RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 63 (1980), p. 61-68

<http://www.numdam.org/item?id=RSMUP_1980__63__61_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 63 (1980)

Some Properties of Positive Line Bundles on 1-Convex Complex Analytic Spaces.

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If X is a complex analytic 1-convex manifold with exceptional set E, and $\mathbf{L} = \{L, X, \pi\}$ is an holomorphic line bundle such that $\mathbf{L}|_E > 0$, then the following «precise» vanishing theorem is proved:

$$H^i(X, \mathbf{L} \otimes \mathbf{K}) = 0 \,, \qquad \text{for } i \geqslant 1 \,,$$

where $\mathbf{K}$ is the canonical bundle of X. Some facts about positive line bundles on complex spaces are proved and a proper embedding of a 1-convex space into $\mathbf{C}^N \times \mathbf{P}_M$ exhibited.

Introduction.

It is well-known that from the point of view of the existence of holomorphic functions on a complex space there are two extreme cases: Stein and compact. These two cases are extreme also for the property of an holomorphic line bundle of being positive: every holomorphic line bundle on a Stein space is positive, but if the base space is compact, it carries a positive line bundle if and only if it is projective algebraic (Grauert's generalization of Kodaira's theorem, [4]). Far less is known in the intermediate case between Stein and compact considered by Andreotti-Grauert: the strongly (p, q)-convex concave case.

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Lavoro eseguito nell'ambito del G.N.S.A.G.A. del C.N.R.

We will prove here that a strongly 1-convex space carrying a line bundle whose restriction to the exceptional compact is positive admits a proper embedding in $\mathbf{C}^N \times \mathbf{P}_M$. This theorem for a smooth base space and for a line bundle positive on the all of it, has been stated in [3].

For what concerns vanishing theorems, for generic (p, q)-convexity concavity, Andreotti-Grauert finiteness theorems are, as firstly noticed in [2], under favorable instances, actually vanishing results.

We will show here that a deeper study of the strongly 1-convex case leads to a refined version of the above vanishing theorems, which is actually «precise» in the smooth case, under the assumption that the exceptional compact is projective algebraic, which is what one could expect as a favourable case.

0. – Preliminaries and notation.

Throughout this paper all the complex analytic spaces considered are reduced and with countable topology.

(0.1) Let G be an open subset of $\mathbf{C}^n$ (with coordinates $(z_1, \ldots, z_n)$). A real valued C^2 function ψ on G is said *strongly q-convex*, if the hermitian matrix $(\partial^2\psi/(\partial z_i\, \partial \bar{z}_j))$ has, at least, $n - q + 1$ positive eigenvalues at every point of G. Let X be a complex analytic space. A function φ on X is said strongly q-convex, if for every $x \in X$ there is an analytic isomorphism τ of an open neighborhood V of x in an analytic subset of an open set G of $\mathbf{C}^n$, and a strongly q-convex function ψ on G such that $\varphi = \psi \circ \tau$.

(0.2) Let φ be a real valued function on X. Suppose a and b are in the image of φ. The sets $\{x \in X: \varphi < b\}$, $\{x \in X: \varphi \leqslant a\}$, $\{x \in X: a < \varphi < b\}$ will be denoted respectively, by B_b, B^a and B_b^a. The property for an analytic sheaf $\mathcal{F}$ to be coherent on X will be denoted by $\mathcal{F} \in \operatorname{Coh}(X)$. An holomorphic map $f: X \to S$, will be also called a morphism of analytic spaces.

(0.3) An holomorphic map $f: X \to S$ is called *strongly* (p, q)-*convex concave* if there is a C^2 real valued function φ on X and there are a_0, b_0 in $\operatorname{Im}(\varphi)$ with $a_0 < b_0$, such that:

(i) $f|B_b^a$ is proper for a, b on $\operatorname{Im}(\varphi)$, $a < b$,

(ii) $\{x \in X: \varphi(x) \leqslant c\} = \bar{B}_c$ for $c > b_0$; $\{x \in X: \varphi(x) \geqslant d\} = \bar{B}^d$ for $d < a_0$,

(iii) φ is strongly p-convex on B_{b_0} and strongly q-convex on B^{a_0}. φ is called exhaustion function and (a_0, b_0) exceptional constants.

(0.4) If $B_{a_0} = \emptyset$, f is called strongly p-convex; if $B_{b_0} = \emptyset$ f is called strongly q-concave.

If S reduces to a point and f is strongly (p, q)-convex concave, X is called *strongly (p, q)-convex concave*; if f is strongly p-convex (resp. strongly q-concave), X is called *strongly p-convex* (resp. strongly *q-concave*). If X is strongly p-convex and the exceptional compact $\bar{B}_{b_0}$ is empty, X is called p-complete, so that X is Stein if and only if it is 1-complete. The following basic results hold:

(0.5) (Andreotti-Grauert, [1]). If X is strongly (p, q)-convex concave and $\mathcal{F} \in \operatorname{Coh}(X)$ then the complex vector spaces $H^r(X, \mathcal{F})$ have finite dimension for $p \leqslant r \leqslant \operatorname{prof}_X \mathcal{F} - q - 1$.

(0.6) (Grauert [4]). X is strongly 1-convex if and only if there is a Stein space S, a finite set of points of S, $\{s_1, \ldots, s_k\}$, and a proper surjective holomorphic map $f: X \to S$ such that $E = f^{-1}(\{s_1, \ldots, s_k\})$ is a maximal compact analytic subset of X, and $f: X - E \to S - - \{s_1, \ldots, s_k\}$ is an analytic isomorphism. E is called the exceptional set of X.

(0.7) Let $\mathbf{L} = \{L, X, \pi\}$ be an holomorphic line bundle.

DEFINITION. $\mathbf{L}$ *is positive if a real valued positive differentiable function φ is given on L^*, such that φ is strongly* 1-*convex outside the zero-section of* $\mathbf{L}^*$.

Let k be an integer. We will denote by $\mathbf{L}^k$ the holomorphic line line bundle k-th symmetric power of $\mathbf{L}$. The sheaf of germs of its holomorphic sections will be denoted by $\mathcal{O}(k)$. If $\mathcal{F}$, $\mathcal{F} \in \operatorname{Coh}(X)$, $\mathcal{F}(k)$ will denote the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}(k)$. One has (see [1] f.i.) a natural injection:

$$(A) \qquad \bigoplus_{k=0}^{+\infty} H^{\cdot}(X, \mathcal{F}(k)) \to H^{\cdot}(L^*, \pi^* \mathcal{F}) .$$

As a direct consequence one obtains:

(0.8) THEOREM, [2]. *Suppose* $\mathbf{L}$ *positive. Then if X is strongly p-convex (resp. strongly q-concave) L^* is strongly p-convex (resp. L is strongly $(q+1)$-concave) so that in view of* (A) *and* (0.5), *for every* $\mathcal{F}$,

$\mathcal{F} \in \mathrm{Coh}(X)$, *there is a positive integer* k_0, $k_0 = k_0(\mathbb{L}, \mathcal{F})$, (*resp. a negative integer* k_1), *such that:*

$$H^r(X, \mathcal{F}(k)) = 0$$

for $k \geqslant k_0$ (*resp.* $k \leqslant k_1$), *and* $p \leqslant r$ (*resp.* $r \leqslant \mathrm{prof}_X \mathcal{F} - q - 2$).

1. – An embedding theorem.

(1.1) Let X and S be analytic spaces and $f\colon X \to S$ be a proper holomorphic map. Consider a holomorphic line bundle $\mathbb{L}\colon \{L, X, \pi\}$. $\mathbb{L}$ is said to be *positive relative to* S if for every $s \in S$ there exist an open neighborhood U of s and an open neighborhood D of $\mathcal{O}_{\mathbb{L}^*}|_{f^{-1}(U)}$ such that $f \circ \pi|_D\colon D \to U$ is a strongly 1-convex map.

(1.2) Let f be as above, and $\mathcal{L}$, $\mathcal{L} \in \mathrm{Coh}(X)$, be locally free of rank 1. Set $\mathcal{E} = f_*(\mathcal{L})$ and suppose that the morphism $f^*(\mathcal{E}) \to \mathcal{L}$ be an epimorphism. From the definition of projective bundle on S defined by $\mathcal{E}$ ([6], exp. 15) we obtain a (canonical) S-morfism $i_{\mathcal{L}}\colon X \to \mathbf{P}(\mathcal{E})$. $\mathcal{L}$ is called *very ample* relatively to S if $i_{\mathcal{L}}$ is a (closed) embedding. $\mathcal{L}$ is called *ample* for f if for every $s \in S$ there exist an open neighborhood U of s and a positive integer k_0 such that $\mathcal{L}^{\otimes k_0}|_{f^{-1}(U)}$ is very ample relatively to U. The map f is called *projective* if an invertible $\mathcal{O}_X$-module $\mathcal{L}$ which is ample for f is given.

(1.3) THEOREM (Knorr-Schneider, [7]). *Let* $f\colon X \to S$ *be proper,* $\mathbb{L} = \{L, X, \pi\}$ *be an holomorphic line bundle positive relative to* S. f *is then projective.*

(1.4) Let X be a complex analytic space of bounded complex dimension N which is strongly 1-convex with exhaustion φ and exceptional compact E, and $\mathbb{L} = \{L, X, \pi\}$ be a holomorphic line bundle. We can now prove the following:

THEOREM. *If* $\mathbb{L}|_E$ *is positive then there is a proper embedding of* X *into* $\mathbf{C}^{2N+1} \times \mathbf{P}_{N+1}$.

PROOF. (α) It follows from a theorem of Lieberman-Rossi, [8] that the total space L^* is strongly 1-convex; $\mathbb{L}$ is then positive, accordingly to def. (0.7). Let $f\colon X \to S$ the Remmert reduction of X to a Stein space S. f is a proper modification accordingly to (0.6). We claim that $\mathbf{L}$ is positive relative to S. Indeed, suppose $\mathbf{L}$ is given on the covering $\{U_i\}$ by the transitions functions $\{g_{ij}\}$ and let $\{h_i\}$ an hermitian metric along the fibers of $\mathbf{L}^*$ so that $h_i = |g_{ij}|^{-2} h_j$ on $U_i \cap U_j$ and $-\partial\bar{\partial} h_i > 0$ at every point of U_i (this is equivalent to $\mathbf{L}$ being positive in view of [4]). Let t_i the fiber coordinate of L^* over U_i; then $t_j t_i^{-1} = g_{ij}$ in $U_i \cap U_j$. The function $\chi(z, t)$ equal to $h_i |t_i|^2$ on $\pi^{-1}(U_i)$ is then well defined on L^* an it is strongly 1-convex on $L^* - 0_{\mathbf{L}^*}$ (apply the same argument as in [4], Satz 1). The map $f \circ \pi|_{\{\chi \leq c\}}$ is proper, because so is $\pi|_{\{\chi \leq c\}}$, so the claim is proved

(β) It follows from (1.3) that $\mathcal{O}(1)$ is ample for f and that, for every open set $U \subset S$ there is a positive integer k_0 and an S-embedding of $f^{-1}(U) \to \mathbf{P}(\pi_* \mathcal{O}(k_0))$, and, if U is such that $\pi_*(\mathcal{O}(k_0))$ is a quotient of $\mathcal{O}_S^{N+1}|_U$, there is proper embedding j of $f^{-1}(U) \to \mathbf{P}_N \times U$, since $\mathbf{P}(\mathcal{O}_S^{N+1}) = \mathbf{P}_N \times S$, such that $pr_2 \circ j = f|_{\pi^{-1}(U)}$. But S being Stein $\pi_*(\mathcal{O}(k_0))$ is a global quotient of $\mathcal{O}_S^{N+1}$ hence we have a proper embedding $J\colon X \to \mathbf{P}_N \times S$. Let $F\colon S \to \mathbf{C}^{2N+1}$ be the proper embedding of S. The map $F \circ J$ is then the required embedding.

Conversely one can show:

(1.5) PROPOSITION. *Let X be a closed analytic subvariety in $\mathbf{C}^r \times \mathbf{P}_s$. X then carries a positive holomorphic line bundle.*

PROOF. Let $(z_1, \ldots, z_r)$ be coordinates in $\mathbf{C}^r$ and $(\eta_0, \ldots, \eta_s)$ homogeneous coordinates in $\mathbf{P}_s$. Consider the divisor $\mathbf{C}^r \times E$, where E is an hyperplane in $\mathbf{P}_s$. Let $U_i = \{\eta \in \mathbf{P}_s \colon \eta_i \neq 0\} \times \mathbf{C}^r$, $i = 0, \ldots, s$. Let $\mathbf{L}$ the line bundle corresponding to $\mathbf{C}^r \times E$. $\mathbf{L}|_{U_i}$ is trivial and $\mathbf{L}$ is defined by the transitions functions $\eta_i \eta_j^{-1}$. Consider coordinates in U_i given by

$$x_{r+1} = \eta_0 \eta_i^{-1}, \ldots, x_{r+i} = \eta_{i-1} \eta_i^{-1}, \qquad x_{r+i+1} = \eta_{i+1} \eta_i^{-1}, \ldots, x_s = \eta_s \eta_i^{-1}.$$

and $(x_1, \ldots, x_r) = (z_1, \ldots, z_r)$. Set

$$h_i = \left(\sum_{k=1}^{s} |\eta_k|^2\right) \cdot |\eta_i|^{-2} \cdot \exp\left(\sum_{l=1}^{r} |z_l|^2\right);$$

then $h_i = |\eta_i \eta_j^{-1}|^2 h_j$ and $-\partial\bar{\partial} h_i > 0$. $\mathbf{L}$ is then positive in view of [4] and so it is $\mathbf{L}|_X$.

2. – Vanishing theorems.

If a line bundle satisfies the assumption in theorem (2.4), it follows at once from the theorem of Liebermann-Rossi quoted in the proof of (1.4) and the injection (A) in (0.7) that $H^r(X, \mathcal{F}(k)) = 0$ for k large enough. The proof of the above mentioned results depends on the existence of a metric in the whole of L^* (which is equivalent in view of [4], Satz 1, to positivity). We start this section by giving a proof the vanishing theorem stated above, directly from definition (0.7).

Throughout this section X will be a strongly 1-convex analytic space with exceptional compact E and $\mathbb{L} = \{L, X, \pi\}$ an holomorphic line bundle.

There is an integer k_0 such that for every k, $k \geqslant k_0$, the following holds:

(2.1) Theorem. *Suppose* $\mathbb{L}|_E$ *is positive. Then* $H^r(X, \mathcal{F}(k)) = 0$ *for* $r \geqslant 1$.

Proof. In general, if F is any closed subvariety on an analytic space S and φ is a strongly 1-convex function on F, there is an open neighborhood U of F and a strongly 1-convex function $\tilde{\varphi}$ on U such that $\tilde{\varphi}|_F = \varphi$, see f.i. [8]. And, granted that, there is an open neighborhood N of E such that $\mathbb{L}/_N$ is positive; moreover, we can take $N = B_c = \{\theta < c\}$, where θ is the exhaustion function giving the 1-convexity of X, (see [8, 9]). Suppose that φ is the function that gives the positivity of $\mathbb{L}/_E$; arguing as in [4], Satz 1, the function $\hat{\varphi}(v) = (2\pi)^{-1}\int_0^{\pi} \tilde{\varphi}(e^{it}v)\,dt$ gives a strongly 1-convex metric along the fibers of $\pi^{-1}(N)$. Therefore N is a strongly 1-convex space with exhaustion $\psi = (c - \theta)^{-1}$. The total space $\pi^{-1}(N)$ is strongly 1-convex with exhaustion $\psi\pi + \lambda(\hat{\varphi})$, where λ, $\lambda\colon \mathbb{R} \to \mathbb{R}$, is positive, increasing, convex, (see [2], Prop. 1), hence, by (0.8), $H^r(B_c, \mathcal{F}(k)) = 0$, for $r \geqslant 1$ and k large enough. The conclusion follows at once from a result of Andreotti-Grauert, [1], that gives the isomorphisms: $H^r(X, \mathcal{G}) \to H^r(B_c, \mathcal{G})$ for $r \geqslant 1$.

(2.2) As a consequence of the vanishing theorem it is not difficult to obtain the following ampleness result:

THEOREM. *Suppose* $\mathbb{L}|_E$ *positive; then for every* $\mathcal{F}$, $\mathcal{F} \in \mathrm{Coh}(X)$, *there is* k_0 *positive integer, such that the stalk* $\mathcal{F}(k)$ *is generated by global sections in* $\Gamma(X, \mathcal{F}(k))$ *for every* $k \geq k_0$ *and* $x \in X$.

PROOF. If $x \in E$, one argues as in the compact case, using Theorem (2.1) instead of Kodaira's vanishing theorem. If $x \in X - E$, let us consider the Remmert reduction $f: X \to S$ to the Stein space S. f is biholomorphic on $X - E$. Let $s = f(x)$. Since S is Stein, the stalk $\big(\mathcal{R}^0 f_*(\mathcal{F}(k))\big)_s$ is generated by global sections; but f is biholomorphic on $X - E$ so that $\big(\mathcal{R}^0 f_*(\mathcal{F}(k))\big)_s$ can be identified with $(\mathcal{F}(k))_x$. A theorem of Grauert-Remmert ([5]) gives the isomorphism

$$\Gamma(X, \mathcal{F}(k)) \simeq \Gamma\big(S, \mathcal{R}^0 f_*(\mathcal{F}(k))\big),$$

hence the conclusion.

(2.3) The main result of this section is a «precise vanishing» when X is smooth, which is a consequence of the following theorem of Sommese, [11],

let $f: X \to S$ be a proper morphism of complex spaces and suppose X smooth. Let $\mathbb{L} = \{L, X, \pi\}$ be a holomorphic line bundle positive relative to S; then, if $\mathbb{K}_X$ is the canonical bundle of X and $\mathcal{F} \in \mathrm{Coh}(S)$, the direct images sheaves $\mathcal{R}^p f_*(\mathcal{O}(\mathbb{L} \otimes \mathbb{K}_X) \otimes f^* \mathcal{F})$ are zero per $p \geq 1$. It follows then:

THEOREM. *If* X *is a smooth strongly* 1*-convex space, and* $\mathbb{L}|_E$ *is positive, then* $H^r(X, \mathbb{L} \otimes \mathbb{K}_X) = 0$ *for* $r \geq 1$.

Let $f: X \to S$ be the Remmert reduction of X. $\mathbb{L}$ is then positive relative to S as in (α), Theorem (1.4). Since S is Stein and f surjective, for every $\mathcal{F} \in \mathrm{Coh}(S)$ and $\mathcal{G} \in \mathrm{Coh}(X)$ Grauert's direct images theorem gives the global isomorphisms

$$\Gamma(S, \mathcal{R}^p f^*(\mathcal{G}) \otimes \mathcal{F}) \simeq H^p(X, \mathcal{G} \otimes f^* \mathcal{F})$$

for $p \geq 0$. For $p \geq 1$ and $\mathcal{G} = \mathcal{O}(\mathbb{K}_X \otimes \mathbb{L})$ we have then the conclusion in view of Sommese's theorem.

Added in proof. – V. V. TAN in Trans. A.M.S., **256** (1979), pp. 185-198, has proven independently by different methods, theorems (1.4) and (2.1).

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Manoscritto pervenuto in redazione il 13 luglio 1979.