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**On some asymptotic minimum problems**

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# On Some Asymptotic Minimum Problems.

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## 1. Introduction.

In this work minimum problems on unbounded sets are considered, and existence theorems are given, together with variational approximation properties of the original problem by suitably related problems on bounded sets. More precisely we consider:

(*a*) optimal control problems on unbounded intervals;

(*b*) classical calculus of variations problems (for simple and multiple integrals) on unbounded regions.

Known results about problems (*a*) deal with semilinear cases only (see [3] and [10]), in the following sense: state equations are linear, and a convex functional is minimized. An existence theorem is proved in more general situations, and an approximation theorem is given for semilinear state equations in [12]. In the present paper we extend the above results: we show existence of optimal controls for a general problem over unbounded intervals, and we prove that suitable restrictions of the given problem to bounded time intervals converge variationally as the duration of such restricted processes diverges.

As a particular case, an existence and variational approximation theorem is obtained for the simplest problem of the calculus of variations on unbounded interval, so extending known results ([6] and [8]).

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Similar existence and approximation theorems are given for multiple integrals: as a corollary a convergence theorem is proved for solutions of some non linear Dirichlet problems on bounded regions exhausting an unbounded one.

All above mentioned results are obtained as applications of some properties of the variational convergence as defined in [14].

## 2. Optimal control problems on unbounded intervals.

In this section $P_\infty$ denotes the following optimal control problem: minimize

$$\int_{t_1}^{+\infty} f(s, x, u)\, ds ,$$

on the set of pairs $(u, x)$, $u$ measurable, $x$ locally absolutely continuous on some interval $[t_1, +\infty]$, such that

(1) $\dot{x}(t) = g(t, x(t), u(t))$ a.e. in $(t_1, +\infty)$;

(2) $(t_1, x(t_1)) \in B$; $(t, x(t)) \in A$ for every $t \geq t_1$;

(3) $u(t) \in V(t, x(t))$ a.e. in $(t_1, +\infty)$.

Let us denote with $P_k$ ($k$ a positive integer) the following problem: minimize

$$\int_{t_1}^{t_2} f(s, x, u)\, ds ,$$

on the set of pairs $(u, x)$, $u$ measurable, $x$ absolutely continuous on some interval $[t_1, t_2]$, such that (1), (2), (3) are verified in $[t_1, t_2]$, and

$$k \leq t_2 \leq k + 1 . \tag{4}$$

We are given a sequence $\{a_k\}$ of non negative numbers, with $a_k \to 0$, and a sequence of pairs $\{(u_k, x_k)\}$, defined in $[t_{1k}, t_{2k}]$, satisfying (1), (2), (3) in $(t_{1k}, t_{2k})$, such that for every $k$

(5) $$\int_{t_{1k}}^{t_{2k}} f(s, x_k, u_k)\,ds \leqslant \inf P_k + a_k\,.$$

(*i*) *Notations, conventions and preliminaries.*

In the above problem $P_\infty$, $\int_{t_1}^{+\infty} f(s, x, u)\,ds$ is meant in the improper sense (that is, $s \to f(s, x(s), u(s)) \in L^1(a, b)$ for every $t_1 \leqslant a < b$, and $\int_{t_1}^{t} f(s, x, u)\,ds$ converges when $t \to +\infty$); $A$, $B$, $V(t, x)$ are non empty given sets; $f$, $g$ are given functions; the state variable $x \in R^n$, the control variable $u \in R^m$.

A pair $(u, x)$ as above is called admissible for $P_\infty$ if (1), (2), (3) hold and $\int_{t_1}^{+\infty}(s, x, u)\,ds$ converges. We assume that some admissible pair for $P_\infty$ exists.

Set

$\bar{t}_1 = \inf\{t_1$: there exists $(u, x)$, admissible for $P_\infty$, defined in $[t_1, +\infty)\}$;

$$Q(t, x) = \{(z, g(t, x, u)) : z \geqslant f(t, x, u),\ u \in V(t, x)\}\,;$$
$$L^1_{\text{loc}} = L^1_{\text{loc}}(\bar{t}_1, +\infty)\,; \qquad L^1 = L^1(\bar{t}_1, +\infty)\,.$$

For $1 \leqslant i \leqslant n$ we denote by

$$A_i = \{(t, x_i) : (t, x_1, \ldots, x_n) \in A\}\,.$$

The graph of $(t, x) \to V(t, x)$ is the set

$$\{(t, x, u) : (t, x) \in A,\ u \in V(t, x)\}\,.$$

As is termed a normal set if it is connected and $(t, x) \in A$ implies $(s, x) \in A$ for every $s > t$ (see [6]). $d(x, B)$ is the distance from $x$ to the set $B$.

A given set-valued map $x \to F(x)$, $x \in R^n$, in termed regular if

(6) $$F(x) = \bigcap_{\delta > 0} \overline{\text{co}}\, F(x, \delta)\,,$$

where

$$F(x, \delta) = \bigcup \{F(y): |y - x| \leqslant \delta\},$$

and $\overline{\text{co}}$ denotes the closed convex hull.

If $(u, x)$ is an admissible pair for $P_k$, defined in $[t_1, t_2]$, we extend $x$ in $[\bar{t}_1, +\infty)$ by constancy and continuity, and $u$ by setting $u = 0$ there (with the same notations).

Subsequences are denoted as the original sequences. Let us collect some results from [14] about variational convergence we will need later.

Suppose $X$ a topological space, $S_k \subset X$, $g_k: S_k \to (-\infty, +\infty)$, $k = 0, 1, 2, \ldots$

DEFINITION. $\{g_k\}$ is variationally convergent to $g_0$ if for every $k \geqslant 1$ there exist $u_k \in S_k$, $b_k \geqslant 0$, such that

(I) $g_k(u_k) \leqslant \inf g_k(S_k) + b_k$, $k \geqslant 1$; $b_k \to 0$;

(II) there exists $u_0 \in S_0$ such that $u_k \to u_0$,

$$g_k(u_k) \to g_0(u_0) = \min g_0(S_0)\,.$$

THEOREM 0. Let $X$, $g_k$, $S_k$, $u_k$, $b_k$ be as in the definition above· Suppose $u_k \to \bar{u}$ for some subsequence. Then $g_k \to g_0$ for some subsequence if

(*a*) $y_k \in S_k$, $y_k \to y_0$ for some subsequence implies $y_0 \in S_0$ and $\liminf g_k(y_k) \geqslant g_0(y_0)$;

(*b*) for every $x \in S_0$ there exists $z_k \in S_k$, $k \geqslant 1$, such that $\limsup g_k(z_k) \leqslant g_0(x)$.

REMARKS ABOUT THEOREM 0. It is sufficient that (*a*) holds for $\{y_k\}$ with the same properties as $\{u_k\}$. Moreover if (*b*) holds, then $\inf g_0(S_0) > -\infty$ implies $\sup_k \inf g_k(S_k) < +\infty$. Finally if $\bar{u} \in S_0$, then in (*a*) it suffices $y_k \to y_0 \in S_0$.

(ii) *Results.*

THEOREM 1. With the above notations $P_\infty$ has some solution $(u_0, x_0)$'

defined in $[t_{10}, +\infty)$, moreover for a subsequence we have

$$\dot{x}_k \rightharpoonup \dot{x}_0 \quad \text{in } L^1_{\text{loc}},$$

$$x_k \to x_0 \quad \text{uniformly on compact sets in } [\bar{t}_1, +\infty),\ t_{1k} \to t_{10},$$

$$\int_{t_{1k}}^{t_{2k}} f(s, x_k, u_k)\, ds \to \min P_\infty = \int_{t_{10}}^{+\infty} f(s, x_0, u_0)\, ds,$$

if we assume that

(7) $f$ is Borel measurable, $g$ is measurable in $t$, $f$ lower semicontinuous, $g$ continuous, in $(x, u)$, and for some constant $D \geq 0$ and $C \in L^1_{\text{loc}}$

$$|g(t, x, u)| \leq C(t) + D|u|;$$

(8) $$f(t, x, u) \geq \phi(t, |u|), \qquad \lim_{z \to +\infty} \frac{\phi(t, z)}{z} = +\infty$$

uniformly in $t$ on bounded sets, $\inf_{z \geq 0} \phi(t, z) \in L^1_{\text{loc}}$; moreover $f(t, x, u) \geq \geq \sum_{i=1}^{n} Z_i(t, x_i)\, g_i(t, x, u) + \varphi(t)$, where $\int_{\bar{t}_1}^{+\infty} \varphi\, ds$ converges, $Z_i$ and $Z_{ix_i}$ are continuous such that $\lim_{t \to +\infty} Z_i(t, x_i) = 0$ for a.e. $x_i$, $\iint_{A_i} |Z_{ix_i}|\, dt\, dx_i$ converges (for every $i = 1, \dots, n$);

(9) $V$ has closed values and closed graph; $Q(t, \cdot)$ is regular for a.e.t; $A$ is normal and closed, $B$ compact.

With the above assumptions, $P_k$ has optimal solutions for every $k$. For the proof we need three lemmas.

LEMMA 1. Given $(u, x)$ admissible for $P_\infty$, defined in $[t_1, +\infty)$, there exists $(v_k, y_k)$ admissible for $P_k$ (for large $k$), defined in $[s_{ik}, s_{2k}]$, such that

$$\limsup \int_{s_{1k}}^{s_{2k}} f(s, y_k, v_k)\, ds \leq \int_{t_1}^{+\infty} f(s, x, u)\, ds.$$

PROOF. Take $P_k$ such that $k > t_1$, and define $(v_k, y_k)$ as the restriction of $(u, x)$ on $[t_1, k]$.

LEMMA 2. Some subsequence of $\{\dot{x}_k\}$ is weakly convergent in $L^1_{\text{loc}}$.

PROOF. Let $(u, x)$ be admissible for $P_\infty$, defined in $[t_1, +\infty)$. Then, using (8), for any $t_2 > t_1$

$$\int_{t_1}^{t_2} f(s, x, u)\, ds \geqslant \sum_{i=1}^{n} \int_{t_1}^{t_2} Z_i(s, x_i)\, \dot{x}_i\, ds + \int_{t_1}^{t_2} \varphi\, ds\,.$$

From the inequality

$$(*) \qquad \left|\int_{t_1}^{t_2} Z_i(s, x_i)\, \dot{x}_i\, ds\right| \leqslant \iint_{A_i \cap \{s \geqslant t_1\}} |Z_{ix_i}|\, ds\, dx_i\,, \qquad i = 1, \ldots, n,$$

(see [6], p. 414), we see that $\inf P_k > -\infty$ for all $k$, $\inf P_\infty > -\infty$, so that, from lemma 1, $\sup_k \inf P_k < +\infty$ (see remarks about theorem 0). Given a fixed compact interval $[a, b] \subset [t_{1k}, t_{2k}]$ for all large $k$, let $E$ be a measurable subset of $[a, b]$. Set

$$L(s, x, u) = \sum_{i=1}^{n} Z_i(s, x_i)\, g_i(s, x, u) + \varphi(s)\ ;$$

then for any $k$, using (8) and $(*)$

$$\int_E [\phi(t, |u_k|) - L(s, x_k, u_k)]\, ds \leqslant \int_E [f(s, x_k, u_k) - L(s, x_k, u_k)]\, ds \leqslant$$

$$\leqslant \int_{t_{1k}}^{t_{2k}} [f(s, x_k, u_k) - L(s, x_k, u_k)]\, ds \leqslant \sup_k (\inf P_k + a_k) + H\,,$$

so that for some constant $H$

$$(10) \qquad \sup_k \int_E \phi(t, |u_k|)\, dt \leqslant H\,.$$

Moreover

$$\sup_k \int_{t_{1k}}^{t_{2k}} f(s, x_k, u_k)\, ds < +\infty .$$

Given $\varepsilon > 0$, from (8) there exists $\delta > 0$ such that

$$\phi(t, z) \geqslant H \frac{z}{\varepsilon} \qquad \text{if } z \geqslant \delta \text{ and } t \in E .$$

Set

$$E_{1k} = \{t \in E\colon |u_k(t)| \leqslant \delta\} , \qquad E_{2k} = E \setminus E_{1k} .$$

Then for all $k$, using (10),

$$\int_E |u_k|\, ds = \Big(\int_{E_{1k}} + \int_{E_{2k}}\Big) |u_k|\, ds \leqslant \delta \text{ meas } E + \frac{\varepsilon}{H} \int_E \phi(s, |u_k|)\, ds \leqslant \delta \text{ meas } E + \varepsilon$$

so that

$$\sup_k \int_E |u_k|\, ds \to 0 \qquad \text{as meas } E \to 0 .$$

From (7)

$$\int_E |\dot{x}_k|\, dt \leqslant \int_E C\, dt + D \int_E |u_k|\, dt ,$$

therefore

$$\lim_{\text{meas} E \to 0} \; \sup_k \int_E |\dot{x}_k|\, ds = 0 ;$$

moreover if $t_{1k} \leqslant a < b \leqslant t_{2k}$

$$\int_a^b |\dot{x}_k|\, dt \leqslant \int_a^b C\, dt + D \int_a^b |u_k|\, dt , \tag{11}$$

and

$$\varphi(t, |u_k|) \geqslant |u_k| \quad \text{ if } |u_k| \geqslant \delta \text{ and } a \leqslant t \leqslant b \text{ for some } \delta > 0,$$

so that

$$(12)\qquad \int_a^b |u_k|\,dt \leqslant \delta(b-a) + \int_{t_{1k}}^{t_{2k}} f(s, x_k, u_k)\,ds + H \quad \text{for some constant } H,$$

therefore from (11) and (12) we get

$$\sup_k \int_a^b |\dot{x}_k|\,dt < +\infty,$$

and lemma 2 is proved.

LEMMA 3. Let $(v_k, y_k)$ be admissible for $P_k$, $k \neq \infty$, defined in $[s_{1k}, s_{2k}]$. Assume that (for a subsequence)

(i) $$\sup_k \int_{s_{1k}}^{s_{2k}} f(s, y_k, v_k)\,ds < +\infty;$$

(ii) $y_k \to y$ uniformly on compact sets of $[\bar{t}_1, +\infty)$;

(iii) $\dot{y}_k \rightharpoonup \bar{g}$ in $L^1_{\text{loc}}$.

Then there exists $v$ such that $(v, y)$, defined in $[t_1, +\infty)$, is admissible for $P_\infty$, moreover

$$\bar{g}(s) = g(s, y(s), v(s)) \qquad \text{for a.e. } s \geqslant t_1,$$

$$\liminf \int_{s_{1k}}^{s_{2k}} f(s, y_k, v_k)\,ds \geqslant \int_{t_1}^{+\infty} f(s, y, v)\,ds \qquad \text{(for a subsequence)}.$$

PROOF. Taking some subsequence, we assume $s_{1k} \to t_1$, and (by (8) together with (*) in the proof of lemma 2)

$$\left\{\int_{s_{1k}}^{s_{2k}} f(s, y_k, v_k)\,ds\right\} \quad \text{convergent to its lim inf.}$$

Following [1], there exists a subsequence $\{\dot{y}_{k_j}\}$ such that (from a theorem of Mazur) a sequence of convex combinations converges strongly in $L^1_{\text{loc}}$ to $\bar{g}$, say

$$\sum_{i=1}^{p} \alpha_{ij}\, \dot{y}_{q_j+i} \to \bar{g} \quad \text{in } L^1_{\text{loc}} \text{ and for a.e. } t > \bar{t}_1,$$

where $p = p(j)$,

$$\alpha_{ij} \geqslant 0\,, \qquad \sum_{i=1}^{p} \alpha_{ij} = 1\,, \qquad q_{j+1} > q_j + p(j)\,.$$

Given $E$ a compact subinterval of $[t_1, +\infty)$, and $\varepsilon > 0$, we get $|y_k(t) - y(t)| < \varepsilon$ for large $k$ and all $t \in E$. Therefore, with the same $k'$s, and a.e.t,

$$\big(f(t, y_k(t), v_k(t)),\, g(t, y_k(t), v_k(t))\big) \in Q\big(t, y(t), \varepsilon\big)\,.$$

For large $j$ and a.e. $t \in E$,

$$(**) \quad \Big(\sum_{i=1}^{p} \alpha_{ij} f(t, y_{q_j+i}(t), v_{q_j+i}(t)),\, \sum_{i=1}^{p} \alpha_{ij}\, g(t, y_{q_j+i}(t), v_{q_j+i}(t))\Big) \in \operatorname{co} Q\big(t, y(t), \varepsilon\big).$$

Set

$$\bar{f}(t) = \liminf_{j\to\infty} \sum_{i=1}^{p} \alpha_{ij} f(t, y_{q_j+i}, v_{q_j+i})$$

(different a.e. from $-\infty$ by (8)).

A.e. in $E$, by (8)

$$\sum_{i=1}^{p} \alpha_{ij} f(t, y_{q_j+i}(t), v_{q_j+i}(t)) \geqslant \sum_{i=1}^{p} \alpha_{ij} \sum_{r=1}^{n} Z_r(t, y_{rq_j+i}(t))\, \dot{y}_{rq_j+i}(t) + \varphi(t)\;;$$

$$\sum_{r=1}^{n} \sum_{i=1}^{p} \alpha_{ij} Z_r(t, y_{rq_j+i})\, \dot{y}_{rq_j+i} = \sum_{r=1}^{n} \sum_{i=1}^{p} \alpha_{ij} [Z_r(t, y_{rq_j+i}) - Z_r(t, y_r)]\, \dot{y}_{rq_j+i} + \\ + \sum_{r=1}^{n} Z_r(t, y_r) \sum_{i=1}^{p} \alpha_{ij}\, \dot{y}_{rq_j+i}\,,$$

but given $\gamma > 0$, for all large $j$

$$\int_E \sum_{r=1}^{n} \sum_{i=1}^{p} \alpha_{ij} |Z_r(t, y_{rq_j+i}) - Z_r(t, y_r)|\, |\dot{y}_{rq_j+i}|\, dt \leqslant \gamma \int_E \sum_{r=1}^{n} \sum_{i=1}^{p} \alpha_{ij} |\dot{y}_{rq_j+i}|\, dt\,,$$

and this shows that (for some subsequence)

$$\sum_{r=1}^{n}\sum_{i=1}^{p}\alpha_{ij}[Z_r(t, y_{q_j+i}) - Z_r(t, y_r)]\,\dot y_{rq_j+i} \to 0 \qquad \text{a.e.},$$

therefore (a.e. in $E$)

$$\bar f(t) \geqslant L_0(t) \equiv \sum_{r=1}^{n} Z_r(t, y_r(t))\,\bar g(t) + \varphi(t)\,,$$

and

$$\sum_{i=1}^{p}\alpha_{ij}\,L(\cdot, y_{q_j+i}, v_{q_j+i}) \to L_0 \qquad \text{in } L^1_{\text{loc}};$$

then

$$\int_E (\bar f - L_0)\,dt \leqslant \liminf \sum_{i=1}^{p}\alpha_{ij}\int_{s_{1q_j+i}}^{s_{2q_j+i}}[f(t, y_{q_j+i}, v_{q_j+i}) - L_0]\,dt < +\infty\,,$$

so that $\bar f$ is a.e. different from $+\infty$, because $L_0 \in L^1_{\text{loc}}$.

Taking a limit in (∗∗) along some subsequence (depending on $t$) we get $(\bar f(t), \bar g(t)) \in \overline{\text{co}}\, Q(t, y(t), \varepsilon)$ for all $\varepsilon$ and a.e. $t \in E$. From regularity of $Q(t, \cdot)$

$$(\bar f(t), \bar g(t))\ \varepsilon\ Q(t, y(t)) \qquad \text{a.e. in } [t_1, +\infty).$$

Given a bounded subinterval $I$ of $[t_1, +\infty)$, by (7) $f$ has a lower semicontinuous, and $g$ continuous, restriction to sets $F \times R^{n+m}$, with meas $F$ arbitrarily near to meas $I$. (see [15]).

Therefore the measurable implicit function theorem of [11] (see [1] for details) gives existence of $v$, measurable in $[t_1, +\infty)$, such that a.e.

$$v(s) \in V(s, y(s))\,, \qquad \bar f(s) \geqslant f(s, y(s), v(s))\,, \tag{13}$$
$$\dot y(s) = \bar g(s) = g(s, y(s), v(s))\,.$$

From (9) we see that $(t, y(t)) \in A$ if $t \geqslant t_1$, and $(t_1, y(t_1)) \in B$. Remember that $s_{1k} \to t_1$ and

$$\sum_{i=1}^{p}\alpha_{ij}\int_{s_{1q_j+i}}^{s_{2q_j+1}} f(s, y_{q_j+i}, v_{q_j+i})\,ds \to \liminf \int_{s_{1k}}^{s_{2k}} f(s, y_k, v_k)\,ds\,,$$

moreover

$$\int_{s_{1k}}^{s_{2k}} L(s, y, v)\, ds \to \limsup \int_{t_1}^{+\infty} L(s, y, v)\, ds$$

(by (*) in the proof of lemma 2).

Take $t > t_0 > t_1$. Then using Fatou's lemma $(L = L(s, y.\, v))$

$$\int_{t_0}^{t} [f(s, y, v) - L]\, ds \leqslant \int_{t_0}^{t} (\bar{f} - L)\, ds \leqslant \liminf \sum_{i=1}^{p} \int_{t_0}^{t} \alpha_{ij}[f(s, y_{q_j+i}, v_{q_j+i}) - L]\, ds \leqslant$$

$$\leqslant \liminf \sum_{i=1}^{p} \alpha_{ij} \int_{s_{1q_j+i}}^{s_{2q_j+i}} [f(s, y_{q_j+i}, v_{q_j+i}) - L]\, ds =$$

$$= \liminf \int_{s_{1k}}^{s_{2k}} f(s, y_k, v_k)\, ds - \limsup \sum_{i=1}^{p} \alpha_{ij} \int_{s_{1q_j+i}}^{s_{2q_j+i}} L\, ds\,.$$

It follows that

$$\int_{t_1}^{t} f(s, y, v)\, ds \leqslant \int_{t_1}^{t} L\, ds + \liminf \int_{s_{1k}}^{s_{2k}} f(s, y_k, x_k)\, ds - \int_{t_1}^{+\infty} L\, ds\,. \tag{14}$$

Set

$$F = \limsup_{t \to +\infty} \int_{t_1}^{t} f(s, y, v)\, ds < +\infty \qquad (\text{from } (14))\,.$$

Given $\varepsilon > 0$ there exists arbitrarily large $T$ such that

$$\int_{t_1}^{T} f(s, y, v)\, ds \geqslant F - \varepsilon\,.$$

From (*) (in the proof of lemma 2) we get

$$\int_{T}^{t} f(s, y, v)\, ds \geqslant -\varepsilon \qquad \text{for } t > T \text{ and large } T,$$

therefore

$$F - 2\varepsilon \leqslant \int_T^t f(s, y, v)\, ds \leqslant F + \varepsilon \qquad \text{for } t > T \text{ and large } T,$$

so that $\int_{t_1}^{+\infty} f(s, y, v)\, ds$ exists, and taking a limit along a suitable sequence of $t$'s in (14) we conclude

$$\liminf \int_{s_{1k}}^{s_{2k}} f(s, y_k, v_k)\, ds \geqslant \int_{t_1}^{+\infty} f(s, y, v)\, ds\ .$$

Proof of theorem 1. From theorem 0 we verify that $(u_k, x_k)$ satisfy lemma 3 (this follows from boundedness of $B$ and lemma 2), so that (for a subsequence) $t_{1k} \to t_{10}$, $\dot{x}_k \rightharpoonup \dot{x}_0$ in $L^1_{\text{loc}}$, $x_k \to x_0$ uniformly on compacta, $\int_{t_{1k}}^{t_{2k}} f(s, x_k, u_k)\, ds \to \int_{t_{10}}^{+\infty} f(s, x_0, u_0)\, ds = \min P_\infty$. Existence of solutions for $P_k$ follows from a slight extension of the results of [1], q.e.d.

Remark 1. From theorem 1 we set that $P_\infty$ can be always variationally approximated by $P_k$ using not necessarily optimal pairs $(u_k, x_k)$: an approximation to an optimal trajectory $(u_0, x_0)$ for $P_\infty$ can be obtained from knowledge of a « quasi-minimizing » sequence only (that is, $\int_{t_{1k}}^{t_{2k}} f(s, x_k, u_k)\, ds - \inf P_k \to 0$) under natural assumptions on $P_\infty$. Compactness of $B$ (see (9)) can be avoided if the projection of $B$ on the $t$ — axes is bounded and (i) $\int_{t_1}^{t_2} f(s, x, u)\, ds$ is unbounded with $\max_s |x(s)|$, or (ii) the projection of $A$ on $R^n$ is bounded, or (iii) admissible pairs for $P_k$, $k \neq \infty$, can be extended in a half line being admissible for $P_\infty$ too, and there exists a compact set $\Omega \subset R^{n+1}$ such that given any trajectory $x$, admissible for $P_\infty$, there exists $t^*$ with $(t^*, x(t^*)) \in \Omega$.

If an uniqueness theorem holds for $P_\infty$, then from theorem 1 we see that the original sequence $\{P_k\}$ converges variationally to $P_\infty$ (the same remark applies to the results in the following sections). From existence theory of optimal control is well known that, as far as non-linear state equations (1) are concerned, no convergence property of $\{u_k\}$ to $u_0$ can be hoped for.

About some convergence property of controls when $g$ is linear (at least in $u$) and $f$ is convex, see [12].

## 3. Problems of the calculus of variations on unbounded intervals.

In this section $P_\infty$ denotes the following problem: minimize

$$\int_{t_1}^{+\infty} f(s, x, \dot{x})\, ds$$

on the set of locally absolutely continuous $x$ in some interval $[t_1, +\infty)$ such that

(15) $$(t, x(t)) \in A \quad \text{if } t \geqslant t_1, \qquad (t_1, x(t_1)) \in B .$$

Let $P_k$ denote the following problem: minimize

$$\int_{t_1}^{t_2} f(s, x, \dot{x})\, ds ,$$

on the set of absolutely continuous $x$ in $[t_1, t_2]$ satisfying there (15), and

$$k \leqslant t_2 \leqslant k + 1 .$$

We are given $a_k \to 0$, $x_k$ admissible for $P_k$ ($k \neq \infty$), defined in $[t_{1k}, t_{2k}]$, such that for every $k$

$$\int_{t_{1k}}^{t_{2k}} f(s, x_k, \dot{x}_k)\, ds \leqslant \inf P_k + a_k .$$

From theorem 1 with $g(t, x, u) = u$, $V(t, x) = R^n = R^m$ we get.

COROLLARY 1. Assume $A$, $B$ as in theorem 1, and that

(16) $f$ is Borel measurable, lower semicontinuous in $x$ uniformly with respect to $u$ on compacta, convex in $u$;

(17) $$f(t, x, u) \geqslant \phi(t, |u|) , \qquad \text{with } \varphi \text{ as in theorem 1;}$$

(18) $$f(t, x, u) \geqslant \sum_{i=1}^{n} Z_i(t, x_i)\, u_i + \varphi(t) , \qquad \text{with } Z_i \text{ and } \varphi \text{ as in theorem 1.}$$

Then $P_k$ has solutions for every $k$, and there exists a solution $x_0$ of $P_\infty$, defined for $t \geqslant t_{10}$, such that (for a subsequence) $\dot{x}_k \rightharpoonup \dot{x}_0$ in $L^1_{\mathrm{loc}}$, $x_k \to x_0$ uniformly on compact intervals, $t_{1k} \to t_{10}$, $\int_{t_{1k}}^{t_{2k}} f(s, x_k, \dot{x}_k)\, ds \to$ $\to \int_{t_{10}}^{+\infty} f(s, x_0, \dot{x}_0)\, ds = \min P_\infty$.

PROOF. We need verify regularity of $Q(t, \cdot)$ only. We have

$$Q(t, x) = \{(z, u) : z \geqslant f(t, x, u),\ u \in R^n\} .$$

Given $(t, x)$, take any $(z, u) \in \bigcap_{\varepsilon > 0} \overline{\mathrm{oc}}\, Q(t, x, \varepsilon)$. Then, for every $\varepsilon > 0$, by Caratheodory's theorem we can find $\alpha_{\varepsilon ik} \geqslant 0$, $\sum_{i=1}^{n+1} \alpha_{\varepsilon ik} = 1$, $z_{\varepsilon ik}$, $u_{\varepsilon ik}$, such that $z_{\varepsilon ik} \geqslant f(t, y_{\varepsilon ik}, u_{\varepsilon ik})$,

$$|y_{\varepsilon ik} - x| \leqslant \varepsilon , \qquad z = \lim_k \sum_{i=1}^{n+1} \alpha_{\varepsilon ik} z_{\varepsilon ik} , \qquad u = \lim_k \sum_{i=1}^{n+1} \alpha_{\varepsilon ik} u_{\varepsilon ik} .$$

From (17) we have

$$z_{\varepsilon ik} \geqslant \phi(t, |u_{\varepsilon ik}|)$$

but $\sup_{\varepsilon, i, k} z_{\varepsilon ik} < +\infty$, therefore

$$\sup_{\varepsilon, i, k} |u_{\varepsilon ik}| < +\infty .$$

Therefore given $\delta > 0$, for small $\varepsilon$ $f(t, y_{\varepsilon ik}, u_{\varepsilon ik}) \geqslant f(t, x, u_{\varepsilon ik}) - \delta$, so that $\sum_{i=1}^{n+1} \alpha_{\varepsilon ik} z_{\varepsilon ik} \geqslant -\delta + \sum_{i=1}^{n+1} \alpha_{\varepsilon ik} f(t, x, u_{\varepsilon ik}) \geqslant -\delta + f(t, x, \sum_{i=1}^{n+1} \alpha_{\varepsilon ik} u_{\varepsilon ik})$ by (16); as $k \to \infty$ we get $z \geqslant -\delta + f(t, x, u)$ ($f(t, x, \cdot)$ is a continuous function, being convex on $R^n$) for all $\delta > 0$, therefore $(z, u) \in Q(t, x)$, q.e.d.

REMARK 2. Corollary 1 extends the existence result in [6] (allowing $f$ to be discontinuous in $x$) and gives the further variational approximation property of $P_\infty$ by means of $P_k$ (which can be considered as a « stability » property of $P_\infty$ in a variational sense).

## 4. Problems with asymptotic conditions.

Very often problems of the above type are encountered, especially in many applications, with pointwise constraints at infinity on the state. Such asymptotic conditions increase in a essential way the complexity of the problem: for example we can solve the problem

$$\min \int_0^{+\infty} \dot{x}^2\, dt\,, \qquad x(0) = 0$$

but no solution exists for the above problem together with the asymptotic constraint

$$\lim_{t\to+\infty} x(t) = 1$$

(see [5], page 253).

In this section we denote by $P_\infty$ the following optimal control problem: minimize

$$\int_a^{+\infty} f(s, x, u)\, ds$$

on the set of pairs $(x, u)$, with $u$ measurable, $x$ locally absolutely continuous on $[a, +\infty)$, such that

$$\text{(19)} \qquad \begin{cases} \dot{x}(t) = g\big(t, x(t), u(t)\big) & \text{for a.e. } t \geqslant a\,, \\ x(a) = 0\,, \end{cases}$$

where a is a fixed number,

$$\text{(20)} \qquad \big(t, x(t)\big) \in A \quad \text{for all } t \geqslant a\,; \qquad \lim_{t\to+\infty} x(t) \in B\,;$$

$$\text{(21)} \qquad u(t) \in V\big(t, x(t)\big) \quad \text{for a.e. } t \geqslant a.$$

$\{c_k\}$ is a given decreasing sequence, $c_k \to 0$, and we set

$$B_k = \{z \in R^n \colon d(z, B) \leqslant c_k\}\,.$$

$P_k$ denotes the following problem: minimize

$$\int_a^b f(s, x, u)\, ds$$

on pairs $(u, x)$ defined in $[a, b]$, $u$ measurable, $x$ absolutely continuous, such that (19), (20), (21) hold in $[a, b]$, and moreover

$$(22) \qquad k \leqslant b \leqslant k+1\,; \qquad x(t) \in B_p \text{ if } p \leqslant t \leqslant p+1 \leqslant k,\ x(t) \in B_k \text{ if } k \leqslant t \leqslant b.$$

We are given $a_k \geqslant 0$, $a_k \to 0$, $(u_k, x_k)$, defined in $[a, b_k]$, such that for every $k$

$$\int_a^{b_k} f(s, x_k, u_k)\, ds \leqslant \inf P_k + a_k\,.$$

THEOREM 2. For a subsequence, $P_k$ has solutions. There exists a solution $(u_0, x_0)$ of $P_\infty$ such that $\dot{x}_k \rightharpoonup \dot{x}_0$ in $L^1$, $x_k \to x_0$ uniformly on $[a, +\infty)$,

$$\int_a^{b_k} f(s, x_k, u_k)\, ds \to \int_a^{+\infty} f(s, x_0, u_0)\, ds = \min P_\infty\,,$$

if the assumptions of theorem 1 holds, $B$ is compact and moreover

$$(23) \qquad f(t, x, u) \geqslant T|u| + q(t)\,, \qquad \text{for a.e.t, all } x \text{ and } u,$$

$T > 0$, $\int_a^{+\infty} q\, ds$ and $\int_a^{+\infty} C\, ds$ convergent.

PROOF. If $(u, x)$ is a fixed admissible pair for $P_\infty$, then, suitably restricted on a bounded interval, it will be admissible for $P_k$, with large $k$, for a subsequence, and for such $k$ $P_k$ has solutions (as remarked at the end of the proof of theorem 1). Therefore we can assume without loss of generality $a_k = 0$ for every $k$.

Following the proof of theorem 1, if we verify that

$$(24) \qquad \lim_{z \to +\infty} \sup_k \int_z^{+\infty} |u_k|\, ds = 0\,,$$

then $\{x_k\}$ will be equicontinuous in $[a, +\infty)$ (by (7)), also uniformly bounded because we know that

$$\lim_{\text{meas}\, E\to 0}\ \sup_k \int_E |u_k|\, ds = 0 \tag{25}$$

if $E \subset (a, b)$, a fixed bounded interval, and therefore (by (24)) without restrictions on $E$: from (23)

$$T \sup_k \int_a^{+\infty} |u_k|\, ds \leqslant \sup_k \int_a^{+\infty} f(s, x_k, u_k)\, ds - \int_a^{+\infty} q\, ds < +\infty\,, \tag{26}$$

so that IV. 13.54 in [7] will be used to get $\dot{x}_k \rightharpoonup \dot{x}_0$ in $L^1(a, +\infty)$.

Let us verify (24), that is (by (26)) the following: given $\varepsilon > 0$, to find $i$ such that $\sup_{k>i} \int_z^{+\infty} |u_k|\, ds < \varepsilon$ if $z > i$. Choose $h, k, z$ such that $a \leqslant \leqslant b_h \leqslant z \leqslant h + 1 \leqslant b_{h+1} \leqslant b_k$: then (by (23))

$$T\int_z^{b_k} |u_k|\, ds \leqslant \int_z^{b_k} [f(s, x_k, u_k) - q]\, ds \leqslant \min P_k - \int_a^z f(s, x_k, u_k)\, ds - \int_z^{b_k} q\, ds \leqslant \min P_k - \min P_h - \int_z^{b_k} q\, ds\,.$$

Given $\epsilon > 0$, we can find $i$ such that if $k > i$, $h > i$, $z > i$

$$|\min P_k - \min P_h| < \varepsilon\,, \qquad \left|\int_z^{b_k} q\, ds - \int_z^{+\infty} q\, ds\right| < \varepsilon\,, \qquad \left|\int_z^{+\infty} q\, ds\right| < \varepsilon\,,$$

since $\{\min P_k\}$ is nondecreasing and bounded.

Therefore we can find $i$ such that

$$\int_z^{+\infty} |u_k| ds < \varepsilon \quad \text{if } k > i \text{ and } z > i\,,$$

and (24) is true. By (24) and (7), given $\varepsilon > 0$ there exists some $\delta > 0$

such that for any $t', t'' > \delta$

$$\sup_k |x_k(t') - x_k(t'')| < \varepsilon$$

so that $\lim_{t\to+\infty} x_k(t)$ exists, and (by compactness of $B$) we assume that $\{\lim_{t\to+\infty} x_k(t)\}$ converges. Since $x_k \to x_0$ uniformly on $[a, +\infty)$, form Moore's theorem on iterated limits (or directly from the above inequality) there exists

$$\lim_{t\to+\infty} x_0(t) = \lim_{t\to+\infty} \lim_k x_k(t) = \lim_k \lim_{t\to+\infty} x_k(t) \in B$$

since $B$ is a closed set, q.e.d.

About the simplest free problem of the calculus of variations on unbounded intervals with asymptotic conditions, (therefore $g(t, x, u) = u$, $V(t, x) = R^n$) we deduce

COROLLARY 2. The free problem $P_\infty$ has a solution $x_0$, and for some subsequence we have $\dot{x}_k \rightharpoonup \dot{x}_0$ in $L^1(a, +\infty)$, $x_k \to x_0$ uniformly in $[a, +\infty)$,

$$\int_a^{b_k} f(s, x_k, \dot{x}_k)\, ds \to \min P_\infty = \int_a^{+\infty} f(s, x_0, \dot{x}_0)\, ds ,$$

if the assumptions of corollary 1 together with (23) of theorem 2 hold. The proof of corollary 2 can be obtained from theorem 2 in the same way corollary 1 was deduced from theorem 1.

REMARK 3. (23) of theorem 2 cannot completely be removed, as the example at the beginning of this section shows. Theorem 2 can be generalized to not fixed initial conditions as $(a, x(a)) \in B_1$, $B_1$ compact. Obviously (23) implies (18).

Given $P_\infty$ as in section 3, $P_k$ can be defined also with fixed end times (that is, $t_2 = k$), to get the corresponding variational approximations results.

## 5. Multiple integrals on unbounded regions.

In this section we denote by $W^{1,s}(\Omega)$ the Banach space of functions $u \in L^s(\Omega)$ such that the first distrbutional partial derivatives belong to $L^s(\Omega)$, equipped with the norm $\|u\|_{L^s} + \|u_x\|_{L^s}$ ($u_x$ is the gradient of $u$). Moreover $W_0^{1,s}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,s}(\Omega)$. Integrands $f$ are considered as functions of $(x, u, p)$. If $\Omega^*$ is an open bounded subset of $\Omega$, and $u \in W_0^{1,s}(\Omega^*)$, we sometimes assume $u$ defined on $\Omega \setminus \Omega^*$ by putting $u(x) = 0$ if $x \notin \Omega^*$. As before, integrals over unbounded sets are meant in the improper sense.

In this section $P$ denotes the following problem: minimize

$$\int_\Omega f(x, u, u_x)\, dx ,$$

on

$$W_0^{1,s}(\Omega) ,$$

$\Omega$ being an open unbounded set in $R^n$, $s > 1$.

Given a sequence $\{\Omega_k\}$ of open bounded subsets exhausting $\Omega$ (that is, $\Omega_k \subset \Omega$ for all $k$, and given $\Omega^*$ a bounded subset of $\Omega$, then $\Omega^* \subset \Omega_k$ for large $k$), $P_k$ will denote the following problem: minimize

$$\int_{\Omega_k} f(x, u, u_x)\, dx$$

on

$$W_0^{1,s}(\Omega_k) .$$

Given $u_k$ admissible for $P_k$, and $c_k \to 0$, $c_k \geqslant 0$, such that

$$\int_{\Omega_k} f(x, u_k, u_{kx})\, dx \leqslant \inf P_k + c_k \quad \text{for all } k ,$$

we can prove

THEOREM 3. *$P$ has a solution $u_0$ (and $P_k$ has solutions for all $k$) such that, for a subsequence,*

$$u_k \rightharpoonup u_0 \quad \text{in } W_0^{1,s}(\Omega) ,$$

$$\int_{\Omega_k} f(x, u_k, u_{kx})\, dx \to \int_\Omega f(x, u_0, u_{0x})\, dx = \min P ,$$

if we assume that

(27) $f$ is Borel measurable, lower semicontinuous with respect to $u$ uniformly in $p$ on compact sets, convex in $p$; $\int_\Omega f(x, 0, 0)\,dx < +\infty$;

(28) for every open $\Omega^* \subset \Omega$ $\int_{\Omega^*} f(x, u, u_x)\,dx \to +\infty$ as $\|u\|_{W_0^{1,s}(\Omega^*)} \to +\infty$

moreover $f(x, u, p) \geqslant c(x)$ for all $u$, $p$ and a.e. $x \in \Omega$, with $\int_\Omega c\,dx$ convergent.

PROOF. We assume, without loss of generality, $f(x, 0, 0) = 0 \leqslant \leqslant f(x, u, p)$ for a.e. $x$, all $u$ and $p$. From slight extensions of well known existence theorems (see [13]) we get existence for $P_k$.

Given $u$ admissible for $P_k$, set

$$\bar{u}(x) = \begin{cases} u(x)\,, & \text{if } x \in \Omega_k\,, \\ 0\,, & \text{if } x \in \Omega \setminus \Omega_k\,, \end{cases}$$

and denote by $\|\cdot\|$ the norm either in $W_0^{1,s}(\Omega)$ or in $W_0^{1,s}(\Omega_k)$. If $v$ is admissible for $P$, then 0 is admissible for $P_k$ for any $k$, and

$$\limsup \int_{\Omega_k} f(x, 0, 0)\,dx \leqslant \int_\Omega f(x, v, v_x)\,dx\,. \tag{29}$$

Since for all $k$

$$\int_\Omega f(x, \bar{u}_k, \bar{u}_{kx})\,dx = \int_{\Omega_k} f(x, u_k, u_{kx})\,dx$$

we see, by (28), that $\sup \|\bar{u}_k\| < +\infty$, therefore $\sup \|u_k\| < +\infty$, so that there exists $u_0 \in W_0^{1,s}(\Omega)$, such that (for a subsequence)

$$u_k \rightharpoonup u_0 \quad \text{in } W_0^{1,s}(\Omega).$$

By theorem 0 and (29) we need only prove that

(30) $v_k$ admissible for $P_k$, $v$ for $P$, $\sup_k \int_{\Omega_k} f(x, v_k, v_{kx})\,dx < +\infty$,

$$v_k \rightharpoonup v \quad \text{in } W_0^{1,s}(\Omega) \text{ implies}$$

$$\liminf \int_{\Omega_k} f(x, v_k, v_{kx})\,dx \geqslant \int_\Omega f(x, v, v_x)\,dx\,.$$

As in the proof of Corollary 1, (27) implies that

$$u \to Q(x, u) = \{(z, p) \in R^{n+1}: z \geqslant f(x, u, p)\}$$

is a regular multifunction.

A.e. in $\Omega$ and for all $k$

$$(31) \qquad \big(f(x, v_k(x), v_{kx}(x)), v_k(x)\big) \in Q\big(x, y_k(x)\big)\,.$$

Moreover, by (28), we can assume (taking again a perhaps new subsequence)

$$(32) \qquad \sum_{j=1}^{r} \alpha_{ij} \int_{\Omega_{q_j+i}} f(x, v_{q_j+i}, v_{q_j+ix})\, dx \to \liminf \int_{\Omega_k} f(x, v_k, v_{kx})\, dx\,.$$

By a theorem of Mazur, given $j$ we can find $r = r(j)$, $q = q(j)$, numbers $\alpha_{ij} \geqslant 0$, $\sum_{j=1}^{r} \alpha_{ij} = 1$, $q_{j+i} > q_j + r(j)$, such that

$$\sum_{j=1}^{r} \alpha_{ij} v_{q_j+i} \to v \quad \text{in } W_0^{1,s}(\Omega) \text{ and a.e. in } \Omega.$$

Therefore by Egorov-Severini theorem, given $\varepsilon > 0$ and $\Omega^*$ an open bounded subset of $\Omega$, there exists $\Omega_\varepsilon \subset \Omega^*$ such that

$$(33) \qquad \operatorname{meas}(\Omega^* \setminus \Omega_\varepsilon) < \varepsilon\,, \qquad \sum_{i=1}^{r} \alpha_{ij} v_{q_j+i} \to v \text{ uniformly on } \Omega_\varepsilon.$$

If follows that given $\delta > 0$, for large $k$ and a.e. $x \in \Omega_\varepsilon$

$$(34) \qquad Q\big(x, v_k(x)\big) \subset Q\big(x, v(x), \delta\big) \subset \overline{\text{co}}\, Q\big(x, v(x), \delta\big)\,.$$

Setting

$$f^*(x) = \liminf_i \sum_{j=1}^{r} \alpha_{ij} f\big(x, v_{q_j+i}(x), v_{q_j+ix}(x)\big)\,,$$

we see that $f^*(x)$ is a.e. finite and a.e. in $\Omega_\varepsilon$

$$(35) \qquad \big(f^*(x), v(x)\big) \in Q\big(x, v(x), \delta\big)$$

by (27), (31), (33), (34). $\varepsilon$ being arbitrary we have

$$(36) \qquad f^*(x) \geqslant f(x, v(x), v_x(x)) \quad \text{a.e. in } \Omega.$$

Given $\Omega^*$ any bounded open subset of $\Omega$, by (36), remembering (32) and Fatou's lemma

$$\int_{\Omega^*} f(x, v, v_x)\, dx \leqslant \int_{\Omega^*} f^*\, dx \leqslant \liminf \int_{\Omega_k} f(x, v_k, v_{kx})\, dx ,$$

so that (30) is proved. q.e.d.

In the next corollary we set

$$L(u) = \sum_{i,j=1}^{n} f_{p_i p_j}(x, u, u_x) u_{x_i} u_{x_j} + \sum_{i=1}^{n} f_{p_i u}(x, u, u_x) u_{x_i} + \\ + \sum_{i=1}^{n} f_{p_i x_i}(x, u, u_x) - f_u(x, u, u_x) = \sum_{i=1}^{u} \frac{d}{dx_i} f_{p_i}(x, u, u_x) - f_u(x, u, u_x) .$$

$u$ is a (weak) solution of $L(u) = 0$ in $\Omega$, $u = 0$ in $\partial\Omega$, if $u \in W_0^{1,s}(\Omega)$ and

$$\int_\Omega \Big[\sum_{i=1} f_{p_i}(x, u, u_x) z_{x_i} + f_u(x, u, u_x) z\Big]\, dx = 0 \qquad \text{for every } z \in C_0^\infty(\Omega).$$

COROLLARY 3. With the same assumptions of Theorem 3, suppose moreover that $\Omega$ has the cone property, that $f_{p_i p_k}$, $f_{p_i u}$, $f_{uu}$, $f_u$, $f_p$ are continuous (for all $i$, $k$), $f$ satisfies the assumptions (2.1) with $m = s > 1$, (2.4), (2.5), (2.6) on pages 324-325 of [9], and

$$(37) \qquad \sum_{i,j=1}^{n} f_{p_i p_j}(x, u, p)\, \lambda_i\, \lambda_j + 2 \sum_{i=1}^{n} f_{p_i u}\, \lambda_i\, \lambda_0 + f_{uu}\, \lambda_0^2 > 0$$

if $\lambda \neq 0$, for a.e. $x \in \Omega$, every $u$ and $p$.

Then the Dirichlet problem

$$L(u) = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \partial\Omega$$

has a unique solution $u_0$, which can be obtained as weak limit in $W_0^{1,s}(\Omega)$

of the unique solution $u_k$ of

$$L(u) = 0 \quad \text{in } \Omega_k, \qquad u = 0 \quad \text{in } \partial\Omega_k,$$

as $k \to \infty$.

PROOF. $L(u) = 0$ in $\Omega_k$ and $u \in W_0^{1,s}(\Omega_k)$ iff $\int_{\Omega_k} f(x, u, u_x)\,dx = \min P_k$, and the same statement is true for $\Omega$ (see [9]), moreover (37) implies uniqueness of solutions of the Dirichlet problems, q.e.d.

REMARK 4. For linear elliptic problems (but not necessarily Euler's equations) a convergence theorem, analogous to corollary 3, is proved in [2].

REMARK 5. In corollary 3 with $s = 2$ one has $u_k \to u_0$ in $W^{1,2}(\Omega)$ if $f$ is strongly convex in $(u, p)$, and this obtains if there exists $\alpha > 0$ such that for every $\lambda$

$$\sum_{i,k=1}^{n} f_{r_i v_k} \lambda_i \lambda_k + 2 \sum_{j=1}^{n} f_{v_i u} \lambda_i \lambda_0 + f_{uu} \lambda_0^2 \geqslant \alpha |\lambda|^2 .$$

Assuming that, in corollary 3, there exists a normal convex integrand $g$ such that

$$f(x, u, p) \geqslant g(x, p) \gg |\mathrm{p}|$$

(see Berliocchi-Lasry in Bull. Soc. Math. France 101 (1973), and in C. R. Acad. Sc. Paris, 274 (1972)), we see that if $z = (u, p) \in R^{n+1}$, $z \to \varphi(x, z) = f(x, u, p)$ is a positive normal integrand, $\bar{u}_k \rightharpoonup u_0$ in $W^{1,1}(\Omega)$ (same notations as in the proof of theorem 3), therefore we get

$$\int_{\Omega_k} |u_k - u_0|\,dx + \int_{\Omega_k} |u_{kx} - u_{0x}|\,dx \to 0 \qquad \text{as } k \to \infty .$$

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