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A Generalization of Signorini's Perturbation Method Suggested by Two Problems of Grioli.

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SUMMARY - Two problems proposed by Grioli induce us to generalize the perturbation method for traction problems of non-linear elasticity discussed in [1], so as to apply to cases when either the loads depend on the solution or the perturbation parameter is interpreted as a material modulus whose vanishing narrows the solution class.

1. Introduction.

Recently Professor Grioli has proposed two problems in hyperelasticity which, though very different the one from the other, have one feature in common: both are meant to be studied *via* a perturbation process, but do not fit within the perturbation scheme of our paper [1].

In that paper we start with a formal analysis of a functional equation

$$- T(p) = l ,$$

where T is a mapping from a Banach space $\mathfrak{X}$, where the solution p is sought, to a Banach space $\mathfrak{Y}$, where the datum l is assigned. We suppose that l is developable as a power series of a real parameter ε and, as is usual in perturbation processes, we seek a solution which

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is also expressed as a power series

$$p(\varepsilon) = p_0 + \sum_{n=1}^{\infty} \varepsilon^n u_n . \tag{1.1}$$

Then we apply the results to the traction problem with dead loads in hyperelasticity.

As we said, the two traction problems of Grioli do not fit into the scheme: the first one (involving an « almost rigid » hyperelastic body), because the operator T itself depends on the parameter ε and, furthermore, T is singular for $\varepsilon = 0$; the second one (involving a heavy hyperelastic body immersed in a fluid), because the loads are « live », *i.e.* they, depend on the solution.

Here we generalize our approach to cover both problems, again considering at first, formally, a functional equation

$$T(p, \varepsilon) = 0 , \tag{1.2}$$

and seeking solutions of the type (1.1). Then we turn our attention to the two problems.

It seems to us that our developments are not simply a formal exercise: apart from some modest specific contributions, they let the suggestions implicit in the two special problems emerge, and trigger new ideas in the field of traction problems with « live » loads for hyperelastic bodies subject to internal constraints.

2. The perturbation technique.

As our developments are strictly local, we require the operator T to be defined only in a set $\mathcal{N} \times \mathcal{E}$, where $\mathcal{N}$ is a neighbourhood of the element p_0 of $\mathcal{X}$ and $\mathcal{E}$ is a neighbourhood of the origin in $\boldsymbol{R}$, possibly deprived of the origin itself. We suppose that: (i) a development of $T(p, \varepsilon)$ in a Laurent series in ε exists for all $p \in \mathcal{N}$; (ii) 0 is at most a simple pole for T:

$$T(p, \varepsilon) = \frac{1}{\varepsilon} T_{-1}(p) + T_0(p) + \varepsilon T_1(p) + \ldots; \tag{2.1}$$

(iii) each one of the operators $T_n(p)$ is analytic within $\mathcal{N}$; (iv) p_0 is a solution of (1.2) for $\varepsilon = 0$ in the sense that

$$\lim_{\varepsilon \to 0} T(p_0, \varepsilon) = 0\,, \tag{2.2}$$

or

$$T_{-1}(p_0) = 0\,, \qquad T_0(p_0) = 0\,. \tag{2.3}$$

For $\varepsilon \neq 0$, we seek solution expressed by a power series (1.1). Adapting notation from Section 2 of [1], we put, for all indices n,

$$T_n(p(\varepsilon)) = T_n\Big(p_0 + \sum_{k=1}^{\infty} \varepsilon^k u_k\Big) = T_n(p_0) + \sum_{m=1}^{\infty} \varepsilon^m T_{n,m}(p_0;\, \{u_k\}_1^m)\,; \tag{2.4}$$

$$\begin{aligned} T_{n,m}(p_0;\, \{u_k\}_1^m) &= \frac{1}{m!}\,\frac{d^m}{d\varepsilon^m}\, T_n(p(\varepsilon))\Big|_0 = \\ &= L_n(p_0)\, u_m + \mathfrak{L}_{n,m}(p_0;\, \{u_k\}_1^{m-1})\,, \end{aligned} \tag{2.5}$$

with

$$L_n(p_0) := \nabla T_n|_{p_0}\,, \tag{2.6}$$

and with

$$\begin{cases} \mathfrak{L}_{n,1} \equiv 0\,, \qquad \mathfrak{L}_{n,2} = \tfrac{1}{2}\,(\nabla^{(2)} T_n|_{p_0}[u_1])[u_1]\,, \\ \mathfrak{L}_{n,m} = \displaystyle\sum_{r=2}^{m} \frac{1}{r!} \sum_{\boldsymbol{P}_r^m} ((\dots))\nabla^{(r)} T_n|_{p_0}[u_{\alpha_1}])[u_{\alpha_2}]) \dots)\,[u_{\alpha_r}]) = \\ \qquad = (\nabla^{(2)} T_n|_{p_0}[u_1])\,]u_{m-1}] + \mathcal{M}_{n,m}(p_0;\, \{u_k\}_1^{m-2})\,, \qquad m > 2\,, \end{cases} \tag{2.7}$$

where $\boldsymbol{P}_r^m$ is the set of all permutations $(\alpha_1, \alpha_2, \dots, \alpha_r)$ of the numbers $1, 2, \dots, m$, taken r at a time with repetitions and such that $\sum_{s=1}^{r} \alpha_s = m$.

Thus,

$$\begin{aligned} T(p(\varepsilon), \varepsilon) = T_{-1,1}(p_0;\, u_1) &+ \varepsilon\big(T_{-1,2}(p_0;\, u_1, u_2) + \\ &+ T_{0,1}(p_0;\, u_1) + T_1(p_0)\big) + \varepsilon^2\big(T_{-1,3}(p_0;\, u_1, u_2, u_3) + \\ &+ T_{0,2}(p_0;\, u_1, u_2) + T_{1,1}(p_0;\, u_1) + T_2(p_0)\big) + \dots\,. \end{aligned} \tag{2.8}$$

The equation for the n-th order approximation u_n of the solution p can be written as follows

$$(2.9) \qquad - L_{-1} u_n = l_n^\star ,$$

where the n-th order «load» $l_n^\star$ depends at most on the approximations of order lower than n. More specifically,

$$(2.10) \qquad \begin{cases} l_1^\star = 0 , \quad l_2^\star = \frac{1}{2}(\nabla^{(2)} T_{-1}|_{p_0}[u_1])[u_1] + L_0 u_1 + T_1(p_0) , \\ l_n^\star = \mathfrak{L}_{-1,n} + L_0 u_{n-1} + \mathfrak{L}_{0,n-1} + \ldots + T_{n-1}(p_0) , \quad n > 2 . \end{cases}$$

If the kernel of the operator L_{-1}^* adjoint to L_{-1} is non-trivial, then the «loads» must satisfy conditions of compatibility

$$(2.11) \qquad \langle l_n^\star , u^* \rangle = 0 , \quad \text{for each } u^* \in \ker L_{-1}^*, \text{ and for each } n > 1 .$$

In the cases that interest us, either the dimension of $\ker L_{-1}^*$ is non-zero, finite and equal to the dimension of $\ker L_{-1}$, or L_{-1} is null.

In the former case, equation (2.9) determines u_n only to within a linear combination of functions forming a basis for $\ker L_{-1}$; but then, often, the compatibility conditions on $l_{n+1}^\star$ remove the indetermination. Details are in the next section.

In the latter case, T_{-1}, and hence L_{-1}, is null. Then (2.9) is substituted by

$$(2.12) \qquad - L_0 u_n = \tilde{l}_n^\star ,$$

where

$$(2.13) \qquad \begin{cases} \tilde{l}_1^\star = T_1(p_0) , \quad \tilde{l}_2^\star = \frac{1}{2}(\nabla^{(2)} T_0|_{p_0}[u_1])[u_1] + L_1 u_1 + T_2(p_0) , \\ \tilde{l}_n^\star = \mathfrak{L}_{0,n} + L_1 u_{n-1} + \mathfrak{L}_{1,n-1} + \ldots + L_{n-1} u_1 + T_n(p_0) , \quad n > 2 . \end{cases}$$

The compatibility conditions become

$$(2.14) \qquad \langle \tilde{l}_n^\star , u^* \rangle = 0 , \quad \text{for each } u^* \in \ker L_0^* \text{ and for each } n \geqslant 1 .$$

Notice that the first condition is now a direct restriction on T:

$$(2.15) \qquad \langle T_1(p_0), u^* \rangle = 0 , \quad \text{for each } u^* \in \ker L_0^* .$$

The case studied in [1] can be considered as a special subcase, where not only T_{-1} is null, but also, for $n \geqslant 1$, all the operators T_n do not depend on p (« dead loads »).

3. Conditions of orthogonality.

Firstly, suppose that $\ker L_{-1}$ and $\ker L^*_{-1}$ have both dimension r and suppose that $\{v_i\}_1^r$ is a basis for $\ker L_{-1}$ and $\{v_i^*\}_1^r$ a basis for $\ker L^*_{-1}$. Then, u_1 is given by

$$u_1 = \sum_{i=1}^r \gamma_1^i v_i ,$$

where γ_1^i $(i = 1, 2, ..., r)$ are, for the moment, arbitrary coefficients, and condition (2.11) for $n = 2$ can be written

$$\text{(3.1)} \qquad \Big\langle \tfrac{1}{2}\Big(\nabla^{(2)} T_{-1}|_{p_0}\Big[\sum_{i=1}^r \gamma_1^i v_i\Big]\Big)\Big[\sum_{j=1}^r \gamma_1^j v_j\Big] + \nabla T_0|_{p_0}\Big[\sum_{i=1}^r \gamma_1^i v_i\Big] + T_1(p_0), v_k^*\Big\rangle = 0 , \qquad k = 1, 2, ..., r ,$$

i.e., it takes the form of an algebraic system in the coefficients γ_1^i:

$$\text{(3.2)} \qquad \sum_{i,j=1}^r \mathfrak{a}_{kij}\gamma_1^i\gamma_1^j + \sum_{i=1}^r \mathfrak{b}_{ki}\gamma_1^i + \mathfrak{c}_{k1} = 0 , \qquad k = 1, 2, ..., r ,$$

where

$$\text{(3.3)} \qquad \begin{cases} \mathfrak{a}_{kij} = \tfrac{1}{2}\langle(\nabla^{(2)} T_{-1}|_{p_0}[v_i])[v_j], v_k^*\rangle , \\ \mathfrak{b}_{ki} = \langle \nabla T_0|_{p_0}[v_i], v_k^*\rangle , \\ \mathfrak{c}_{k1} = \langle T_1(p_0), v_k^*\rangle . \end{cases}$$

If (3.2) admits a real solution $\{\bar{\gamma}_1^i\}_1^r$, the first-order approximation u_1 is correspondingly specified and one can proceed to the next system, whose solution is again determined to within a linear combination of the functions v_i:

$$u_2 = \bar{u}_2 + \sum_{i=1}^r \gamma_2^i v_i ;$$

$\overline{u}_2$ is any solution of the non-homogeneous equation for u_2 and the coefficients γ_2^i are restricted by the second condition (2.11), which can be written as follows

$$\sum_{i=1}^{r}\Big(2\sum_{j=1}^{r}\mathfrak{a}_{kij}\bar{\gamma}_1^j+\mathfrak{b}_{ki}\Big)\gamma_2^i+\mathfrak{c}_{k2}=0\,, \tag{3.4}$$

where the coefficients $\mathfrak{a}_{kij}$ and $\mathfrak{b}_{ki}$ are the same as before and $\mathfrak{c}_{k2}$ need not be given explicitly here. This system is linear in γ_2^i and the matrix is the jacobian matrix of the left-hand side of (3.2) calculated for $\gamma_1^j=\bar{\gamma}_1^j$.

Thus, when the determinant of the system is different from zero (so that the solution $\bar{\gamma}_1^j$ of (3.2) is isolated), a unique solution of (3.4) can also be found. The process can be carried forward, all successive systems being linear with the same matrix of coefficients.

The situation appears formally to be very similar in the second case considered in Section 2, if the assumptions are accepted on L_0 which were made previously on L_{-1}; however, a trait of distinction is that the compatibility condition (2.15) must be satisfied now by the first order loads. The coefficients $\tilde{\mathfrak{a}}_{kij}$ in the system that takes the place of (3.2) have an obvious definition; the coefficients $\tilde{\mathfrak{b}}_{ki}$ and $\tilde{\mathfrak{c}}_{k1}$ are more complex because u_1 already is not simply a linear combination of the functions v_i, but rather

$$u_1=\overline{u}_1+\sum_{i=1}^{r}\gamma_1^i v_i;$$

as a consequence,

$$\begin{aligned}
\tilde{\mathfrak{a}}_{kij}&=\tfrac{1}{2}\langle(\nabla^{(2)}T_0|_{p_0}[\tilde{v}_i])[\tilde{v}_j],\tilde{v}_k^*\rangle\,,\\
\tilde{\mathfrak{b}}_{ki}&=\langle(\nabla^{(2)}T_0|_{p_0}[\tilde{v}_i])[\overline{u}_1]+\nabla T_1|_{p_0}[\tilde{v}_i],\tilde{v}_k^*\rangle\,,\\
\tilde{\mathfrak{c}}_{k1}&=\langle\tfrac{1}{2}(\nabla^{(2)}T_0|_{p_0}[\overline{u}_1])[\overline{u}_1]+\nabla T_1|_{p_0}[\overline{u}_1]+T_2(p_0),\tilde{v}_k^*\rangle\,.
\end{aligned}$$

We have written these expressions explicitly also to allow a comparison with formulae (3.11) of [1].

4. The traction problem in elasticity.

We identify a continuous body with a given regular region $\mathcal{B}_0$ of a three-dimensional Euclidean space; $\mathcal{B}_0$ has interior part $\mathring{\mathcal{B}}_0$ and boundary $\partial\mathcal{B}_0$ with outer unit normal n_0. We let p_0 denote the position vector of a point x of $\mathcal{B}_0$ with respect to a fixed origin q:

$$p_0 = x - q; \tag{4.1}$$

furthermore, we let p be the position vector of the same point in the current placement $\mathcal{B}$ of $\mathcal{B}_0$, so that

$$u = p - p_0 \tag{4.2}$$

is the displacement vector,

$$H = \nabla u \tag{4.3}$$

is the displacement gradient,

$$F = \nabla p = 1 + H \tag{4.4}$$

is the position gradient.

We assume that $\mathcal{B}_0$ is hyperelastic, with stored-energy function $\sigma = \sigma(F)$, and write the constitutive equation for S, the Piola-Kirchhoff stress tensor, in the form

$$S = \mathcal{F}(F)\,, \qquad \text{with } \mathcal{F}(F) := \nabla\sigma|_F\,. \tag{4.5}$$

We formulate a traction problem of equilibrium as follows:

$$\begin{cases} \operatorname{Div} S + b = 0 & \text{in } \mathring{\mathcal{B}}_0\,, \\ S n_0 - s = 0 & \text{in } \partial\mathcal{B}_0\,, \end{cases} \tag{4.6}$$

where b is the body force and s the surface traction. The loads b and s may depend on p and F; moreover, they and the function $\mathcal{F}$

involve a parameter ε:

$$b = b(p, F, \varepsilon)\,, \qquad s = s(p, F, \varepsilon)\,; \tag{4.7}$$

$$S = \mathcal{F}(F, \varepsilon)\,; \tag{4.8}$$

the dependence on ε is such that $\big(cf.\ (2.2)\big)$

$$\begin{cases} \lim\limits_{\varepsilon\to 0} \big(\mathrm{Div}\, \mathcal{F}(1, \varepsilon) + b(p_0, 1, \varepsilon)\big) = 0 \quad \text{in } \mathring{\mathcal{B}}_0\,, \\ \lim\limits_{\varepsilon\to 0} \big(\mathcal{F}(1, \varepsilon) n_0 - s(p_0, 1, \varepsilon)\big) = 0 \quad \text{in } \partial\mathcal{B}_0\,. \end{cases} \tag{4.9}$$

Thus, the problem can be considered as a special case of the general problem (1.2), with

$$T := (\mathrm{Div}\, S + b,\ S n_0 - s)\,, \tag{4.10}$$

the terms between brackets being defined in $\mathring{\mathcal{B}}_0$ and $\partial\mathcal{B}_0$, respectively.

We do not write here the consequences of the perturbation procedure when applied to (4.6) (for the case when $\mathcal{F}$ does not depend directly on ε, and b, s do not depend on p and F, such consequences are detailed in Sections 5 and 6 of [1]). Rather, in the following two sections we study the two problems posed by Grioli: in the first one, $\mathcal{F}$ has a simple pole for $\varepsilon = 0$; in the second one, $\mathcal{F}$ does not depend explicitly on ε, but s depends on p and F.

5. The first problem.

The first problem proposed by Grioli raises the question as to how a process of successive approximations can be set up for a traction problem in hyperelasticity, under the presumption that the body can be considered as rigid to within the first approximation.

We interpret Grioli's proposal as follows. We assume that the constitutive equation itself depends on the parameter ε in the same way as was stipulated for the operator T in Section 2 $\big(cf.$ equation (2.1)$\big)$:

$$\mathcal{F}(F, \varepsilon) = \frac{1}{\varepsilon}\, \mathcal{F}_{-1}(F) + \mathcal{F}_0(F) + \varepsilon \mathcal{F}_1(F) + \dots\,. \tag{5.1}$$

As to the loads, we take them to be infinitesimal with ε and, for simplicity, of the dead type:

$$b = \varepsilon b_1(p_0)\,, \qquad s = \varepsilon s_1(p_0)\,. \tag{5.2}$$

Then we seek a solution of problem (4.7) of the type (1.1), with finite stress for $\varepsilon = 0$ (so that we must require $\mathcal{F}_{-1}(1) = 0$).

By substitution in (5.1), we get series expansions which are the direct counterparts of (2.4), (2.5) and (2.8), respectively:

$$\mathcal{F}_n(F(\varepsilon)) = \mathcal{F}_n(1) + \sum_{m=1}^{\infty} \varepsilon^m \mathcal{F}_{n,m}(\{H_k\}_1^m)\,, \tag{5.3}$$

with

$$\mathcal{F}_{n,m} = \boldsymbol{S}_n H_m + \mathcal{S}_{n,m}(\{H_k\}_1^{m-1})\,, \qquad \boldsymbol{S}_n := \nabla\mathcal{F}_n|_1; \tag{5.4}$$

$$\begin{aligned} S(\varepsilon) &= S_0 + \varepsilon S_1 + \dots \\ &= \boldsymbol{S}_{-1}H_1 + \mathcal{F}_0(1) + \varepsilon(\boldsymbol{S}_{-1}H_2 + \mathcal{S}_{-1,2}(H_1) + \boldsymbol{S}_0 H_1 + \mathcal{F}_1(1)) + \dots\,. \end{aligned} \tag{5.5}$$

The successive displacement gradients H_k satisfy the equations

$$\begin{cases} \operatorname{Div}(\boldsymbol{S}_{-1}H_1 + \mathcal{F}_0(1)) = 0 & \text{in } \mathring{\mathcal{B}}_0\,, \\ (\boldsymbol{S}_{-1}H_1 + \mathcal{F}_0(1))\,n_0 = 0 & \text{in } \partial\mathcal{B}_0; \end{cases} \tag{5.6}$$

$$\begin{cases} -\operatorname{Div}(\boldsymbol{S}_{-1}H_2) = b_1 + \operatorname{Div}(\mathcal{S}_{-1,2}(H_1) + \boldsymbol{S}_0 H_1 + \mathcal{F}_1(1)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathring{\mathcal{B}}_0\,, \\ (\boldsymbol{S}_{-1}H_2)\,n_0 = s_1 - (\mathcal{S}_{-1,2}(H_1) + \boldsymbol{S}_0 H_1 + \mathcal{F}_1(1))\,n_0 \quad \text{in } \partial\mathcal{B}_0\,, \end{cases} \tag{5.7}$$

etc.

Thus, we have set up a perturbation process which is similar to the one sketched in Section 2 and we can make use of the corresponding formal developments.

Suppose for instance that we specify $\mathcal{F}(F, \varepsilon)$ so that (5.6) admits as solutions only the infinitesimal isometries

$$u_1(x) = w_1 + W_1 p_0(x)\,, \tag{5.8}$$

with w_1 some arbitrary constant vector and W_1 some arbitrary constant skew tensor. Then we may interpret our body as rigid to the first order of approximation; in principle, the stress field in $\mathcal{B}_0$ can be determined by solving a sequence of linear systems, each of which is usually simpler than the one of classical elasticity.

The first system is obtained by inserting W_1 for H_1 in the right-hand side of (5.7). W_1 does not remain arbitrary in general, as it must satisfy the orthogonality condition

$$W\cdot\Big(-\int_{\mathcal{B}_0} p_0\otimes b_1-\int_{\partial\mathcal{B}_0} p_0\otimes s_1+\int_{\mathcal{B}_0}\big(\mathcal{S}_{-1,2}(W_1)+S_0W_1+\mathcal{F}_1(1)\big)\Big)=0\,, \tag{5.9}$$

for each skew tensor W.

Notice that the successive approximations S_0, S_1, ... of the stress field in $\mathcal{B}_0$ are influenced by increasingly many of the functions $\mathcal{F}_n$ which specify the choice of the complete constitutive equation (4.5).

EXAMPLE. The so-called St. Venant-Kirchhoff material (*cf.* [3], Section 94) corresponds to the following prescription for the stress

$$\begin{aligned}S=-\tfrac{1}{2}\mu\big((F\cdot F)^2-FF^T\cdot FF^T\big)F^{-T}+\big(\tfrac{1}{2}\lambda(F\cdot F-3)+\\+\mu(F\cdot F-1)\big)F+\mu(\det F)^2F^{-T}F^{-1}F^{-T}\,,\end{aligned} \tag{5.10}$$

where the two material moduli λ and μ have constant values. If we set

$$\varepsilon=\frac{1}{\mu}\,,$$

then we have from (5.1) and (5.10) that

$$\begin{cases}\mathcal{F}_{-1}(F)=-\tfrac{1}{2}\big((F\cdot F)^2-FF^T\cdot FF^T\big)F^{-T}+(F\cdot F-1)F+\\ \qquad\qquad+(\det F)^2F^{-T}F^{-1}F^{-T}\,,\\ \mathcal{F}_0(F)\;=\tfrac{1}{2}\lambda(F\cdot F-3)F\,,\\ \mathcal{F}_n(F)\;=0\,,\quad\text{for all } n\geqslant 1\,,\end{cases} \tag{5.11}$$

with

$$\mathcal{F}_{-1}(1)=\mathcal{F}_0(1)=0\,. \tag{5.12}$$

Performing the differentiations indicated in (5.4), we obtain

$$(5.13)\qquad \begin{cases} \boldsymbol{S}_{-1}\boldsymbol{H}_1 \;\; = \nabla\mathcal{F}_{-1}|_1[\boldsymbol{H}_1] = 2\,\mathrm{sym}\,\boldsymbol{H}_1\,, \\ \mathcal{S}_{-1,2}(\boldsymbol{H}_1) = \frac{1}{2}\,(\nabla^{(2)}\mathcal{F}_{-1}|_1[\boldsymbol{H}_1])[\boldsymbol{H}_1] \\ \boldsymbol{S}_0\boldsymbol{H}_1 \;\; = \lambda(1\cdot\boldsymbol{H}_1)1\,. \end{cases}$$

Thus, in view of $(5.11)_3$ and (5.5), we have that problem (5.6) now reads

$$(5.14)\qquad \mathrm{Div}\,\boldsymbol{E}(\boldsymbol{u}_1) = 0 \quad \text{in } \mathring{\mathcal{B}}_0\,, \qquad \boldsymbol{E}(\boldsymbol{u}_1)\boldsymbol{n}_0 = 0 \quad \text{in } \partial\mathcal{B}_0\,,$$

with

$$(5.15)\qquad \boldsymbol{E}(\boldsymbol{u}_1) := \tfrac{1}{2}\,(\nabla\boldsymbol{u}_1 + \nabla\boldsymbol{u}_1^T)\,, \qquad \boldsymbol{H}_1 := \nabla\boldsymbol{u}_1\,.$$

An easy argument based on integration by parts shows that $\boldsymbol{u}_1$ solves (5.14) only if it is an infinitesimal isometry of the form (5.8). Indeed, (5.14) coincides with the homogeneous traction problem of classical isotropic elasticity for $\lambda/\mu = 0$ (1), whose solutions are of course rigid infinitesimal displacement fields.

Moreover, in view also of $(5.13)_{2,3}$

$$\boldsymbol{S}_0\boldsymbol{H}_1 = \mathcal{S}_{-1,2}(\boldsymbol{H}_1) = 0 \qquad \text{for } \boldsymbol{H}_1 \text{ a skew tensor}\,,$$

so that problem (5.7) becomes simply

$$-\,\mathrm{Div}\,\boldsymbol{E}(\boldsymbol{u}_2) = \boldsymbol{b}_1 \quad \text{in } \mathring{\mathcal{B}}_0\,, \qquad \boldsymbol{E}(\boldsymbol{u}_2)\boldsymbol{n}_0 = \boldsymbol{s}_1 \quad \text{in } \partial\mathcal{B}_0$$

(and (5.9) is automatically satisfied).

Remark. To follow Grioli's original suggestion, we have mentioned sofar only cases where the body is « approximately rigid ». However, the procedure lends itself to the analysis of other constraints. One can study, for instance, bodies which are « approximately incompressible ».

We do not dwell on the matter in general; rather, we illustrate in a very special case the type of developments one encounters. Set

(1) Recall that the St. Venant - Kirchhoff material is isotropic.

$\varepsilon = 1/\lambda$ in (5.10). Then, in view of (5.1), and in place of (5.11), we now have

$$(5.\mathrm{B})\quad \begin{cases} \mathcal{F}_{-1}(F) = \frac{1}{2}(F\cdot F - 3)F\,, \\ \mathcal{F}_0(F) \;= \mu\Big(-\frac{1}{2}\big((F\cdot F)^2 - FF^T\cdot FF^T\big)F^{-T} + \\ \qquad\qquad + (F\cdot F - 1)F + (\det F)^2 F^{-T}F^{-1}F^{-T}\Big)\,, \\ \mathcal{F}_n(F) \;= 0\,, \quad \text{for all } n \geqslant 1\,. \end{cases}$$

As

$$S_{-1}H_1 = (1\cdot H_1)1\,,$$

the traction problem of lowest order becomes

$$\nabla(1\cdot H_1) = 0 \quad \text{in } \mathring{\mathcal{B}}_0\,,$$

$$1\cdot H_1 = 0 \quad \text{in } \partial\mathcal{B}_0\,.$$

As

$$1\cdot H_1 = \operatorname{Div} u_1\,,$$

the conclusion is that the first-order displacement field u_1 is solenoidal over $\mathcal{B}_0$.

6. - The second problem.

Consider a hyperelastic heavy body immersed in a homogeneous incompressible fluid. If e is a unit vector pointing downwards, ϱ_B is the reference density of the fluid, and ε is the acceleration due to gravity, the loads in (4.8) become

$$(6.1)\qquad b = \varepsilon\varrho_B e\,, \qquad s = -\,\varepsilon\varrho_F(p\cdot e)(\det F)F^{-T}n_0\,.$$

An easy computation shows that the successive terms in the series for s are:

$$(6.2)\quad \begin{cases} s_1 = -\,\varrho_F(p_0\cdot e)n_0\,, \\ s_2 = -\,\varrho_F\mathbf{s}[u_1]\,, \quad \text{with} \quad \mathbf{s}[\cdot] := \Big(\big((p_0\cdot e)\operatorname{Div}(\cdot) + \\ \qquad\qquad + e\cdot(\cdot)\big)1 - (p_0\cdot e)\nabla(\cdot)^T\Big)n_0\,, \\ s_n = -\,\varrho_F\mathbf{s}[u_{n-1}] + \varrho_F\sigma_n\big(\{u_k\}_1^{n-2}\big) \quad \text{for } n > 2\,. \end{cases}$$

On the other hand, the successive terms in the series for S are (*cf.* [1], Section 5):

$$(6.3)\qquad \begin{cases} S_0 = \mathcal{F}(1)\,, \\ S_1 = \boldsymbol{S}H_1\,, \quad \text{with} \quad \boldsymbol{S} := \nabla\mathcal{F}|_1 = \nabla^{(2)}\sigma|_1\,, \\ S_n = \boldsymbol{S}H_n + \mathfrak{S}_n(\{H_k\}_1^{n-1}) \quad \text{for } n > 1\,. \end{cases}$$

Let now the body be at ease in the reference placement (so that $S_0 \equiv 0$ in $\mathcal{B}_0$), and set (*cf.* (2.11) and (2.12), respectively)

$$(6.4)\qquad L_0(\cdot) = \big(-\operatorname{Div}\boldsymbol{S}\nabla(\cdot),\ (\boldsymbol{S}\nabla(\cdot))n_0\big)\,;$$

$$(6.5)\qquad \begin{cases} l_1^* = \big(\varrho_B e,\ -\varrho_F(p_0\cdot e)n_0\big)\,, \\ l_n^* = \big(\operatorname{Div}\mathfrak{S}_n(\{H_k\}_1^{n-1})\,, \\ -\mathfrak{S}_n(\{H_k\}_1^{n-1})n_0 - \varrho_F\boldsymbol{s}[u_{n-1}] - \varrho_F\sigma_n(\{u_k\}_1^{n-2})\big)\,, \quad \text{for } n > 1\,. \end{cases}$$

Thus, a sequence of linear problems of type (2.11) is obtained, each with dead loads: although the original problem is one of live loads, the approximating problems have a simpler character (²).

If we assume further that $\ker L_0$ coincide with the set of infinitesimal isometries, then Proposition 1 in Section 7 of [1] can be taken over as it stands: the set of Fredholm conditions can be summarized in the two equilibrium equations for the body in the present placement, *i.e.*,

$$(6.6)\qquad \int_{\mathcal{B}_0}(\varrho_B - \varrho_F\iota) = 0\,, \qquad \iota := \det F\,,$$

and

$$(6.7)\qquad e \times \int_{\mathcal{B}_0}(\varrho_B - \varrho_F\iota)p = 0\,.$$

These equations split into the compatibility conditions on the first-order loads (*cf.* (2.14)):

$$(6.8)\qquad \int_{\mathcal{B}_0}(\varrho_B - \varrho_F) = 0\,, \qquad e \times \int_{\mathcal{B}_0}(\varrho_B - \varrho_F)p_0 = 0\,,$$

(²) The linear problem considered in [2] is relevant for techniques of successive approximations only if the loads are not infinitesimal with ε.

and the linear conditions (3)

$$\int_{\mathcal{B}_0} \iota_k = 0\,, \quad \left(\text{where } \iota = 1 + \sum_{k=1}^{\infty} \varepsilon^k \iota_k\right), \tag{6.9}$$

$$e \times \int_{\mathcal{B}_0} \left((\varrho_B - \varrho_F) u_k - \varrho_F \iota_k p_0 - \varrho_F \sum_{s=1}^{k=1} \iota_{k-s} u_s\right) = 0\,. \tag{6.10}$$

The formal coincidence with the developments of Section 7 of [1] should not blur the differences in substance: here the first equilibrium equation is not automatically valid and, consequently, (6.9) impose restrictions which are not easily satisfied (*e.g.*, at the first order one must require that

$$\int_{\mathcal{B}_0} \operatorname{Div} u_1 = 0;$$

the fact that an infinitesimal rotation is left undetermined in the solution of the first-order system does not help here).

(3) The conditions here are all linear, as the hypotheses accepted on the material behaviour (hyperelasticity and contents of $\ker L_0$) imply that the coefficients a_{kij} vanish.

REFERENCES

[1] G. Capriz - P. Podio-Guidugli, *The role of Fredholm conditions in Signorini's perturbation method*, Arch. Rat. Mech. An., **70** (1979), pp. 261-288.
[2] G. Capriz - P. Podio-Guidugli, *Duality and stability questions for the linearized traction problem with live loads in elasticity*, Proc. IUTAM Symposium on Stability, Nümbrecht (1981), pp. 70-78.
[3] C. Truesdell - W. Noll, *The Non-linear Field Theories of Mechanics*, PH III/3, S. Flügge Ed., Berlin - Heidelberg - New York, Springer, 1965.

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