RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

C. J. HIMMELBERG

Precompact contraction of metric uniformities, and the continuity of $F(t,x)$

Rendiconti del Seminario Matematico della Università di Padova, tome 50 (1973), p. 185-188

<http://www.numdam.org/item?id=RSMUP_1973__50__185_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 50 (1973)

Precompact Contraction of Metric Uniformities, and the Continuity of $F(t, x)$.

C. J. HIMMELBERG (*)

In this note we use the easily proven fact that every metric uniformity for a separable metrizable space contains a topologically equivalent precompact metric uniformity to further generalize for multifunctions the result of Scorza Dragoni [*S*] on the continuity of $F|_{T_\varepsilon \times X}$ for some closed $T_\varepsilon \subset T$ with $\mu(T - T_\varepsilon) < \varepsilon$, when $F\colon T \times X \to E$ is a function measurable in t and continuous in x. Results of this type have been obtained for multifunctions by Castaing [C-1] and Himmelberg and Van Vleck [HV]. In [C-1], it is assumed that F is a multifunction with compact values; in [HV] F need not have compact (or even closed) values, but E is taken to be Euclidean space. Here, F need not have closed values, and E will be separable metric.

THEOREM 1. Let (E, d) be a separable metric space. Then there exists a metric ϱ topologically equivalent to d such that:

a) (E, ϱ) is precompact, and

b) The uniformity on E defined by ϱ is smaller than the uniformity defined by d. (I.e., the inclusion $(X, d) \subset (X, \varrho)$ is uniformly continuous.)

REMARK. (E, ϱ) is not the precompact reflection of (E, d) in the category of uniform spaces. For example, take $E = N =$ the positive

---

(*) Indirizzo dell'A.: Dept. of Mathematics, The University of Kansas - Lawrence, Kan. 66044, U.S.A.

The research in this paper was supported by University of Kansas Research Fund Grant 3918-5038.

integers, with the discrete uniformity. Then the precompact reflection pN of N is not metrizable, since the completion of pN is the Cech compactification of N.

PROOF. It is clearly sufficient to embed (E, d) in the product of countably many unit intervals by a uniformly continuous homeomorphism. This will be done by a simple modification of the usual embedding construction. We may (and do) assume $0 < \operatorname{diam} E \leqslant 1$. Let D be a countable dense subset of E, and let $\mathfrak{B}$ be the set of all ordered pairs (U, V) of concentric open balls in E with center in D and distinct rational radii such that $U \subset V$ and $E - V \neq \emptyset$. $\mathfrak{B}$ is a countable set. For each $(U, V) \in \mathfrak{B}$, define $f_{UV}\colon E \to I = [0, 1]$ by $f_{UV}(x) = d(x, U)/\big(d(x, U) + d(x, E - V)\big)$. It is easily checked that each f_{UV} is uniformly continuous (in fact, if r is the difference of the radii of U and V, then $|f_{UV}(x) - f_{UV}(y)| < (2/r^2)\, d(x, y)$), and that the collection $F = \{f_{UV} | (U, V) \in \mathfrak{B}\}$ separates points and closed sets. It follows that the embedding $e\colon E \to I^F$ defined in the customary way by $e(x)(f_{UV}) = f_{UV}(x)$, is a uniformly continuous homeomorphism.

Now let T be a locally compact Hausdorff space with Radon measure μ, let X be a Polish (= complete separable metric) space, and let E be a separable metric space with metric d. Define the Hausdorff pseudometric H_d on the set $S(E)$ of all non-empty subsets of E by

$$H_d(A, B) = \operatorname{lub}\{d(x, B), d(y, A) | x \in A, y \in B\}\,.$$

H_d may take on infinite values, but this causes no difficulties. We define a multifunction $G\colon X \to E$ (i.e., for each $x \in X$, $G(x)$ is a non-empty subset of E) to be *continuous* iff G is continuous as a function from X to $S(E)$, when $S(E)$ is topologized by H_d. G is *upper* (*lower*) *semicontinuous* iff $G^{-1}(B) = \{x | G(x) \cap B \neq \emptyset\}$ is closed (open) for each closed (open) subset B of E. Recall that, if E is compact metric and G has closed values, then G is continuous iff G is both upper and lower semicontinuous. If the multifunction G is from T to E instead of from X to E, then G is *measurable* (*weakly measurable*) iff $G^{1-}(B)$ is μ-measurable for each closed (open) subset B of E.

THEOREM. With T, X, E as above, let $F\colon T \times X \to E$ be a multifunction such that $t \to F(t, x)$ defines a measurable multifunction for each $x \in X$, and $x \to F(t, x)$ defines a continuous multifunction for each $t \in T$. Then for each $\varepsilon > 0$ there exists a closed subset T_ε of T such that $\mu(T - T_\varepsilon) < \varepsilon$ and $F|_{T_\varepsilon \times X}$ is lower semicontinuous. If, in addition, F is assumed to have closed values, then $F|_{T_\varepsilon\times X}$ has closed graph and is lower semicontinuous. (If F has compact values, then $F|_{T_\varepsilon\times X}$ is continuous [C-1, Remark 2].)

PROOF. Let ϱ be the totally bounded metric for E given by Theorem 1. It follows easily that the inclusion map $(S(E), H_d)\subset(S(E), H_\varrho)$, where H_d, H_ϱ are the Hausdorff pseudometrics defined by d, ϱ, respectively, is continuous, in fact, uniformly continuous with the same modulus of uniform continuity as the inclusion $(E, d)\subset(E, \varrho)$. Thus if E is metrized by ϱ, it remains true that $F\colon T\times X\to E$ is measurable in t and continuous in x. For the remainder of the proof we assume that E is metrized by ϱ. The argument is the same as in [HV, Theorem 1], but we include it here for completeness.

Let $\bar{E}$ be the completion of E and define $\bar{F}\colon T\times X\to\bar{E}$ by $\bar{F}(t, x) = \overline{F(t, x)}$, where here and throughout this proof all closures are with respect to $\bar{E}$. Note that $\bar{E}$ is compact metric.

Then $\bar{F}$ is weakly measurable (and hence measurable, by [C-2, Theorem 1.1]) in t for each x, since for each open subset B of $\bar{E}$, we have

$$\{t|\overline{F(t, x)}\cap B\neq\emptyset\} = \{t|F(t, x)\cap B\neq\emptyset\}\ .$$

Also $\bar{F}(t, x)$ is continuous in x for each t with respect to the Hausdorff metric $\bar{H}_\varrho$ on the set $\mathcal{C}(\bar{E})$ of all non-empty compact subsets of $\bar{E}$, since $\bar{H}_\varrho(\overline{F(t, x)}, \overline{F(t, y)}) = H_\varrho(F(t, x), F(t, y))$.

It follows by [C-1, Theorem] that for each $\varepsilon>0$ there exists a closed subset T_ε of T such that $\mu(T-T_\varepsilon)<\varepsilon$ and $\bar{F}|_{T_\varepsilon\times X}$ is continuous in t and x jointly. Equivalently, $\bar{F}|_{T_\varepsilon\times X}\colon T_\varepsilon\times X\to E$ is both upper and lower semicontinuous.

But lower semicontinuity for $\bar{F}|_{T_\varepsilon\times X}$ is equivalent to lower semicontinuity for $F|_{T_\varepsilon\times X}$: So $F|_{T_\varepsilon\times X}$ is lower semicontinuous.

Finally, if F has closed values, then Graph $F|_{T_\varepsilon\times X} = (T_\varepsilon\times X\times E)\cap$ $\cap$ Graph $\bar{F}|_{T_\varepsilon\times X}$, and the latter set is closed since $\bar{F}|_{T_\varepsilon\times X}$ is upper semicontinuous.

REFERENCES

[C-1] C. CASTAING, *Sur le graphe d'une multi-application Souslinienne (Mesurable)*, Faculté des Sciences de Montpellier, Mathematiques (1969).

[C-2] C. Castaing, *Sur les multi-applications mesurables*, Revue Française d'Informatique et de Recherche Operationelle, **1** (1967), 91-126.

[HV] C. Himmelberg - F. S. Van Vleck, *Lipschitzian generalized differential equations*, Rend. Sem. Mat. Univ. Padova, to appear.

[S] G. Scorza-Dragoni, *Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile*, Rend. Sem. Mat. Univ. Padova, **17** (1948), 102-106.

Manoscritto pervenuto in redazione il 1° febbraio 1973.