

RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

S. ZAIDMAN

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Rendiconti del Seminario Matematico della Università di Padova,
tome 46 (1971), p. 191-198

http://www.numdam.org/item?id=RSMUP_1971__46__191_0

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ON A CERTAIN APPROXIMATION PROPERTY FOR FIRST-ORDER ABSTRACT DIFFERENTIAL EQUATIONS

S. ZAIDMAN *)

Introduction.

This paper is closely related with some previous work of ours [1], [2]. There we insisted mainly on the so called « global existence theorems » and a main tool in the proof was a certain density result. Here we shall concentrate ourselves on density (or approximation) theorem; what we prove is essentially a generalization of Lemma 6-bis in [1]. The condition S in [1] is here replaced by condition S' below; the proofs are essentially those indicated in our Lecture Notes [2]; the condition S' is implied by S (Lemma 1 and Lemma 5 in [1]) and this indicates how our Theorem below is (slightly) more general than Lemma 6-bis in [1].

§ 1. Let H be a hilbert space; $(,)$ and $||$ are the notations for scalar product and for the norm.

Consider in H a linear closed operator A with domain D_A and let A^* be the adjoint of A .

For any open interval $a < t < b$ of the real line, we define a class of test-functions (vector-valued); precisely

$$K_{A^*}(a, b) = \{ \varphi(t), a < t < b \rightarrow D_{A^*}; \varphi \in C^2(a, b; H); \\ A^* \varphi \in C(a, b; H); \text{supp } \varphi \text{ is compact in } (a, b) \}.$$

*) Indirizzo dell'A.: Université de Montréal, Case Postale 6128, Montréal, 101 Canada.

This research is supported by a grant of the N.R.C. of Canada.

Similarly is defined K_{A^*} , by taking (a, b) = whole real axis, and more generally the classes K_B , $K_B(a, b)$, for any linear closed operator B in H .

Let us give now a function $f(t)$, $a < t < b \rightarrow H$, belonging to $L^2(a, b; H)$, i.e. square Bochner integrable in (a, b) , and a function $u(t)$ with the same property.

We say that $u(t)$ is weak solution on $a < t < b$ of the differential equation

$$(1.1) \quad u'(t) = Au(t) + f(t)$$

when the integral identity

$$(1.2) \quad \int_a^b (u(t), \varphi'(t) + (A^*\varphi)(t)) dt = - \int_a^b (f(t), \varphi(t)) dt$$

is verified, $\forall \varphi \in K_{A^*}(a, b)$.

Let us take now an arbitrary positive number $T > 0$, and denote by V_T the set of weak solutions in $(-T, T)$ of the homogeneous differential equation

$$(1.3) \quad u'(t) = Au(t);$$

also call V_∞ the set of weak solutions of (1.3) on the whole real axis¹⁾. We give now some more definitions;

DEFINITION 1. The abstract differential operator $\frac{d}{dt} - A$ has the *approximation property* if for any pair of positive numbers $T_1 < T_2$, the set V_∞ is dense in V_{T_2} , in the norm of $L^2(-T_1; T_1; H)$.

DEFINITION 2. The abstract differential operator $\frac{d}{dt} + A^*$ has the *support property* if:

¹⁾ Precisely, $u \in V_\infty$ if $u \in L^2_{loc}(-\infty, \infty; H)$ and the relation

$$\int_{-\infty}^{\infty} (u(t), \varphi'(t) + (A^*\varphi)(t)) dt = 0 \text{ holds, } \forall \varphi \in K_{A^*}.$$

- i) $u \in L^2_{\text{loc}}(-\infty, \infty; H)$, $\text{supp } u$ -compact
- ii) $f \in L^2_{\text{loc}}(-\infty, \infty; H)$, $\text{supp } f \subset [a, b]$
- iii) $u' + A^*u = f$ on $-\infty < t < \infty$ in weak sense, imply
- iv) $\text{supp } u \subset [a, b]$ too.

Let us assume also that the following holds:

PROPERTY (3). For any finite interval $a < t < b$ there is a constant $C_{a,b}$ such that

$$(1.4) \quad \|\Psi\|_{L^2(a,b;H)} \leq C_{a,b} \|\Psi' + A^*\Psi\|_{L^2(a,b;H)}, \quad \forall \Psi \in K_{A^*}(a, b)$$

is verified.

Now we finally define condition S'

DEFINITION 3. The linear closed operator A satisfies condition S' when

$$(1.5) \quad \frac{d}{dt} + A^* \text{ has the support property, and}$$

$$(1.6) \quad \text{property (3) holds.}$$

Then we have

THEOREM. If A verifies condition S' then $\frac{d}{dt} - A$ has the approximation property.

The proof is given below.

§ 2. We have

MAIN LEMMA. If A verifies condition S' , and if we take three positive numbers $T_1 < T_2 < T_3$, then V_{T_3} is dense in V_{T_2} in the norm of $L^2(-T_1, T_1; H)$.

As well-known, it will be enough to prove that:

$$(2.1) \quad \int_{-T_1}^{T_1} (v(t), u(t)) dt = 0 \quad \forall u \in V_{T_3}$$

implies

$$(2.2) \quad \int_{-T_1}^{T_1} (v(t), w(t)) dt = 0 \quad \forall w \in V_{T_2}$$

where $v(t)$ is an arbitrary function in $L^2(-T_1, T_1; H)$.

If we define outside $(-T_1, T_1)$, $\tilde{v}(t) = \theta$, $\tilde{v}(t) = v(t)$, $-T_1 < t < T_1$, then (2.2) equals

$$(2.3) \quad \int_{-T_2}^{T_2} (\tilde{v}(t), w(t)) dt = 0, \quad \forall w \in V_{T_2}.$$

If we take a sequence of mollifiers $(\alpha_n)_{n=1}^\infty \rightarrow \delta$ -the Dirac distribution, and if we form convolutions

$$(2.4) \quad (\tilde{v} * \alpha_n)(t) = \int_{|t-\zeta| < \frac{1}{n}} \tilde{v}(\zeta) \alpha_n(t-\zeta) d\zeta$$

then $\tilde{v} * \alpha_n \in C^\infty(-\infty, \infty; H)$ and $\text{supp}(\tilde{v} * \alpha_n) \subset (-T_2, T_2)$ for n large enough. Furthermore $\|\tilde{v} * \alpha_n - \tilde{v}\|_{L^2(-T_2, T_2; H)} \rightarrow 0$ as $n \rightarrow \infty$.

Let now $\mathfrak{N} \subset L^2(-T_3, T_3; H)$ be defined as the image of $K_{A^*}(a_3, b_3)$ through the operator $\frac{d}{dt} + A^*$, i.e.

$$(2.5) \quad \mathfrak{N} = \{\varphi' + A^*\varphi, \varphi \in K_{A^*}(-T_3, T_3)\}.$$

We have

PROPOSITION 1. $\tilde{v}(t) \in \text{closure } \mathfrak{N}$ (in $L^2(-T_3; T_3; H)$).

As well known it is enough to show the following. For any $h(t) \in L^2(-T_3, T_3; H)$ such that $h(t) \perp \mathfrak{N}$, it follows $h(t) \perp \tilde{v}(t)$ (in $L^2(-T_3, T_3; H)$).

Thus, we assume that

$$(2.6) \quad \int_{-T_3}^{T_3} (h(t), \varphi'(t) + (A^*\varphi)(t)) dt = 0 \quad \forall \varphi \in K_{A^*}(-T_3, T_3).$$

This means precisely that $h(t) \in V_{T_3}$; then

$$\int_{-T_3}^{T_3} (h(t), \tilde{v}(t)) dt = \int_{-T_1}^{T_1} (h(t), v(t)) dt = 0$$

in view of (2.1).

Now, the Proposition means that we have for a sequence

$$(\Psi_n)_{n \rightarrow \infty} \in \mathfrak{N}\mathfrak{C}, \quad \lim_{n \rightarrow \infty} \|\tilde{v} - \Psi_n\|_{L^2(-T_3, T_3; H)} = 0;$$

here $\Psi_n = \varphi'_n + A^* \varphi_n$, where $\varphi_n \in K_{A^*}(-T_3, T_3)$, hence

$$(2.7) \quad \lim_{n \rightarrow \infty} \int_{-T_3}^{T_3} \|\tilde{v}(t) - (\varphi'_n + A^* \varphi_n)\|^2 dt = 0.$$

At this stage we shall apply property (\mathfrak{B}); as the sequence $\varphi'_n + A^* \varphi_n$ is convergent in $L^2(-T_3, T_3; H)$, the sequence $(\varphi_n)_{n \rightarrow \infty}$ will be a Cauchy sequence in $L^2(-T_3, T_3; H)$; let $\Phi(t) = \lim \varphi_n(t)$ in $L^2(-T_3, T_3; H)$; so $\Phi(t)$ is well-defined on $-T_3 < t < T_3$; put also $\Phi(t) = 0$ outside this interval. We have

PROPOSITION 2. The $\text{supp } \Phi(t) \subset [-T_1, T_1]$.

Consider in fact $\frac{d}{dt} \Phi + A^* \Phi$ on $-\infty < t < \infty$ in weak sense, i.e. the integral

$$\int_{-\infty}^{\infty} (\Phi(t), \zeta'(t) - (A\zeta)(t)) dt, \quad \forall \zeta \in K_A$$

(use $A^{**} = A$, for densely defined, linear closed operators in Hilbert spaces).

We have then

$$(2.8) \quad \int_{-\infty}^{\infty} (\Phi(t), \zeta'(t) - (A\zeta)(t)) dt = \int_{-T_3}^{T_3} (\Phi(t), \zeta'(t) - (A\zeta)(t)) dt =$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \int_{-T_3}^{T_3} (\varphi_n(t), \zeta'(t) - (A\zeta)(t)) dt = - \lim_{n \rightarrow \infty} \int_{-T_3}^{T_3} (\varphi'_n(t) + (A^*\varphi_n)(t), \zeta(t)) dt = \\
&= - \int_{-T_3}^{T_3} (\tilde{v}(t), \zeta(t)) dt = - \int_{-\infty}^{\infty} (\tilde{v}(t), \zeta(t)) dt, \quad \forall \zeta \in K_A
\end{aligned}$$

hence $\Phi' + A^*\Phi = \tilde{v}$, in weak sense.

We can apply here the support property; remark that $\text{supp } \tilde{v} \subset [-T_1, T_1]$ hence $\text{supp } \Phi \subset [-T_1, T_1]$ too, as we assumed $\text{supp } \Phi \subset [-T_3, T_3]$ -compact in R^1 .

Now, in order to prove (2.3), it will suffice to show

$$(2.9) \quad \lim_{n \rightarrow \infty} \int_{-T_2}^{T_2} ((\tilde{v} * \alpha_n)(t), w(t)) dt = 0.$$

Let us use now (2.8) and Lemma 2.3 in [2], for $B = A^*$; we get that $(\Phi * \alpha_n)(t) \in D_{A^*}$, $\forall t$, and that

$$(2.10) \quad (\Phi * \alpha_n)'(t) + A^*(\Phi * \alpha_n)(t) = (\tilde{v} * \alpha_n)(t), \quad -\infty < t < \infty.$$

From (2.10) we obtain, for $n \geq n_0$

$$(2.11) \quad \int_{-T_2}^{T_2} (\tilde{v} * \alpha_n, w) dt = \int_{-T_2}^{T_2} ((\Phi * \alpha_n)' + A^*(\Phi * \alpha_n), w) dt = 0,$$

if we remember that $w \in V_{T_2}$, and if remark that $\Phi * \alpha_n \in K_{A^*}(-T_2, T_2)$ for $n \geq n_0$ (because $\text{supp } \Phi \subset [-T_1, T_1]$, hence $\text{supp } (\Phi * \alpha_n) \subset (-T_2, T_2)$ for large n). Hence (2.9) follows.

This will prove Main Lemma.

We can pass now to the final step: take $0 < T_1 < T_2$, a function $u \in V_{T_2}$, and an $\varepsilon > 0$. We must find $u_\varepsilon \in V_\infty$ such that

$$\|u_\varepsilon - u\|_{L^2(-T_1, T_1; H)} < \varepsilon.$$

Let us consider an increasing sequence $T_3 < T_4 < \dots$, where $T_2 < T_3$, and $\lim_{n \rightarrow \infty} T_n = \infty$.

We shall use successively the Main Lemma to triplets (T_1, T_2, T_3) , (T_2, T_3, T_4) , ... etc. We get a function $u_1(t) \in V_{T_3}$ such that

$$\|u - u_1\|_{L^2(-T_1, T_1; H)} < \frac{\varepsilon}{2}.$$

Then a function $u_2(t) \in V_{T_4}$ exists, such that

$$\|u_1 - u_2\|_{L^2(-T_2, T_2; H)} < \frac{\varepsilon}{2^2};$$

continuing this way we shall find $u_n(t) \in V_{T_{n-2}}$, such that

$$\|u_{n-1} - u_n\|_{L^2(-T_n, T_n; H)} < \frac{\varepsilon}{2^n}.$$

Now, on any finite interval $[a, b] \subset R^1$, the sequence (u_n, u_{n+1}, \dots) is well-defined for $n \geq n_0$ depending on $[a, b]$ (when $-T_{n_0} < a < b < T_{n_0}$). Then, we shall have

$$\|u_{j-1} - u_j\|_{L^2(-T_{n_0}, T_{n_0}; H)} < \frac{\varepsilon}{2^j},$$

for $j > n_0$ and this implies that $\lim_{j \rightarrow \infty} u_{n_0+j}$ exists in $L^2(a, b; H)$.

Taking $[a, b]$ successively $=[-1, 1], [-2, 2], \dots$, we shall find functions $u^1(t), u^2(t), \dots$, so that $u^p(t) \in L^2(-p, p; H)$, and

$$u^p(t) = \lim_{\substack{n \rightarrow \infty \\ n \geq n_p}} u_n(t) \text{ in } L^2(-p; p; H).$$

Remark that $u^2(t) = u^1(t)$ a.e. on $-1 \leq t \leq 1$; $u^3(t) = u^2(t)$, a.e. on $-2 \leq t \leq 2$, ... etc. and we may put, $\forall t \in (-\infty, \infty)$

$$u_\varepsilon(t) = u^j(t) \text{ for } -j \leq t \leq j.$$

Remark that $u_\varepsilon(t) \in L^2[a, b; H]$ for any finite interval $[a, b] \subset R^1$, so $u_\varepsilon \in L^2_{loc}(-\infty, \infty; H)$. We see also that

$$\|u_\varepsilon - u\|_{L^2(-T_1, T_1; H)} = \lim_{n \rightarrow \infty} \|u_n - u\|_{L^2(-T_1, T_1; H)}.$$

But

$$\begin{aligned} & \|u_n - u\|_{L^2(-T_1, T_1; H)} \leq \|u - u_1\|_{L^2(-T_1, T_1; H)} + \\ & + \|u_1 - u_2\|_{L^2(-T_1, T_1; H)} + \dots + \|u_{n-1} - u_n\|_{L^2(-T_1, T_1; H)} \leq \\ & \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \dots + \frac{\varepsilon}{2^n} = \frac{\frac{\varepsilon}{2} - \frac{\varepsilon}{2^{n+1}}}{1 - \frac{1}{2}} = \frac{\frac{\varepsilon}{2} - \frac{\varepsilon}{2^{n+1}}}{\frac{1}{2}} = \varepsilon - \frac{\varepsilon}{2^n}. \end{aligned}$$

So

$$\|u_n - u\|_{L^2(-T_1, T_1; H)} \leq \varepsilon - \frac{\varepsilon}{2^n},$$

and

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{L^2(-T_1, T_1; H)} \leq \varepsilon.$$

Finally, it remains to show that $u_\varepsilon \in V_\infty$, i.e. that $\frac{d}{dt}u_\varepsilon - Au_\varepsilon = 0$ in weak sense.

Take then an arbitrary $\varphi \in K_{A^*}$; for $i \geq i_0$ all $u_i(t)$ are defined on an interval $[a, b] \supset \text{supp } \varphi$; moreover, because $u_i(t) \in V_{T_{i+2}}$, we get

$$\int_{-\infty}^{\infty} (u_i, \varphi' + A^*\varphi) dt = \int_a^b (u_i, \varphi' + A^*\varphi) dt = 0.$$

When $i \geq i_0$ tends to ∞ , we get $\int_a^b (u_\varepsilon, \varphi' + A^*\varphi) dt = 0$, and this proves the theorem.

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