \beta_\psi/\beta_\varphi, \tag{7.11}$$

so that an oriented time metric and a spatial metric are determined on Σ_φ up to constant factors.

8. Comparison of the classical and relativistic cases. Additional characterization of them.

In the classical and relativistic cases the inertial (affine) frames determine, up to a constant factor, the (oriented) space time metric on every inertial space Σ, but only in the latter case is the spatial metric on Σ determined up to such a factor.

In the classical case, for every ellipsoid $\mathcal{E} \subset \mathbf{R}^3$, centered at the origin, two mutually joined inertial frames φ and ψ can be choosen for which $(\varphi\psi^{-1})(\mathcal{E})$ is a sphere in $\mathbf{R}^3$, centered at the origin and with radius 1. In the same case (and only in it), in order to characterize the physically isotropic inertial frames (among the inertial affine frames), i.e. the *Galilean* frames, it is necessary to use something outside kinematics, such as dynamic interactions.

Among the interactions above forces at a distance between mass points can be choosen, as is done in [1]. However one of the aims of the present work is an axiomatic treatment of the part of kinematics, wich is common to classical physics and special relativity. Therefore it is preferable to extend it by characterizing Galilean frames in classical physics by means of internal forces. Incidentally it is also natural to consider rigid bodies; but in special relativity they raise difficulties concerning kinematics.

In special relativity Lorentz transformations are often arrived at by postulating the invariance of the speed c of light in vacuum under transformations between Galilean frames relative to the same units.

Now we characterize the relativistic case, within our preceding purely kinematic theory, as the one in which every inertial frame φ can be associated with a scalar speed c_φ in such a way that, if φ and ψ are such frames, every vector velocity $\boldsymbol{v}$ in φ, with $|\boldsymbol{v}| = c_\varphi$, is transformed by $(4.1)_2$ into a ψ-velocity $\boldsymbol{w}$, with $|\boldsymbol{w}| = c_\psi$ (where c_ψ is independent of $\boldsymbol{v}$'s direction).

THEOR. 8.1 (*a*) *We have the relativistic case iff for every inertial frame χ a $c_\chi \in \mathbf{R}$ exists, such that, if φ and ψ are such frames, any moving point (possibly non-material such as a photon) has the scalar velocity c_φ with respect to φ whenever c_ψ is its scalar velocity with respect to ψ.*

(*b*) *In the relativistic case $c_\varphi = 1/\beta_\varphi$ for every inertial frame φ.*

Indeed let the relativistic case hold. Then from ordinary textbooks we know that the condition in φ and ψ considered in (*a*), holds for φ and any frame ψ' joined to ψ and related with φ by means of a Lorentz transformation. By Theor 7.2 the ratio of the velocity units in φ and ψ' is $c_\varphi/c_\psi = \tau/\lambda = \beta_\psi\beta_\varphi^{-1}$. Hence a φ-velocity $\boldsymbol{v}$ with $|\boldsymbol{v}| = 1/\beta_\varphi$ becomes a ψ-velocity $\boldsymbol{w}$ with $|\boldsymbol{w}| = 1/\beta_\psi$, i.e. (*b*) holds, as well as the « only if » part of (*a*).

Now let the condition on φ and ψ considered in (*a*) hold for all inertial frames; and as an hypothesis for reduction ad absurdum assume $\beta_\varphi(=\beta_\psi)=0$. We can chose φ and ψ related by (6.1) with $x^r_s=\delta^r_s$, $x^0_0=1$, and $x^r_0=\tau\delta^r_1$ where $\tau>0$. Then $(6.1)_3$ implies $v^r=$ $=c\pm x^r_0$ for $w^s=\pm c\delta^s_1$ in contrast to the afore-mentioned condition on φ and ψ. Hence the relativistic case holds by Theor 4.2. q.e.d.

THEOR. 8.2. (*a*) [(*b*)] *Assume that the classical [relativistic] case holds, and that φ is an inertial (affine) frame. Then*

(i) *ψ is another such frame if, and only if, it is related to φ by any among the ∞^{17} [∞^{15}] transformations of the form $(6.1)_{1\text{-}2}$ [$T'(z^0_0)^{-1}\Omega L$, where $L=\gamma^{-1}\mathfrak{L}\alpha$—see $(7.9)_2$—is a general Lorentz transformation relative to the units of φ, Ω is a spatial homothety, z^0_0 is a space-time homothety, and T' is a space time translation]* (11). *Furthermore*

(ii) *ψ is joined to φ, iff ψ is related to φ by any of the ∞^{14} [∞^{12}] transformations of the form (5.6) [(5.6) with $x^r_s=\mu\alpha^r_s$, where (α^r_s) is orthogonal]. Thus*

(iii) *the inertial spaces are ∞^3: for every $\boldsymbol{\tau}\in\mathbf{R}^3$ with $|\boldsymbol{\tau}|<1/\beta_\varphi$ there is one inertial space $\Sigma(=\Sigma_\psi)$, whose φ-translation velocity is $\boldsymbol{\tau}$ $(\tau^r=\tau^r_{\psi,\varphi})$; and, conversely, every inertial space has a φ-translation velocity $\boldsymbol{\tau}$, with $|\boldsymbol{\tau}|<1/\beta_\varphi$.*

Indeed, the « only if » part of thesis (i) in part (*a*) [(*b*)] is included in Theor 6.1 (*a*) [Theor 7.1].

In order to deduce the « if » part of (i), we assume that ψ is related to φ by any of the transformations considered in thesis (i) of part (*a*) [(*b*)]; and we set $\beta_\psi=0$ [$\beta_\psi=\pm\beta_\varphi/\Omega_{11}>0$—see the decomposition (7.9) and $(7.8)_{2\text{-}3}$—]. Then, for every conceivable world line $\mathfrak{W}$ in EP,

(α) *$\mathfrak{W}$ has a constant ψ-velocity $\boldsymbol{w}$ with $|\boldsymbol{w}|<1/\beta_\psi$ iff*

(β) *$\mathfrak{W}$ has a constant φ-velocity $\boldsymbol{v}$ with $|\boldsymbol{v}|<1/\beta_\varphi$.*

(11) The general Lorentz transformation $L=\gamma^{-1}\mathfrak{L}\alpha$, which relates φ with, say, $\psi'=L\varphi$, can be determined by giving, first, the φ-translation velocity $\boldsymbol{\tau}$ of $\Sigma_{\psi'}$. If $\boldsymbol{\tau}\neq 0$, $\boldsymbol{\tau}$ determines the rotation α around the origine 0 of ψ, that leaves the plane $(0, \boldsymbol{c}_1, \boldsymbol{\tau})$ fixed, where $\boldsymbol{c}_r$ is the unit vector of the φ-axis $x^r=$ var, and turns $\boldsymbol{c}_1$, into vers $\boldsymbol{\tau}$. For $\boldsymbol{\tau}=0$ set $\alpha=I$. The same rotation turns $\boldsymbol{c}_r$ into $\boldsymbol{c}'_r$ $(r=1,2,3)$. The special Lorentz transformation $\mathfrak{L}$ is determined by φ, α, and $\boldsymbol{\tau}$. Since φ is given, $\boldsymbol{\tau}$ suffices. Since the spatial rotation γ is characterized by 6 scalars, L depends on 9 real parameters; other 6 such parameters are needed to determine Ω, z^0_0, and T'. Thus the transformations of the form $T'(z^0_0)^{-1}\Omega L$ are ∞^{17}.

Furthermore, since the frame φ is inertial, by Def. 2.3, (β) holds iff $\mathcal{W} \in \mathrm{PWI}$—see Def. 2.1 ($b$). Then $\mathcal{W} \in \mathrm{PWI}$ iff (α) holds, so that, by Def. 2.3, ψ is an inertial frame. This completes the deduction of thesis (i).

Thesis (ii) is checked by inspection of (5.6). Then the first part of thesis (iii) follows easily, in order to prove the « converse » part of (iii) let Σ be an inertial space, so that, by definition, $\Sigma = \Sigma_\psi$ for some inertial frame ψ. Then its translation φ-velocity $\boldsymbol{\tau}$ exists—see below (5.1)—and is the φ-velocity of some $\mathcal{W} \in \Sigma_\varphi$. Hence $\mathcal{W} \in \mathrm{PWI}$, so that by Def. 2.3 ($a$), $|\boldsymbol{\tau}| < 1/\beta_\varphi$. q.e.d.

REFERENCES

[1] A. Bressan, *Metodo di assiomatizzazione in senso stretto della Meccanica classica. Applicazione di esso ad alcuni problemi di assiomatizzazione non ancora completamente risolti*, Rend. Sem. Mat. Univ. di Padova, **32** (1962), pp. 55-212.

[2] A. Bressan, *A General Interpreted Modal Calculus*, New Haven, Yale University Press (1972).

[3] A. Bressan, *On the usefulness of modal logic in axiomatizations of physics*, in PSA 1972 (Boston Studies in the Philosophy of Science, Vol. 20), ed. by K. F. Schaffner - R. S. Cohen, Dordrecht and Boston, Reidel (1974).

[4] A. Bressan, *On Physical Possibility*, in « Italian studies in the philosophy of science » (1981), p. 197.

[5] P. K. Feyerabend (a) *Against method. Outline of an anarchistic theory of knowledge* (1975). P. K. Feyerabend (b) *Contro il metodo*, Feltrinelli, Milano (1980).

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