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NORMAL NEIGHBORHOOD SPACES

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The object of this paper is to extend to neighborhood spaces the well-known Urysohn's lemma for topological spaces.

DEFINITION 1. Let X be a set and k a set-valued set-function mapping the power set, of X, to itself. Then (X, k) is said to be a neighborhood space iff,

1. $k\emptyset=\emptyset$
2. $A\subset kA$ for every $A\subset X$ and
3. $kA\subset kB$ if $A\subset B\subset X$.

The neighborhood space (X, k) is said to be directed iff $k(A\cup B)=$ $=kA\cup kB$ for all $A, B\subset X$.

For a subset A of X, write $cA=X-A$.

DEFINITION 2. Let (X, k) be a neighborhood space. Take $i=ckc$. Then a set A is said be a neighborhood of a set B iff $B\subset iA$.

DEFINITION 3. A neighborhood space (X, k) is said to be normal iff $A, B\subset X$ and kA, kB are disjoint imply kA, kB have disjoint neighborhoods.

It is obvious that a neighborhood space (X, k) is normal iff $A, B\subset X$ and $kA\subset iB$ imply there is $C\subset X$ such that $kA\subset iC$ and $kC\subset iB$.

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DEFINITION 4. Let (X, k), (Y, m) be two neighborhood spaces and f a function from X to Y. Then f is said to be continuous at the point x of X iff the inverse, under f, of every neighborhood of $f(x)$ is a neighborhood of x. We will say f is continuous iff f is continuous at each point of X.

It is easily seen that f is continuous iff $fk \subset mf$.

Let R denote the reals and n the closure function of the usual topology for R. Denote by I the closed unit interval $[0, 1]$ and let p be the restriction of n to I.

LEMMA 1. Let (X, k) be a directed neighborhood space and D a dense subset of the positive reals. For each t in D let $S(t)$ be a subset of X such that

1. $\cup\{S(t) : t \in D\} = X$ and
2. $kS(t) \subset iS(u)$ if $t < u$.

Take $f(x) = \inf\{t : x \in S(t)\}$. Then f is a continuous function from (X, k) to (R, n).

PROOF. Let $x \in X$. To prove f is continuous it is enough to show that $f(x) < v$ implies $E = \{y : f(y) < v,\ y \in X\}$ is a neighborhood of x and that $u < f(x)$ implies $F = \{y : f(y) > u,\ y \in X\}$ is a neighborhood of x.

Now $f(x) < v$ implies there are w, z in D such that $f(x) < w < z < v$. Hence $x \in S(w)$. Also $S(z) \subset E$ since y in $S(z)$ implies $f(y) \leq z < v$. Therefore $x \in iS(z)$ and so E is a neighborhood of x.

Also $u < f(x)$ implies there are r, s in D such that $u < r < s < f(x)$. Then $x \in cS(s)$ since $x \in S(s)$ implies $f(x) \leq s$. Next, $y \in cS(r)$ implies $f(y) \geq r > u$ and so $y \in F$; hence $cS(r) \subset F$. Now $kS(r) \subset iS(s)$ and so $kcS(s) \subset icS(r)$. Hence F is a neighborhood of x.

The next lemma can be proved in the same way as the corresponding part of Urysohn's lemma; for instance we can use the method of proof of Lemma 4 on page 115 of Kelley [1].

LEMMA 2. Let (X, k) be a normal directed neighborhood space and A, $B \subset X$ such that kA, kB are disjoint. Then there is a continuous function f from (X, k) to (I, p) such that f is 0 on kA and 1 on kB.

DEFINITION 5. A directed neighborhood space (X, k) is said to be completely normal iff A, $B \subset X$ and kA, kB are disjoint imply there is a continuous function f from (X, k) to (I, p) such that f is 0 on kA and 1 on kB.

The next result now easily follows.

THEOREM 1. A directed neighborhood space is normal iff it is completely normal.

Define a neighborhood space for the reals R as follows. For a real number x let $\mathfrak{R}(x)$ be the family of all subsets N of R such that $\{y : y<v\}\subset N$ for some $v>x$ or $\{y : u<y\}\subset N$ for some $u<x$. For a subset A of the reals, let hA be the set of all points x such that each N in $\mathfrak{R}(x)$ intersects A. then (R, h) is a neighborhood space. Let g be the restriction of h to I.

DEFINITION 6. A neighborhood space (X, k) is said to be completely normal iff A, $B\subset X$ and kA, kB are disjoint imply there is a continuous function f from (X, k) to (I, g) such that f is 0 on kA and 1 on kB.

We then have the following result.

THEOREM 2. A neighborhood space is normal iff it is completely normal.

REFERENCES

[1] KELLEY, J. L.: *General Topology*, Princeton (1968).

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