Rendiconti del Seminario Matematico della Università di Padova

G. De Marco

M. Richter

Rings of continuous functions with values in a non-archimedean ordered field

Rendiconti del Seminario Matematico della Università di Padova, tome 45 (1971), p. 327-336

<http://www.numdam.org/item?id=RSMUP_1971__45__327_0>

RINGS OF CONTINUOUS FUNCTIONS WITH VALUES IN A NON-ARCHIMEDEAN OREDERED FIELD

G. DE MARCO *) and M. RICHTER **)

I. Introduction.

In a recent paper, [DMW], investigations have been made over the ring $C(X, F)$ of continuous functions from a topological space X into a proper subfield F of the field of real numbers. Here we show that the techniques used in [DMW] lead, with suitable modifications, to analogous results if F is a non-archimedean ordered field.

Our main concern here are the residue class fields of $C(X, F)$ and $C^*(X, F)$ (the ring of bounded functions). It does not seem easy to give a full description of these fields by known concepts. The analogue of [DMW, 2.1] reads: If a residue class field of $C^*(X, F)$ contains one new element of a completion of F, it actually contains the whole completion (Theorem 3.3). But here the analogy ends: F is never dense in a residue class field K of $C^*(X, F)$ unless $K=F$ or measurable cardinals are considered (Proposition 3.2, Corollary 3.4).

In II we describe those properties of ordered fields which will be used later. In III and IV residue class fields of $C^*(X, F)$ and $C(X, F)$ are studied and in V we take the special case that F is real closed.

No inquiry is made about the structure spaces of $C(X, F)$ and $C^*(X, F)$ since it is clear that such an analysis can be carried out as in [DMW, 1] without essential modifications.

---

*) Istituto di Matematica Applicata, Università di Padova, Padova, Italy. This work was performed while this co-author held a CNR fellowship at the University of Texas at Austin.

**) Universität Freiburg i. Br. and University of Texas at Austin.

II. General remarks on order fields.

The term « ordered » always means « totally ordered ». Throughout the whole paragraph, the letter K denotes an ordered field. With the order topology, K is a topological field. On its subsets, K induces an order and a topology, which is not, in general, the topology of the induced order.

PROPOSITION 2.1. Let F be a subfield of K. Then F (with the topology of the induced order) is a topological subfield of K if and only if F is cofinal in K. Otherwise, F is a discrete subspace of K.

PROOF. F is cofinal in K if and only if $F^+=\{x\in F : x>0\}$ is coinitial in $K^+=\{y\in K : y>0\}$. And this is clearly equivalent to the fact that the neighborhoods of 0 in F are the sets $V\cap F$, V neighborhood of 0 in K. The last assertion is then also clear.

If $S\subseteq K$, $cl_K(S)$ denotes the closure of S in the order topology of K.

PROPOSITION 2.2. The subset S of K is topologically dense in K if and only if it is order dense in K. If F is a subfield of K, then $cl_K(F)$ is a subfield of K, which is cofinal in K if and only if F is. If F is not cofinal in K, then $cl_K(F)=F$.

PROOF. The last statement follows from 2.1. The remaining are obvious.

Given a non-empty $S\subseteq K$, we denote by $\omega(S)$ its cofinality type (which is an initial ordinal). A cut γ of K is an ordered pair of sets (A, B) such that $K=A\cup B$ and $A<B$ (i.e., $a\in A$, $b\in B$ imply $a<b$). The cut γ is said to be trivial if either one of the following is true: (i) $A=\emptyset$; (ii) $B=\emptyset$; (iii) A has a maximum; (iv) B has a minimum. A cut $\gamma=(A, B)$ is called a Cauchy cut if A and B are non empty and $B-A=\{b-a : a\in A, b\in B\}$ is coinitial in K^+. If F is a subfield of K and $\gamma=(A, B)$ is a cut of F, an element $a\in K$ such that $A\leq a\leq B$ is said to fill γ. Given $a\in K$, put $A_a=\{x\in F : x<a\}$, $B_a=\{y\in F : y>a\}$. The cofinality type of A_a is denoted by $\omega_F^-(a)$, the coinitiality type of B_a by $\omega_F^+(a)$. If $a\notin F$, (A_a, B_a) is a non-trivial cut of F which we denote by $\gamma_F(a)$.

PROPOSITION 2.3. Let F be a cofinal subfield of X, and let a belong to $K\backslash F$. Then the following are equivalent

(i) $a\in cl_K(F)$

(ii) $\gamma_F(a)$ is a Cauchy cut of F.

Consequently, $cl_K(F)=\{a\in K$: a fills a Cauchy cut of $F\}$.

PROOF. (i)$\Rightarrow$(ii). Since F is cofinal in K, A_a and B_a are non empty. Assume that B_a-A_a is not coinitial in F^+, that is $B_a-A_a>\varepsilon>0$ for some $\varepsilon\in F$. Put $V_\varepsilon=\{x\in K: |x|<\varepsilon\}$. By (i), $a\in F+V_\varepsilon$, that is $a=c+u$ for some $c\in F$ and $u\in V_\varepsilon$. If $u>0$, then $c\in A_a$; and $c+\varepsilon>$ $>c+u=a$, hence $c+\varepsilon\in B_a$. But this implies $\varepsilon\in B_a-A_a$, a contradiction. Analogously $u<0$, $c\in B_a$ and $c-\varepsilon\in A_a$ imply $\varepsilon\in B_a-A_a$.

(ii)$\Rightarrow$(i). If $a\notin cl_K(F)$, there is $\varepsilon\in K^+$ such that $(a-\varepsilon, a+\varepsilon)\cap F=\emptyset$. Hence $A_a<a-\varepsilon<a+\varepsilon<B_a$ imply $B_a-A_a>2\varepsilon$ that is, B_a-A_a is not coinitial in K^+; hence B_a-A_a cannot be coinitial in F^+.

Concerning the topology of K, we prove:

PROPOSITION 2.4. K is connected if and only if K is a copy of $\boldsymbol{R}$ (the real numbers field). Otherwise, K admits a base $\mathfrak{N}(K)$ of open-and-closed neighborhoods of 0, $\mathfrak{N}(K)=\{V_\alpha: \alpha\in\omega(K)\}$ such that $\alpha<\beta$ implies $V_\beta\subsetneqq V_\alpha$. If K is non archimedean, the V_α may be chosen as subrings of K.

PROOF. Assume $K\not\simeq\boldsymbol{R}$. If K is archimedean, K is a copy of a proper subfield of $\boldsymbol{R}$, and $\omega(F)=\omega_0$, the first infinite ordinal. Take $a\in\boldsymbol{R}\backslash K$ and put $V_n=\{x\in K: |x|<|a|/n\}$ for each $n\in\omega_0$.

If K is non archimedean, let $\{a_\alpha: \alpha\in\omega(K)\}$ be a subset of K^+, coinitial in K^+, such that $a_\alpha<1$ for all $\alpha\in\omega(K)$. Put $U_\alpha=\{x\in K: |a|<a_{\alpha/n}$ for all positive integers $n\}$. Since K is non archimedean, the U_α are neighborhoods of 0, and they clearly are a base. If $x, y\in U_\alpha$, then

$$|x+y|\leq|x|+|y|\leq a_{\alpha/2n}+a_{\alpha/2n}=a_{\alpha/n}$$

and

$$|x\cdot y|=|x||y|\leq a_{\alpha/n}\cdot 1$$

for all positive integers n. Thus the U_α are subrings of K; hence they are open-and-closed, and the family of all intersection $W_\alpha=\bigcap_{\beta\leq\alpha} U_\beta$ clearly contains a subfamily V_α satisfying the required conditions.

From now on, $\mathcal{N}(K)$ will always denote such a neighborhood base.

As every topological field, K is said to be complete if it is complete with respect to the uniform structure $\mathcal{W}$ generated by the entourages $W_\alpha = \{(x, y) \in K \times K : x - y \in V_\alpha,\ V_\alpha \in \mathcal{N}(K)\}$. The completion $\bar{K}$ of the uniform space $(K, \mathcal{W})$ may be given a structure of an ordered field, whose topology is that of the completion. The field K is obviously topologically dense in $\bar{K}$ (and hence order dense), and $\bar{K}$ is unique up to an order-preserving field isomorphism which is the identity on K. By 2.3 only the Cauchy cuts of K are filled in $\bar{K}$. Clearly, $\bar{K}$ fills all the Cauchy cuts of K. For each $a \in \bar{K}$, $\omega_F^-(a) = \omega_F^+(a) = \omega(\bar{K}) = \omega(K)$.

PROPOSITION 2.5. Let F be a cofinal subfield of K, and let $\bar{F}$ be a completion of F. Then $cl_K(F)$ is isomorphic to a subfield of $\bar{F}$ containing F.

PROOF. Extension theorem for uniformly continuous mappings.

We refer the reader to [GJ, chapter 13] for terminology and basic facts on real closed fields. By [S], if F is real closed, then $\bar{F}$ is real closed.

PROPOSITION 2.6. Let K be a real closed extension of the real closed field F. If every Cauchy cut of F is filled in K, then there exists a copy E of $\bar{F}$ in K, such that $F \subseteq E \subseteq K$.

PROOF. Let Φ be the set of all isomorphisms $\varphi : L \to K$, where L is a real-closed extension of F in $\bar{F}$, which are the identity on F. If $\varphi, \psi \in \Phi$, we write $\varphi \leq \psi$ if ψ is an extension of φ. Let $\varphi \in \Phi$, $\varphi : L \to K$, and $a \in \bar{F} \setminus L$; assume that b fills in K the Cauchy cut $\gamma_F(a)$. Then $b \notin \varphi[L]$, since φ is order-preserving; hence b is transcendental over the real closed field $\varphi[L]$. Thus φ has an extension $\varphi' : L(a) \to \varphi[L](b)$, such that $\varphi'(x) = \varphi(x)$ for $x \in L$, and $\varphi'(a) = b$. By [GJ, 13.12], φ' is order preserving. Hence φ' has an extension ψ from the real closed field L' (which is the algebraic closure of $L(a)$ in $\bar{F}$) to the algebraic closure of $\varphi[L](b)$ in K.

By Zorn's lemma, Φ has a maximal element $\bar{\varphi}$ and we have shown that the domain of $\bar{\varphi}$ is all of $\bar{F}$.

III. Residue class fields.

Let X be a topological space, F a non-archimedean ordered field. We denote by $C=C(X, F)$ the lattice ordered ring of all continuous functions from X to F; $C^*=C^*(X, F)$ will be the subring of C consisting of all $f \in C$ such that $f[X]$ has a lower and an upper bound in F.

If P (resp. P^*) is a prime ideal of C (resp. C^*) the ring C/P (resp. C^*/P^*) is totally ordered under the quotient ordering. (The proofs are analogous to those given in [GJ, chapter 5].) The canonical mapping of C onto C/P (resp. of C^* onto C^*/P^*) maps the constants onto a copy of F, which, again, will be denoted by F.

A set E of idempotents of C such that $\sum_{e \in E} e(x)=1$ for all $x \in X$ is called a partition of unity into idempotents, or simply a partition ($|E|$-partition if we want to emphasize its cardinality). A partition E is said to be strongly contained in the ideal I of C (resp. C^*) if $\sum_{e \in S} e \in I$ for all $S \subset E$ such that $|S|<|E|$.

In what follows, M^* is a maximal ideal of C^* and K is the ordered field C^*/M^*. Observe that F is cofinal in K. The next lemma, analogous to [DMW, Lemma 2.1] is the key step in the study of the residue class fields of C^* and C.

LEMMA 3.1. Let $f \in C^*$. If $a=M^*(f)$ (the residue class of f modulo M^*) does not belong to $f[X]$, then M^* strongly contains either an $\omega_F^-(a)$-partition or an $\omega_F^+(a)$-partition.

PROOF. Assume first $f[X]<a$. We show that, in this case, M^* strongly contains an $\omega_F^-(a)$-partition. Put $A_a=\{x \in F : x<a\}$, and choose a well-ordered cofinal subset of A_a, say $\{a_\alpha : \alpha \in \omega_F^-(a)\}$ (where $a_\alpha<a_\beta$ if $\alpha<\beta$, and for each limit ordinal $\gamma<\omega$ (a), $\sup\{a_\alpha : \alpha<\gamma\}$ does not exist in F) and for each α choose $V_\alpha \in \mathfrak{N}(F)$ (see 2.4) in such a way that $a_{\alpha+1}>a_\alpha+V_\alpha$. Put $L_\alpha=\{x \in F : x<a_\alpha+v$ for some $v \in V_\alpha\}$. Clearly the L_α are open-and-closed, $L_{\alpha+1} \supsetneqq L_\alpha$, and $\bigcup_{\alpha \in \omega_F^-(a)} L_\alpha=A_a$. Thus, $Z_\alpha= =f^{\leftarrow}[L_\alpha \setminus \bigcup_{\alpha<\beta} L_\beta]$ is open and closed in X, the Z_α are pairwise disjoint, and $\bigcup_{\alpha \in \omega_F^-(a)} Z_\alpha=X$. Furthemore, given $\alpha_0 \in \omega_F(a)$, there exists $\alpha \geq \alpha_0$ such that $Z_\alpha \neq \emptyset$, since, otherwise, we would have $f[X]<a_{\alpha_0+1}<a$, which is

impossible. This shows that $\{e_\alpha : \alpha \in \omega_F^-(a)\}$, where e_α is the characteristic function of Z_α, is an $\omega_F^-(a)$-partition. For each $\alpha \in \omega_F^-(a)$, put $g_\alpha = f \vee a_{\alpha+2} - f$. Since $M^*(g_\alpha) = M^*(f \vee a_{\alpha+2}) - M^*(f) = a \vee a_{\alpha+2} - a = 0$, $g_\alpha \in M^*$ Define h_α on X to be $1/g_\alpha$ on $\bigcup_{\beta \leq \alpha} Z_\beta = f^{\leftarrow}[L_\alpha]$, to be zero otherwise. Observe that $|h_\alpha| < (a_{\alpha+2} - a_{\alpha+1})^{-1}$. Hence $h_\alpha \in C^*$, and $\sum_{\beta \leq \alpha} e_\beta = h_\alpha g_\alpha \in M^*$. This proves that $\{e_\alpha : \alpha \in \omega_F^-(a)\}$ is strongly contained in M^*.

In an analogous way it can be shown that if $f[X] > a$, then M^* strongly contains an $\omega_F^+(a)$-partition. It only remains to prove that we may always assume either $f[X] < a$ or $f[X] > a$.

If $a \in F$, simply use $f \wedge a$ (or $f \vee a$) instead of f.

If $a \notin F$, put $B_a = \{y \in F : y > a\}$; A_a, B_a are open-and-closed in F, since F is cofinal in K. Let e be the characteristic function of $f^{\leftarrow}[A_a]$. Then $f = fe + f(1-e)$, and since M^* is prime, exactly one of the two idempotents e, $1-e$ belongs to M^*. The conclusion is now obvious.

PROPOSITION 3.2.

(i) If M^* contains no partitions, then $C^*/M^* = F$

(ii) If M^* contains a countable partition, then some non-Cauchy cut of F is filled in K

(iii) If M^* contains an α-partition, with α non-measurable, then M^* contains a countable partition.

PROOF.

(i) By Lemma 3.1, $M^*(f) \in F$ for all $f \in C^*$.

(ii) Let $\{e_n : n \in \boldsymbol{N}\}$ ($\boldsymbol{N}$ is the set of natural numbers) be a partition contained in M^*. Put $f = \sum_{n \in \boldsymbol{N}} n e_n$. Since F is non archimedean, $f \in C^*$ and $M^*(f)$ fills the non-Cauchy-cut $\gamma = (A, B)$, where $B = \{y \in F : y > \boldsymbol{N}\}$ and $A = F \backslash B$.

(iii) Repeat the proof given in [DMW], theorem 2.2.

THEOREM 3.3. The following statements (i), (ii), (iii) are equivalent and imply (iv) .If F is not complete, then all four are equivalent.

(i) For some $f \in C^*$, $M^*(f) \in cl_K(F) \backslash f[X]$.

(ii) M^* strongly contains an $\omega(F)$-partition.

(iii) M^* contains a unit of $C = C(X, F)$.

(iv) $cl_K(F)$ is a completion of F.

PROOF. (i) implies (ii). Put $a=M^*(f)$. By 2.5, $\omega_F^-(a)=\omega_F^+(a)=$ $=\omega(F)$. By 3.1, M^* strongly contains an $\omega(F)$-partition.

(ii) implies (iii). Let $\{e_\alpha : \alpha\in\omega(F)\}$ be an $\omega(F)$-partition strongly contained in M^* and let $\{a_\alpha : \alpha\in\omega(F)\}$ be a coinitial subset of F, such that $\alpha<\beta$ implies $a_\beta<a_\alpha$. Put $u=\sum_{\alpha\in\omega(F)} a_\alpha e_\alpha$. Then u is a unit of C, and since the partition is strongly contained in M^*, we have $0\leq M^*(u)=$ $=M^*(\sum_{\beta\leq\alpha} a_\beta e_\beta)\leq a_\alpha$ for all $\alpha\in\omega(F)$. Since F is cofinal in K, $M^*(u)=0$.

(iii) implies (i). Apply Lemma 3.1.

(i) implies (iv). By 2.3 and 2.5, we have to show that every Cauchy cut of F is filled in K. Let $\gamma=(A, B)$ be a non-trivial Cauchy cut of F, and let $\{a_\alpha : \alpha\in\omega(F)\}$ be a cofinal subset of A, such that $a_\alpha<a_\beta$ whenever $\alpha<\beta$. Take an $\omega(F)$-partition strongly contained in M^*. It can be verified, arguing as above, that $M^*(\sum_{\alpha\in\omega(F)} a_\alpha e_\alpha)$ fills γ.

That (iv) implies (i) if F is not complete is obvious by 2.3.

COROLLARY 3.4. If $cl_K(F)\setminus F\neq\emptyset$, then $cl_K(F)$ is a completion of F. If $\omega(F)$ is non-measurable, K is a completion of F if and only if $F=K$ and F is complete.

PROOF. The first part is proved in the same way as (iv) was derived from (i) in 3.3. For the second statement, apply 3.3 and 3.2.

IV. $K=C(X, F)/M$.

In this section, M will denote a maximal ideal of $C=C(X, F)$ and K the field C/M.

THEOREM 4.1. The following are equivalent:

(i) M does not contain strongly an $\omega(F)$ partition.

(ii) F is cofinal in K.

(iii) $M\cap C^*$ is maximal in C^*.

Furthermore, if (iii) holds then C/M is isomorphic to $C^*/M\cap C^*$.

PROOF. Assume (i) holds. If F is not cofinal in K, there exists $f\in C$ such that $M(f)>F$. Arguing as in Lemma 3.1, we see that M contains an $\omega(F)$-partition and (ii) is proved.

Now assume (ii) and put $P^{*}=M\cap C^{*}$. The natural mapping j of C^{*}/P^{*} into C/M is one-to-one, and C/M is the field of fractions of $j[C^{*}/P^{*}]$. Let M^{*} be the maximal ideal of C^{*} containing the prime ideal P^{*}. By an argument analogous to [DMW, lemma 2], it can be shown that if $u\geq 0$, $u\in M^{*}\setminus P^{*}$, then $0<u<F^{+}$. But, then, F is not cofinal in K. Hence $M\cap C^{*}$ is maximal in K.

If $M\cap C^{*}$ is a maximal ideal of C^{*} which contains no unit of C, then by Theorem 3.1, $M\cap C^{*}$ does not strongly contain any $\omega(F)$-partition, whence M cannot contain strongly an $\omega(F)$-partition and (i) holds.

V. Real closed fields.

This section is devoted to a brief investigation of the residue class fields of $C(X, F)$ and $C^{*}(X, F)$ with F real-closed. The first natural question is: are these residue class fields also real closed? The answer to this question is affirmative and a proof may be given following [GJ, 13.4], once we have proved that in a real-closed field K the roots of a polynomial depend continuously on the coefficients. This fact is well-known in the case $K=\boldsymbol{R}$, but its proof makes use of Rouché's theorem. Hence we give an elementary proof of this fact. Let K be a real-closed field, and let L be its algebraic closure. The topology and the absolute value on L are defined in the usual way.

Let n be a positive integer. For each $a=(a_0, \ldots, a_{n-1})\in L^n$ we denote by $\rho_1 a, \ldots, \rho_n a$ the « real parts » of the roots (in L) of the polynomial $p_a(t)=\sum_{\nu=0}^{n-1} a_\nu t^\nu+t^n$, (listing each according to its multiplicity) indexed so that $\rho_1 a\leq \ldots \leq\rho_n a$ (see [GJ, 13.3]). Put also

$$\| a \|=\operatorname{Max}\{| a_0 | \ldots | a_{n-1} |\}.$$

THEOREM 5.1. The functions $\rho_1, \ldots, \rho_n$ are continuous functions from L^n to K.

PROOF. The theorem is an immediate consequence of the following:

PROPOSITION 5.2. Let $a\in L^n$ be given, and let $\varepsilon_a>0$ be such that $| x_a-y_a |>\varepsilon_a$ whenever x_a, y_a are distinct roots of $p_a(t)$. Let x_a be a root of $p_a(t)$ of multiplicity r_a. Then, for each ε, $0<\varepsilon\leq\varepsilon_a$, there exists $\delta>0$ such that, if $\|a-b\|<\delta$, then $p_b(t)$ has (counting multiplicities) exactly r_a roots x_b such that $|x_b-x_a|<\varepsilon$.

PROOF. It is enough to show that under the above assumptions, $p_b(t)$ has at least r_a roots in x_a+U_ε. We first show that $p_b(t)$ has at least one root in x_a+U_ε. In fact, for every $b\in L^n$

$$|p_b(x_a)|=|p_b(x_a)-p_a(x_a)|\leq\sum_{\nu=0}^{n-1}|a_\nu-b_\nu|\ |x_a|^\nu\leq\|a-b\|\,(\sum_{\nu=0}^{n-1}|x_a|^\nu)$$

and

$$|p_b(x_a)|=\prod_{j=1}^{n}|x_a-x_j|$$

where the x_j are the roots of $p_b(t)$. Thus, for

$$\delta=\varepsilon^n\cdot(\sum_{\nu=0}^{n-1}|x_a|^\nu)^{-1},\ \text{we have}\ |x_a-x_j|<\varepsilon$$

for at least one j.

Dividing the polynomial $p_a(t)$ by $t-\xi(\xi\in L)$ we obtain

$$p_a(t)=(t-\xi)p_{\varphi(a,\xi)}(t)+p_a(\xi)$$

where $\varphi: L^n\times L\to L^{n-1}$ is a continuous function (its components are polynomials in ξ, having $a_0,\ldots,a_{n-1}$, 1 as coefficients).

We have already seen that the proposition is true for $r_a=1$. Also, the proposition is trivial for $n=1$. Assume that it is true for $n-1$ $(n>1)$, and that $r_a>1$. Then x_a is a root of $p_{\varphi(a,x_a)}(t)$, of multiplicity r_a-1. Given $\varepsilon>0$, we can find $\eta>0$ such that for every $c\in L^{n-1}$ satisfying $\|c-\varphi(a, x_a)\|<\eta$, $p_c(t)$ has exactly r_a-1 roots in x_a+U_ε, by the induction hypothesis. Since φ is continuous, there exists $\delta_1>0$ such that whenever $\|a-b\|$, $|\xi-x_a|<\delta_1$, then

$$\|\varphi(b,\xi)-\varphi(a,x_a)\|<\eta.$$

By what we have above shown, we can find $\delta<\delta_1$ such that $\|a-b\|<\delta$ implies $|x_b-x_a|<\min\{\delta_1,\varepsilon\}$ for at least one root x_b of $p_b(t)$. Thus, if

$\| a-b \| < \delta$, $p_{\varphi(b, x_b)}(t)$ has $r_a - 1$ roots in $x_a + U_\varepsilon$. Hence $p_b(t)$ has at least r_a roots in $x_a + U_\varepsilon$.

Following now the proof of [G J, 13.4] (and observing, in the bounded case, that the roots of $p_a(t)$ are bounded by $n(1+\| a \|)$), we have

THEOREM 5.3. Let F be a real closed field, M (resp. M^*) a maximal ideal of $C=C(X, F)$ (resp. of $C^*=C^*(X, F)$). Then C/M (resp. C^*/M^*) is real-closed.

Also we have

THEOREM 5.4. Let F be a real closed field, M a maximal ideal of $C=C(X, F)$, $K=C/M$. If F is not cofinal in K, then there is a copy E of $\bar{F}$ such that

$$F \subseteq E \subseteq K.$$

PROOF. Since M contains an $\omega(F)$-partition, every Cauchy cut F is filled in K, as is easy to see. Thus Theorem 2.6. applies.

REMARK. This copy of F in K is not unique, unless F is already complete. It could be shown that there are at least $\mathrm{trdeg}_F(\bar{F}) \cdot | I |$ such copies, where $I = \{x \in K : | x | < F^+\}$.

REFERENCES

[D M W] DE MARCO, G. and WILSON R. G.: *Rings of continuous functions with values in an archimedean ordered field,* Rend. Sem. Mat. Padova, to appear.

[G J] GILLMAN, L. and JERISON, M.: *Rings of continuous functions,* Van Nostrand, New York, 1960.

[S] SCOTT, D.: *On completing ordered fields, Applications of Model Theory to Algebra, Analysis and Probability,* Holt, Rinehart and Winston, New York, pp. 274-278.

Manoscritto pervenuto in redazione l'8 gennaio 1971.