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AREA CONTRACTIONS IN THE PLANE

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The Picard-Banach contraction principle admits generalizations in which volume contractions appear instead of distance contractions, as in the original theorem. Our purpose in this note is to obtain such results in the plane case; this will be completely done for bounded domains of definition, the nonbounded case being only touched by Theorems 2 and 3. To generalize these results for the n-dimensional case is routine.

We restrict ourselves to considering area contractions in a quite usual Euclidean plane $\boldsymbol{R}^2$. There, we have a set M and a function $f : M \to M$ which will be called *area contraction* because, for some number $\alpha \in (0, 1)$,

$$A(f(a), f(b), f(c)) \leq \alpha A(a, b, c)$$

for each triple of points $a, b, c \in M$,

$$A : \boldsymbol{R}^2 \times \boldsymbol{R}^2 \times \boldsymbol{R}^2 \to \boldsymbol{R}_+$$

being the triangle-area function on $\boldsymbol{R}^2$.

The sequence of points $\{f^n(u)\}_{n=0}^{\infty}$ is called the *orbit* of $u \in M$, and its set of limit points is denoted by $L(u)$. Put $\mathfrak{L} = \bigcup L(u)$.

A set in $\boldsymbol{R}^2$ will be called *linear* if it is included in some straight line.

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THEOREM 1. *If M is bounded, then $\mathfrak{L}$ is linear.*

PROOF. Suppose $\mathfrak{L}$ is not linear, i.e. there exists a set $\{x, y, z\} \subset \mathfrak{L}$ which is not linear. Let X, Y, Z be neighbourhoods of x, y, z respectively, such that $I>0$, where

$$I = \inf_{\substack{a \in X \\ b \in Y \\ c \in Z}} A(a, b, c).$$

Let

$$J = \sup_{a, b, c \in M} A(a, b, c)$$

and

$$N = \left[\log_\alpha \frac{I}{J} \right] + 1.$$

Suppose $x \in L(u)$, $y \in L(v)$, $z \in L(w)$. Then there exist three natural numbers p, q, $r \geq N$ such that $f^p(u) \in X$, $f^q(v) \in Y$, $f^r(w) \in Z$. It follows

$$A(f^p(u), f^q(v), f^r(w)) \leq \alpha^N A(f^{p-N}(u), f^{q-N}(v), f^{r-N}(w)) \leq \alpha^N J < I,$$

which provides a contradiction.

For M nonbounded, one may still conjecture that $\mathfrak{L}$, if nonempty, is linear. We can only prove here that under certain additional assumptions, $\mathfrak{L}$ is linear indeed.

THEOREM 2. *Il two points u and v from M have bounded orbits and $L(u)$ is disjoint from $L(v)$, then $\mathfrak{L}$ is linear.*

PROOF. Following Theorem 1, $L(u) \cup L(v)$ is linear. Suppose there exists a point w in $\mathfrak{L}$ which does not belong to the line containing $L(u) \cup L(v)$. Then $A(w, x, y) > 0$ for every $x \in L(u)$, $y \in L(v)$. The product $\{w\} \times L(u) \times L(v)$ being compact, the infimum

$$\nu = \inf_{\substack{x \in L(u) \\ y \in L(v)}} A(w, x, y)$$

is attained, hence $\nu > 0$.

Take the open sets W, X, Y respectively including $\{w\}$, $L(u)$, $L(v)$, such that $A(a, b, c)>\nu/2$ for each $a\in W$, $b\in X$, $c\in Y$. For some natural numbers N_1 and N_2, $n\geq N_1$ implies $f^n(u)\in X$, and $n\geq N_2$ implies $f^n(v)\in Y$.

Let $w\in L(z)$. For some natural number N_3, $n\geq N_3$ yields

$$A(f^n(u),\ f^n(v),\ f^n(z))\leq\nu/2.$$

Taking now $n'\geq \max\{N_1, N_2, N_3\}$ such that $f^{n'}(z)\in W$,

$$A(f^{n'}(u),\ f^{n'}(v),\ f^{n'}(z))>\nu/2$$

($f^{n'}(u)$ and $f^{n'}(v)$ respectively lying in X and Y), which contradicts the preceding inequality.

THEOREM 3. *If $\mathfrak{L}$ is linear, x and y are two distinct points in $\mathfrak{L}$, u, $v\in M$ are such that $x\in L(u)$, $y\in L(v)$, and the orbits of u and v are bounded (u and v my coincide), then the line including $\mathfrak{L}$ is a fixed line under the mapping f.*

PROOF. Consider the line $\delta\supset\mathfrak{L}$. Suppose $f(M\cap\delta)\not\subset\delta$. Then there exists a point $z\in M\cap\delta$ such that $f(z)\notin\delta$.

Take the neighbourhoods X of x and Y of y, and the number $\nu>0$ such that

$$A(a,\ b,\ f(z))>\nu$$

for each couple of points $(a, b)\in X\times Y$.

Let Δ be an open set including $L(u)\cup L(v)\cup\{z\}$ such that

$$A(a,\ b,\ c)\leq\nu$$

for each three points a, b, $c\in\Delta$. There is a number N such that $n\geq N$ implies $f^n(u)$, $f^n(v)\in M\cap\Delta$.

Consider now two natural numbers p, $q\geq N+1$ such that $f^p(u)\in X$, $f^q(v)\in Y$. Then

$$A(f^{p-1}(u),\ f^{q-1}(v),\ z)\leq\nu$$

and

$$A(f^p(u),\ f^q(v),\ f(z)) > \nu,$$

which both together contradict the contraction inequality.

COROLLARY. *If $\mathfrak{L}$ is not a single point, and M is bounded, then the line including $\mathfrak{L}$ is a fixed line under the mapping f.*

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