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DISTRIBUTIONAL BOUNDARY VALUES IN $\mathfrak{D}'_{L^p}$. II

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Section I. Introduction.

In Carmichael [1] we have obtained distributional boundary value results concerning the space $\mathfrak{D}'_{L^p}$, which is a subspace of $\mathcal{S}'$, using the topology of $\mathcal{S}'$. In particular we obtained conditions under which an analytic function can be represented by the Fourier-Laplace transform of an element in $\mathfrak{D}'_{L^p}$, $1 \leq p \leq 2$, such that the convergence in $\mathcal{S}'$ of the analytic function to a distribution can be proved. This limit distribution is not necessarily an element of $\mathfrak{D}'_{L^p}$ but is the Fourier transform of such an element.

In the present paper we shall continue our investigation of distributional boundary values in $\mathfrak{D}'_{L^p}$ considered as a subspace of $\mathcal{S}'$. Notation and definitions will be stated in section II; while some theorems which will be used in the proofs of the main results of this paper will be proved in section III. In section IV we define a generalized Cauchy integral of $U \in \mathfrak{D}'_{L^p}$ and obtain conditions under which it equals the Fourier-Laplace transform of an element $V \in \mathcal{S}'$. Under these conditions it will be shown that the generalized Cauchy integral of U converges to U in the topology of $\mathcal{S}'$. We then consider analytic functions of a certain type and give necessary and sufficient conditions in terms of the generalized Cauchy integral that these functions have a distributional boundary value in $\mathfrak{D}'_{L^p}$. Also in section IV we define a generalized Poisson integral for $U \in \mathfrak{D}'_{L^p}$ and relate the boundary value properties of it to those of the

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generalized Cauchy integral. Our results concerning these generalized integrals are reminiscent of some classical theorems of Hille and Tamarkin ([2] and [3]). In section V we shall obtain a generalization of the main results in Carmichael [1] in which we let p range over $1 \leq p < \infty$ instead of just over $1 \leq p \leq 2$. New difficulties arise with this more general assumption. In section VI we show how several of our results, which we shall prove for functions analytic in an octant, can be extended to the more general case of functions analytic in tubular radial domains; and we obtain a theorem concerning boundary values of functions analytic in a tubular cone which is a more general subset of $\mathfrak{C}^n$ than the tubular radial domain.

Tillmann [4] has given conditions under which an element $U \in \mathfrak{D}'_{L^p}$, $1 < p < \infty$, is the boundary value of an analytic function in the topology of $\mathfrak{D}'_{L^p}$. Luszczki and Zieleźny [5] have obtained results similar to Tillmann but for only one dimension. The analytic function which Tillmann uses is the generalized Cauchy integral of U, which he calls the « Indikatrix ». However, by not considering $\mathfrak{D}'_{L^p}$ as a subspace of $\mathcal{S}'$ one loses the powerful tool of Fourier transform, which we use to equate the generalized Cauchy integral and the Fourier-Laplace transform. In fact in the setting of Tillmann one can not relate the generalized Cauchy integral with the Fourier-Laplace transform. Also Tillmann does not consider the generalized Poisson integral, and neither does he consider functions analytic in tubular cones. Furthermore, some of the $\mathfrak{D}'_{L^p}$ boundary value results in this paper, using the topology of $\mathcal{S}'$, concern a much more general class of analytic functions than the class which Tillmann uses in [4] and which characterizes the $\mathfrak{D}'_{L^p}$ boundary value problem in the topology of $\mathfrak{D}'_{L^p}$.

We note further that Beltrami and Wohlers ([6], [7], [8], and [9]) have considered the distributional boundary value problem with respect to $\mathfrak{D}'_{L^2}$ as a subspace of $\mathcal{S}'$. However, they confine their attention to the space $\mathfrak{D}'_{L^p}$ and work only in one dimension.

Section II. Notation and Definitions.

The n dimensional notation used in this paper will be the same as in Carmichael [1]. (See section II in [1]). Let $\sigma = (\sigma_1, \ldots, \sigma_n)$, n being the dimension, where $\sigma_j = \pm 1$, $j = 1, \ldots, n$. Then for each fixed σ,

$B_\sigma=\{z\in \mathbb{C}^n : \sigma_j \operatorname{Im}(z_j)>0,\ j=1, \ldots, n\}$ is called an octant. There are 2^n such octants in $\mathbb{C}^n$. Corresponding to B_σ for each fixed σ we shall associate the set $S_\sigma^A=\{t\in \mathbb{R}^n : -\infty<\sigma_j t_j\leq a_j,\ j=1, \ldots, n\}$, where $A=(a_1, \ldots, a_n)$ is a fixed n-tuple of nonnegative real numbers.

For the definitions and properties of the function spaces $\mathcal{S}$, $\mathcal{D}_{L^p}$, $1\leq p\leq\infty$, $\mathcal{B}$, and $\dot{\mathcal{B}}$ and the generalized function spaces $\mathcal{S}'$ and $\mathcal{D}'_{L^p}$, $1\leq p\leq\infty$, we refer either to Carmichael [1] or directly to Schwartz ([10], pp. 233-248 and pp. 199-205). We call the readers attention especially to the definition of convergence in $\mathcal{S}$ and continuity in $\mathcal{S}'$. The concepts of support of functions and distributions are also defined in Carmichael [1].

The convolution $U*V$ of two generalized functions U and V is defined, when it exists, by

$$\langle U*V, \varphi\rangle=\langle U_\xi, \langle V_\eta, \varphi(\xi+\eta)\rangle\rangle,$$

where φ is an element of the appropriate function space.

The fourier transform of an element $\varphi(t)\in L^1$ will be denoted as $\hat{\varphi}$ or $\mathcal{F}[\varphi(t); x]$ and is defined by

$$\hat{\varphi}(x)=\int_{\mathbb{R}^n} \varphi(t)e^{-2\pi i\langle x, t\rangle}dt.$$

If $\varphi(t)\in L^p$, $1<p\leq 2$, then its Fourier transform is defined by

$$\hat{\varphi}(x)=\underset{N\to\infty}{\text{l.i.m.}} \int_{-N}^{N}\cdots\int_{-N}^{N} \varphi(t)e^{-2\pi i\langle x, t\rangle}dt,$$

where l.i.m. denotes the limit in the L^q norm, $\frac{1}{p}+\frac{1}{q}=1$. If $\varphi(t)\in L^1$ it is known that $\hat{\varphi}(x)$ is continuous and bounded on $\mathbb{R}^n$; while if $\varphi(t)\in L^p$, $1<p\leq 2$, then $\hat{\varphi}(x)\in L^q$, $\frac{1}{p}+\frac{1}{q}=1$. It is well known that the Fourier transform is a continuous isomorphism of $\mathcal{S}$ onto $\mathcal{S}$ with the same being true of $\mathcal{S}'$ under the definition

$$\langle \hat{U}, \varphi\rangle=\langle U, \hat{\varphi}\rangle,$$

where $U \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$. If $\langle V_t, e^{-2\pi i\langle z, t\rangle}\rangle$ exists for $z \in \mathbb{C}^n$ and some generalized function V, $\langle V_t, e^{-2\pi i\langle z, t\rangle}\rangle$ will be called the Fourier-Laplace transform of V. This terminology has been used before by Hörmander ([11], p. 20).

We shall now introduce some terminology concerning cones, which shall be needed in section VI of this paper. $C \subset \mathbb{R}^n$ is a cone (with vertex at zero) if $y \in C$ implies $\lambda y \in C$ for all positive scalars λ. The intersection of C with the unit sphere $\{y : |y| = 1\}$ is called the projection of C and is denoted $\operatorname{pr} C$. Let C' be a cone such that $\operatorname{pr} C' \subset \operatorname{pr} C$; then C' will be called a compact subcone of C. The function

$$u_C(t) = \sup_{y \in \operatorname{pr} C} (-\langle t, y\rangle)$$

is the indicatrix of C. $0(C)$ will denote the convex envelope of C. $T^C = \mathbb{R}^n + iC$, where C is an open connected cone, is a tubular radial domain. Examples of tubular radial domains are the octants B_σ and the forward light cone $\Gamma^+ = \{z : Im(z_1) > (\sum_{j=2}^{n} Im^2(z_j))^{1/2}\}$. In $\mathbb{C}^1$ the upper half plane $\{z : Im(z) > 0\}$ is a tubular radial domain. If C is not connected, then we call T^C a tubular cone. The number

$$\rho_C = \sup_{t \in C_*} \frac{u_{0(C)}(t)}{u_C(t)}$$

where $C_* = \mathbb{R}^n \setminus C^*$ and $C^* = \{t : u_C(t) \leq 0\}$, characterizes the non-convexity of the cone C. A cone is convex if and only if $\rho_C = 1$. (See Vladimirov [12], Lemma 2, p. 220).

Let $f(z)$ be a function of n complex variables $z = (z_1, \ldots, z_n)$. We say that $f(z)$ has a distributional boundary value U in the topology of $\mathcal{S}'$, that is $f(z) \to U$ in $\mathcal{S}'$, if

$$\langle f(z), \varphi(x)\rangle \to \langle U, \varphi(x)\rangle, \quad \varphi \in \mathcal{S},$$

as $y = Im(z) \to 0$ in some specified manner, where by $Im(z) \to 0$ we mean $Im(z_j) \to 0$, $j = 1, \ldots, n$.

Throughout this paper, by $f(x) \in L^p$, $1 \leq p < \infty$, we mean that $f(x) \in L^p(\mathbb{R}^n)$ for some p, $1 \leq p < \infty$; and we have a similar meaning for $U \in \mathcal{D}'_{L^p}$, $1 \leq p \leq \infty$.

Section III. Preliminary Theorems.

It is well known that if f and g are in L^2, then

$$\mathfrak{F}^{-1}[\widehat{\hat{f}\hat{g}}]=f*g, \tag{1}$$

where $\mathfrak{F}^{-1}$ denotes the inverse Fourier transform and $*$ is the usual convolution. Similarly if f and $\hat{g}$ are in L^p, $1<p<2$, and if $\hat{f}$ and g are in L^q, $\frac{1}{p}+\frac{1}{q}=1$, then (1) holds. Furthermore, if $f\in L^2$ and $g\in L^1$, then

$$\mathfrak{F}[f*g]=\hat{f}\hat{g} \tag{2}$$

in the sense of L^2. It is obvious that under the above conditions both $f*g$ and $\hat{f}\hat{g}$ are elements of $\mathcal{S}'$.

Lemma 1. *Let f and $\hat{g}$ be elements of L^p, $1<p<2$; and let $\hat{f}$ and g be elements of L^q, $\frac{1}{p}+\frac{1}{q}=1$. Then $\mathfrak{F}[f*g]=\hat{f}\hat{g}$ in $\mathcal{S}'$.*

Proof. Let $\varphi\in\mathcal{S}$. Then by (1), $\langle f*g, \varphi\rangle=\langle\mathfrak{F}^{-1}[\widehat{\hat{f}\hat{g}}], \varphi\rangle$. But the Fourier transform is a continuous, one to one mapping of $\mathcal{S}'$ onto $\mathcal{S}'$. Thus $\langle\mathfrak{F}[f*g], \varphi\rangle=\langle\hat{f}\hat{g}, \varphi\rangle$ as desired.

Using (1) and (2) and exactly the same proof as in Lemma 1 we have the following.

Lemma 2. *Let $f\in L^2$, and $g\in L^1$ or $g\in L^2$. Then $\mathfrak{F}[f*g]=\hat{f}\hat{g}$ in $\mathcal{S}'$.*

Lemma 3. *Let $f\in L^p$, $1<p\leq 2$. Let $g\in L^q$, $\frac{1}{p}+\frac{1}{q}=1$, and assume that $\mathfrak{F}^{-1}g$ exists and belongs to L^p. Then*

$$\mathfrak{F}^{-1}[f*g]=\mathfrak{F}^{-1}(f)\mathfrak{F}^{-1}(g) \tag{3}$$

in $\mathcal{S}'$.

Proof. Since $f\in L^p$, $1<p\leq 2$, then $\mathfrak{F}^{-1}(f)$ exists and is an element of L^q, $\frac{1}{p}+\frac{1}{q}=1$. It is thus evident that $\mathfrak{F}^{-1}(f)\mathfrak{F}^{-1}(g)\in\mathcal{S}'$. Also $f*g$ exists as a classical convolution, and it is known that $f*g\in L^s$, $\frac{1}{s}=\frac{1}{p}+\frac{1}{q}-1$. Thus $f*g\in\mathcal{S}'$, and hence so does $\mathfrak{F}^{-1}[f*g]\in\mathcal{S}'$. Thus

both sides of (3) are well defined as elements of $\mathcal{S}'$. By a well known result of Schwartz [10], we thus obtain (3) since the inverse Fourier transform in $\mathcal{S}'$ converts convolution into multiplication.

Schwartz ([10], p. 270) has shown that if $U\in\mathcal{D}'_{L^p}$, $1\leq p\leq 2$, and $V\in\mathcal{D}'_{L^p}$, $1\leq q\leq 2$, then $\mathcal{F}[U*V]=\widehat{U}\widehat{V}$. We now obtain a variation of this result, where we let one of the distributions be an element of $\mathcal{D}'_{L^p}$ such that q does not have to lie between 1 and 2. We shall also obtain the converse result that $\mathcal{F}[UV]=\widehat{U}*\widehat{V}$ in $\mathcal{S}'$. In the proofs of these results we shall use the characterization theorem of Schwartz for $\mathcal{D}'_{L^p}$ ([10], p. 201): if $U\in\mathcal{D}'_{L^p}$, then $U=\sum_{|\alpha|\leq m} D^\alpha f_\alpha$, $f_\alpha\in L^p$. We recall by D^α, α being an n-tuple of nonnegative integers, we mean $D^\alpha=D_1^{\alpha_1}\dots D_n^{\alpha_n}$, where $D_j=\frac{1}{2_{\pi i}}\frac{\partial}{\partial t_j}$, $j=1,\dots,n$.

THEOREM 1. *If* $U\in\mathcal{D}'_{L^p}$, $1<p\leq 2$, *and* $V\in\mathcal{D}'_{L^q}$, $\frac{1}{p}+\frac{1}{q}=1$, *such that* $\widehat{V}=\sum_{|\beta|\leq r} x^\beta \widehat{g}_\beta(x)$, $\widehat{g}_\beta(x)\in L^p$, *then* $U*V\in\mathcal{S}'$ *and* $\mathcal{F}[U*V]=\widehat{U}\widehat{V}$ *in* $\mathcal{S}'$. *Conversely, if* $\widehat{U}\in\mathcal{D}'_{L^p}$, $1<p\leq 2$, *and* $\widehat{V}\in\mathcal{D}'_{L^p}$, $\frac{1}{p}+\frac{1}{q}=1$, *such that* $\widehat{V}=\sum_{|\beta|\leq r} D^\beta g_\beta(t)$ *and* $V=\sum_{|\beta|\leq r}(-1)^{|\beta|}x^\beta\mathcal{F}^{-1}[g_\beta(t);\,x]$, *where* $\mathcal{F}^{-1}[g_\beta(t);\,x]$ *is assumed to exist and* $\mathcal{F}^{-1}[g_\beta(t);\,x]\in L^p$, *then* $\mathcal{F}[UV]=\widehat{U}*\widehat{V}$ *in* $\mathcal{S}'$.

PROOF. If $U\in\mathcal{D}'_{L^p}$, $1<p\leq 2$, and $V\in\mathcal{D}'_{L^q}$, $\frac{1}{p}+\frac{1}{q}=1$, then by the representation theorem of Schwartz, $U=\sum_{|\alpha|\leq m} D^\alpha f_\alpha$, $f_\alpha\in L^p$, $1<p\leq 2$, and $V=\sum_{|\beta|\leq r} D^\beta g_\beta$, $g_\beta\in L^q$, $\frac{1}{p}+\frac{1}{q}=1$. Now as in the proof of Lemma 3, $f_\alpha * g_\beta$ exists and is an element of L^s, $\frac{1}{s}=\frac{1}{p}+\frac{1}{q}-1$. Thus $f_\alpha * g_\beta\in\mathcal{S}'$, and hence also the distributional derivative $D^{\alpha+\beta}(f_\alpha * g_\beta)\in\mathcal{S}'$. Let $\varphi\in\mathcal{S}$. Then

$$
\begin{aligned}
(4)\quad \Big\langle \sum_{|\alpha|\leq m}\sum_{|\beta|\leq r} D^{\alpha+\beta}(f_\alpha * g_\beta),\ \varphi\Big\rangle &= \sum_{|\alpha|\leq m}\sum_{|\beta|\leq r}(-1)^{|\alpha|+|\beta|}\langle f_\alpha * g_\beta,\ D^{\alpha+\beta}\varphi\rangle\\
&=\Big\langle \sum_{|\alpha|\leq m}\sum_{|\beta|\leq r}(-1)^{|\alpha|+|\beta|}\langle (f_\alpha)_\xi,\ \langle (g_\beta)_\eta,\ D^{\alpha+\beta}\varphi(\xi+\eta)\rangle\rangle\\
&=\Big\langle \sum_{|\alpha|\leq m} D^\alpha(f_\alpha)_\xi,\ \Big\langle \sum_{|\beta|\leq r} D^\beta(g_\beta)_\eta,\ \varphi(\xi+\eta)\Big\rangle\Big\rangle\\
&=\langle U*V,\ \varphi\rangle.
\end{aligned}
$$

Thus $U * V \in \mathcal{S}'$, and as a consequence $\mathfrak{F}[U * V] \in \mathcal{S}'$. Using (4), the assumption $\widehat{g_\beta}(x) \in L^p$, and the Lemmas 1 and 2 we have

$$
(5) \qquad \langle \mathfrak{F}[U * V],\ \varphi(x) \rangle = \langle \sum_{|\alpha| \leq m} \sum_{|\beta| \leq r} D^{\alpha+\beta}(f_\alpha * g_\beta),\ \widehat{\varphi}(t) \rangle
$$

$$
= \sum_{|\alpha| \leq m} \sum_{|\beta| \leq r} \langle f_\alpha * g_\beta,\ \mathfrak{F}[x^{\alpha+\beta}\varphi(x);\ t] \rangle
$$

$$
= \sum_{|\alpha| \leq m} \sum_{|\beta| \leq r} \langle \mathfrak{F}[f_\alpha * g_\beta],\ x^{\alpha+\beta}\varphi(x) \rangle
$$

$$
= \langle \sum_{|\alpha| \leq m} \sum_{|\beta| \leq r} x^{\alpha+\beta} \widehat{f_\alpha}(x) \widehat{g_\beta}(x),\ \varphi(x) \rangle .
$$

Now

$$
(6) \qquad \langle \widehat{U},\ \varphi(x) \rangle = \langle \sum_{|\alpha| \leq m} D^\alpha f_\alpha(t),\ \widehat{\varphi}(t) \rangle
$$

$$
= \langle \sum_{|\alpha| \leq m} f_\alpha(t),\ \mathfrak{F}[x^\alpha \varphi(x);\ t] \rangle
$$

$$
= \langle \sum_{|\alpha| \leq m} x^\alpha \widehat{f_\alpha}(x),\ \varphi(x) \rangle ,
$$

where $\widehat{f_\alpha}(x) \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$, since $f_\alpha(t) \in L^p$, $1 < p \leq 2$. By assumption $\widehat{V} = \sum_{|\beta| \leq r} x^\beta \widehat{g_\beta}(x)$, $\widehat{g_\beta}(x) \in L^p$. Thus by this assumption and (6), we see from (5) that

$$
\langle \mathfrak{F}[U * V],\ \varphi \rangle = \langle \widehat{U}\widehat{V},\ \varphi \rangle
$$

as desired. To prove the converse it is sufficient to show that

$$
(7) \qquad \mathfrak{F}^{-1}[\widehat{U} * \widehat{V}] = UV
$$

in $\mathcal{S}'$; for the desired result follows immediately by taking the Fourier transform of both sides of (7). It is evident that (7) can be proved by the same method as in the first part of this theorem where we use Lemma 3 instead of Lemmas 1 and 2. We leave he straighforward details to the interested reader.

We note that by using the same calculation as in (6) above, one can show that if $U \in \mathfrak{D}'_{L^p}$, $1 \leq p \leq 2$, then $\widehat{U} \in \dot{\mathcal{S}}'$ has the form $\widehat{U} = \sum_{|\alpha| \leq m} x^\alpha \widehat{f_\alpha}(x)$, where $\widehat{f_\alpha}(x)$ is continuous and bounded if $p = 1$ and

$\hat{f}_\alpha(x) \in L^q$, $\frac{1}{p}+\frac{1}{q}=1$, if $1<p\leq 2$. Schwartz ([10], p. 256) first recognized this result. We note further that in the case of $p=2$ in Theorem 1, some of our assumptions are redundant. For example in the first part, if $p=2$ then $q=2$ and the form of V can be proved as in the calculation (6). In the conserve part the form of V can be proved if $p=2$; and one does not have to assume that $\mathfrak{F}^{-1}[g_\beta(t); x]$ exists, for it automatically does by the Plancherel theory and is an element of L^2. But the stated assumptions are needed for $1<p<2$.

Theorem 1 will be used to obtain one of our basic results in section IV. We shall also need the following theorem, the proof of which can be found in Carmichael [13]. See also Lemma 1 in Carmichael [1]; the proof for any octant B_σ is the same as for the octant $B_{(1,\dots,1)}$.

THEOREM 2. *Let* $V \in \mathcal{S}'$ *such that* $\operatorname{supp}(V) \subseteq S_\sigma{}^A$. *Let* $\varphi \in \mathcal{S}$. *Then*

$$\langle\langle V, e^{-2\pi i\langle z, t\rangle}\rangle, \varphi(x)\rangle = \langle V, e^{2\pi\langle y, t\rangle}\hat{\varphi}(t)\rangle, \quad z \in B_\sigma .$$

Section IV. Distributional Boundary Values in $\mathfrak{D}'_{L^p}$.

Hille and Tamarkin ([2] and [3]) have obtained Paley-Wiener type theorems for functions analyic in the half plane $Im(z)>0$. They are concerned with obtaining necessary and sufficient conditions that a function $f(x) \in L^p$, $1\leq p<\infty$, be the non-tangential limit function of $f(z)$, which is analytic in $Im(z)>0$ and which is represented by its Cauchy integral. One such necessary and sufficient condition for the above to be true is that

$$(2\pi i)^{-1}\int_{-\infty}^{\infty} \frac{f(t)}{t-\bar{z}}\,dt=0, \quad \bar{z}=x-iy, \ y>0.$$

In other results of this type Hille and Tamarkin relate the Cauchy and Poisson integrals of an analytic function with each other and with the Fourier transform of $f(x)$, which is assumed to exist for all p, $1\leq p<\infty$.

The results of Hille and Tamarkin are closely associated with the classical H^p spaces. In fact if $f(z) \in H^p$, $p\geq 1$, in $Im(z)>0$, then $f(z)$ has non-tangential limits which are in L^p at almost every point of the real axis, and $f(z)$ can be represented by its Poisson integral in $Im(z)>0$ (see Hoffman [14], Theorem, p. 128; see also Titchmarsh [15], Chapter V and Hille [16], Chapter 8). But then by a result of Hille and Tamarkin ([2], Theorem 2), we have for $1\leq p<\infty$ that $f(z)$ is represented by its Cauchy integral since it is represented by its Poisson integral. Thus the results of Hille and Tamarkin hold for the H^p spaces. Zygmund ([17] and [18]) has extended the classical H^p results to n dimensions.

In this section we shall obtain distributional counterparts to the results of Hille and Tamarkin and to the classical H^p theorems. The use of $\mathfrak{D}'_{L^p}$ as a subspace of $\mathcal{S}'$ with the topology of $\mathcal{S}'$ is the correct setting in which to obtain these distributional theorems; and because of the $\mathcal{S}'$ topology, the analytic functions which we consider here are in a less restrictive class of functions than in the classical case. They are of polynomial growth instead of having to be in an H^p space. Also the boundary values of these analytic functions will be in $\mathfrak{D}'_{L^p}$ and not just in L^p.

Tillmann [4] has recognized that if $U\in\mathfrak{D}'_{L^p}$, $1<p<\infty$, then the function

$$C(U;\ z\in B_\sigma)=(2\pi i)^{-n}(\prod_{j=1}^{n}\operatorname{sgn} y_j)<U_t,\ \prod_{j=1}^{n}\frac{1}{t_j-z_j}>,$$

where $z_j=x_j+iy_j$, $j=1, \ldots, n$, and $\operatorname{sgn} y_j$ represents the sign of y_j, exists and is analytic for $z\in B_\sigma$. We call $C(U;\ z\in B_\sigma)$ the generalized Cauchy integral of U for $z\in B_\sigma$. The following result gives conditions under which $C(U;\ z\in B_\sigma)$ equals the Fourier-Laplace transform of an element $V\in\mathcal{S}'$ and converges in $\mathcal{S}'$ to U.

THEOREM 3. *Let $U\in\mathfrak{D}'_{L^p}$, $1<p\leq 2$, such that $U=\hat{V}$, where $V\in\mathcal{S}'$ and* $\operatorname{supp}(V)\subseteq S_\sigma^0=\{t:\ -\infty<\sigma_j t_j\leq 0,\ j=1, \ldots, n\}$ *for some fixed σ. Then $V=\sum_{|\beta|\leq m}(-1)^{|\beta|}t^\beta h_\beta(t)$, $h_\beta(t)\in L^q$, $\frac{1}{p}+\frac{1}{q}=1$; $C(U;\ z\in B_\sigma)$ is analytic in B_σ;*

$$C(U;\ z\in B_\sigma)=\langle V,\ e^{-2\pi i\langle z,t\rangle}\rangle,\ z\in B_\sigma, \tag{8}$$

as elements of $\mathcal{S}'$; and $C(U;\ z\in B_\sigma)\to U\in\mathfrak{D}'_{L^p}$ in the topology of $\mathcal{S}'$ as $Im(z)\to 0$.

PROOF. Since $\hat{V}=U$, then $V=\mathfrak{F}^{-1}(U)$, where $U=\sum_{|\beta|\leq m}D^\beta g_\beta$, $g_\beta\in L^p$,

$1<p\leq 2$, by the Schwartz representation theorem for $\mathfrak{D}'_{L^p}$. Then for $\varphi\in\mathcal{S}$,

$$\langle V,\ \varphi(t)\rangle=\langle \sum_{|\beta|\leq m} D^{\beta}g_{\beta}(x),\ \mathfrak{F}^{-1}[\varphi(t);\ x]\rangle \tag{9}$$
$$= \sum_{|\beta|\leq m}(-1)^{|\beta|}\langle g_{\beta}(x),\ \mathfrak{F}^{-1}[t^{\beta}\varphi(t);\ x]\rangle$$
$$=\langle \sum_{|\beta|\leq m}(-1)^{|\beta|}t^{\beta}\mathfrak{F}^{-1}[g_{\beta}(x);\ t],\ \varphi(t)\rangle.$$

Since $g_{\beta}\in L^p$, $1<p\leq 2$, then $h_{\beta}(t)=\mathfrak{F}^{-1}[g_{\beta}(x);\ t]\in L^q$, $\frac{1}{p}+\frac{1}{q}=1$; and $V=\sum_{|\beta|\leq m}(-1)^{|\beta|}t^{\beta}h_{\beta}(t)$. The fact that $C(U;\ z\in B_{\sigma})$ is analytic in B_{σ} has been shown by Tillmann [4] as stated above.

Now let $\alpha(t)\in\mathcal{E}$, the set of infinitely differentiable functions, such that $\alpha(t)=1$ on S_{σ}^{0}, $0\leq\alpha(t)\leq 1$, and supp $(\alpha(t))\subseteq S_{\sigma}^{(\varepsilon,\dots,\varepsilon)}$ for some fixed $\varepsilon>0$. Then $\langle V,\ e^{-2\pi i\langle z,t\rangle}\rangle=\langle V,\ \alpha(t)e^{-2\pi i\langle z,t\rangle}\rangle$ exists and is analytic for $z\in B_{\sigma}$ (see Carmichael [13]); and by Theorem 2 we have

$$\langle\langle V,\ e^{-2\pi i\langle z,t\rangle}\rangle,\ \varphi(x)\rangle=\langle V,\ \alpha(t)e^{2\pi\langle y,t\rangle}\hat{\varphi}(t)\rangle. \tag{10}$$

Define $H(t)$ by $H(t)=1$ on S_{σ}^{0}, and $H(t)=0$ otherwise. Since supp$\,(V)\subseteq S_{\sigma}^{0}$ and $V=\sum_{|\beta|\leq m}(-1)^{|\beta|}t^{\beta}h_{\beta}(t)$, then supp $(h_{\beta}(t))\subseteq S_{\sigma}^{0}$ for each β. Thus

$$\begin{aligned}\langle V,\ \alpha(t)e^{2\pi\langle y,t\rangle}\hat{\varphi}(t)\rangle&=\langle H(t)V,\ \alpha(t)e^{2\pi\langle y,t\rangle}\hat{\varphi}(t)\rangle\\&=\langle \mathfrak{F}[H(t)\alpha(t)e^{2\pi\langle y,t\rangle}V],\ \varphi(x)\rangle.\end{aligned} \tag{11}$$

By Theorem 1 we then have that

$$\begin{aligned}\langle \mathfrak{F}[H(t)\alpha(t)e^{2\pi\langle y,t\rangle}V],\ \varphi(x)\rangle&=\langle U*\mathfrak{F}[H(t)\alpha(t)e^{2\pi\langle y,t\rangle}],\ \varphi(x)\rangle\\&=\langle U*(-2\pi i)^{-n}\prod_{j=1}^{n}\frac{\operatorname{sgn} y_j}{z_j},\ \varphi(x)\rangle=\\&=\langle(2\pi i)^{-n}(\prod_{j=1}^{n}\operatorname{sgn} y_j)\langle U,\ \prod_{j=1}^{n}\frac{1}{t_j-z_j}\rangle,\ \varphi(x)\rangle.\end{aligned} \tag{12}$$

Combining (10), (11), and (12) we obtain (8), where the equality is interpreted as equality in $\mathcal{S}'$. It is straightforward to show that $\alpha(t)e^{2\pi\langle y,t\rangle}\hat{\varphi}(t)\rightarrow\alpha(t)\hat{\varphi}(t)$ in the topology of $\mathcal{S}$ as $y=Im(z)\rightarrow 0$. Since

$V \in \mathcal{S}'$ then by (10) we have

$$(13) \qquad \langle\langle V, e^{-2\pi i\langle z, t\rangle}\rangle, \varphi(x)\rangle \to \langle V, \hat{\varphi}(t)\rangle = \langle U, \varphi(x)\rangle$$

as $Im(z) \to 0$. Thus from (8) and (13) we see that $C(U; z \in B_\sigma) \to U \in \mathcal{D}'_{L^p}$, $1 < p \leq 2$, as $Im(z) \to 0$; and the proof is complete.

We note that p can't be 1 in Theorem 3 because $\prod_{j=1}^{n} \frac{1}{t_j - z_j}$ is not an element of $\dot{\mathcal{B}}$, whose dual space is $\mathcal{D}'_{L^1}$. Thus if $p=1$, then $C(U; z \in B_\sigma)$ is not well defined.

In Carmichael [13] we have considered functions $f(z)$ of n complex variables which are analytic in B_σ and satisfy

$$(14) \quad |f(z)| \leq C_\delta(1+|z|)^N \exp[2\pi\langle A, (|y_1|, \ldots, |y_n|)\rangle], \ y = Im(z),$$

in $\{z : \sigma_j Im(z_j) \geq \delta_j > 0, \ j=1, \ldots, n\}$ where $\delta = (\delta_1, \ldots, \delta_n)$. Here N is any fixed real number, C_δ is a constant which may depend on δ, and $A = (a_1, \ldots, a_n)$ is an n-tuple of fixed, nonnegative real numbers. We shall denote such functions by G_σ^A. The following result is proved in Carmichael [13].

THEOREM 4. *Let $f(z) \in G_\sigma^A$, and let $f(z)$ converge in $\mathcal{S}'$ to an element U as $Im(z) \to 0$. Then $U \in \mathcal{S}'$; and there exists an element $V \in \mathcal{S}'$ having support in S_σ^A such that $U = \hat{V}$ and $f(z) = \langle V, e^{-2\pi i\langle z, t\rangle}\rangle$, $z \in B_\sigma$.*

Using Theorems 3 and 4 we obtain following.

COROLLARY 1. *Let $U \in \mathcal{D}'_{L^p}$, $1 < p \leq 2$, be the boundary value in $\mathcal{S}'$ of a function $f(z) \in G_\sigma^0$ as $Im(z) \to 0$. Then $f(z) = C(U; z \in B_\sigma)$, where U is the Fourier transform of an element $V = \sum_{|\beta| \leq m} (-1)^{|\beta|} t^\beta h_\beta(t) \in \mathcal{S}'$, $h_\beta \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$, and* supp $(V) \subseteq S_\sigma^0$.

PROOF. By Theorem 4 there exists an element $V \in \mathcal{S}'$ with supp $(V) \subseteq S_\sigma^0$ such that $U = \hat{V}$ and $f(z) = \langle V, e^{-2\pi i\langle z, t\rangle}\rangle$, $z \in B_\sigma$. Calculate as in (9) we obtain that $V = \sum_{|\beta| \leq m} (-1)^{|\beta|} t^\beta h_\beta(t)$, $h_\beta \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$. But now by (8), $C(U; z \in B_\sigma) = \langle V, e^{-2\pi i\langle z, t\rangle}\rangle$ in $\mathcal{S}'$. Thus $f(z) = C(U; z \in B_\sigma)$ in $\mathcal{S}'$ as desired.

We now give necessary and sufficient conditions that an element $U\in\mathfrak{D}'_{L^p}$ be the $\mathcal{S}'$ boundary value of a function in G_σ^0.

THEOREM 5. *$U\in\mathfrak{D}'_{L^p}$, $1<p\leq 2$, is the $\mathcal{S}'$ boundary value of a function $f(z)\in G_\sigma^0$ for fixed σ as $Im(z)\to 0$ if and only if $C(U;\ z\in B_{\tilde\sigma})=0$, where $B_{\tilde\sigma}$ is any of the other 2^n-1 octants.*

PROOF. Calculating as in (9) we have that $V=\mathfrak{F}^{-1}(U)=$ $=\sum_{|\beta|\leq m}(-1)^{|\beta|}t^\beta h_\beta(t)$, where $h_\beta\in L^q$, $\frac{1}{p}+\frac{1}{q}=1$, and $V\in\mathcal{S}'$. Now assume that U is the boundary value of $f(z)\in G_\sigma^0$ for fixed σ. Then by Theorem 4, $\operatorname{supp}(V)\subseteq S_\sigma^0$. Let $\tilde\sigma$ be any of the other 2^n-1 choices for σ, and let $z\in B_{\tilde\sigma}$. For any choice of $\tilde\sigma$ let $\alpha_{\tilde\sigma}(t)$ be defined as was $\alpha(t)$ in the proof of Theorem 4. Define $H_{\tilde\sigma}(t)$ by $H_{\tilde\sigma}(t)=1$ on $\{t:\ -\infty<\tilde\sigma_j t_j<0,$ $j=1, \ldots, n\}$; and $H_{\tilde\sigma}(t)=0$ otherwise. Then by exactly the same calculation as in obtaining (8) of Theorem 3 we have

$$C(U;\ z\in B_{\tilde\sigma})=\langle H_{\tilde\sigma}(t)V,\ \alpha_{\tilde\sigma}(t)e^{-2\pi i\langle z,t\rangle}\rangle \tag{15}$$
$$=\langle V,\ H_{\tilde\sigma}(t)\alpha_\sigma(t)e^{-2\pi i\langle z,t\rangle}\rangle.$$

But $\operatorname{supp}(V)\subseteq S_\sigma^0$. Thus $\operatorname{supp}(V)\cap\operatorname{supp}(H_{\tilde\sigma}(t)\alpha_{\tilde\sigma}(t)e^{-2\pi i\langle z,t\rangle})=\varphi$, which implies that $\langle V,\ H_{\tilde\sigma}(t)\alpha_{\tilde\sigma}(t)e^{-2\pi i\langle z,t\rangle}\rangle=0$. Thus by (15), $C(U;\ z\in B_{\tilde\sigma})=0$ as desired.

Conversely, suppose that $C(U;\ z\in B_{\tilde\sigma})=0$ where $B_{\tilde\sigma}$ is any of the 2^n-1 octants other than B_σ, for fixed σ. Since $V=\sum_{|\beta|\leq m}(-1)^{|\beta|}t^\beta h_\beta(t)$, then $V=H_\sigma(t)V+\sum_{\tilde\sigma}H_{\tilde\sigma}(t)V$ almost everywhere, where there are 2^n-1 terms in $\sum_{\tilde\sigma}$. Hence for $\varphi\in\mathcal{S}$ we have

$$\langle V,\ \varphi\rangle=\langle H_\sigma(t)V+\sum_{\tilde\sigma}H_{\tilde\sigma}(t)V,\ \varphi\rangle$$

and

$$\langle U,\ \varphi\rangle=\langle V,\ \varphi\rangle=\langle\mathfrak{F}[H_\sigma(t)V],\ \varphi\rangle+\sum_{\tilde\sigma}\langle\mathfrak{F}[H_{\tilde\sigma}(t)V],\ \varphi\rangle. \tag{16}$$

By hypothesis and a calculation as in the proof of Theorem 3 we have for each octant $B_{\tilde\sigma}$ that

$$0=C(U;\ z\in B_{\tilde\sigma})=\langle H_{\tilde\sigma}(t)V,\ \alpha_{\tilde\sigma}(t)e^{-2\pi i\langle t,z\rangle}\rangle; \tag{17}$$

and using the same proof as in (13) of Theorem 3 we obtain

$$(18) \qquad \langle H_{\tilde{\sigma}}(t)V,\ \alpha_{\tilde{\sigma}}(t)e^{-2\pi i\langle z,t\rangle}\rangle \to \mathfrak{F}[H_{\tilde{\sigma}}(t)V]$$

in $\mathcal{S}'$ as $Im(z) \to 0$. From (17) and (18) we see that $\mathfrak{F}[H_{\tilde{\sigma}}(t)]=0$ for each choice of $\tilde{\sigma}$. Returning to (16) we thus that $\langle U, \varphi\rangle = \langle \hat{V}, \varphi\rangle = \langle \mathfrak{F}[H_\sigma(t)V], \varphi\rangle$. Since the Fourier transform is a continuous, one-to-one mapping of $\mathcal{S}'$ onto $\mathcal{S}'$, then $\hat{V}=\mathfrak{F}[H_\sigma(t)V]$ implies $V=H_\sigma(t)V$ in $\mathcal{S}'$. Thus supp $(V)=$ supp $(H_\sigma(t)V)\subseteq S_\sigma^0$. We now consider the function $f(z)=\langle V, \alpha_\sigma(t)e^{-2\pi i\langle z,t\rangle}\rangle z\in B_\sigma$, which exists. Since supp $(V)\subseteq S_\sigma^0$ then by (8) in Theorem 3, $f(z)=\langle V, \alpha_\sigma(t)e^{-2\pi i\langle z,t\rangle}\rangle = C(U;\ z\in B_\sigma)$; and $f(z)$ is analytic in B_σ. Thus by Theorem 3, $f(z)\to U\in\mathcal{D}'_{L^p}$, $1<p\leq 2$, in the topology of $\mathcal{S}'$ as $Im(z)\to 0$. We have proved in Carmichael [13] that if $V\in\mathcal{S}'$ and supp $(V)\subseteq S_\sigma^0$, then $f(z)=\langle V, \alpha_\sigma(t)e^{-2\pi i\langle z,t\rangle}\rangle$ satisfies the boundedness condition (14) where $A=(0, \ldots, 0)$. Thus $f(z)\in G_\sigma^0$ for the fixed σ. Hence $f(z)=\langle V, \alpha_\sigma(t)e^{-2\pi i\langle z,t\rangle}\rangle$, $z\in B_\sigma$, is the desired function, and the proof is complete.

In view of the boundary value results obtained, it is natural to ask if any given $U\in\mathcal{D}'_{L^p}$ can be expressed as a sum of boundary values of functions analytic in the 2^n octants. Tillmann [4] has shown that this is indeed true. We wish now to obtain this result using the techniques and theorems obtained in this paper; for by our techniques we are able to say more about the elements in the decomposition of U than Tillmann has done. Not only is each element in the decomposition the boundary value of an analytic function, but it is also the Fourier transform of an element in $\mathcal{S}'$ having support in a specified subset of $\boldsymbol{R}^n$. We also include the case $p=1$, which Tillmann has not done.

THEOREM 6. *Let* $U\in\mathcal{D}'_{L^p}$, $1\leq p\leq 2$. *Then* $U=\sum_\sigma W_\sigma$ *in* $\mathcal{S}'$, *where* W_σ *is the boundary value in the* $\mathcal{S}'$ *topology of a function* $f_\sigma(z)\in G_\sigma^0$; *and* $W_\sigma=\hat{V}_\sigma$, *where* $V_\sigma\in\mathcal{S}'$ *and* supp $(V_\sigma)\subseteq S_\sigma^0$.

PROOF. By a calculation as in (9), $V=\mathfrak{F}^{-1}(U)=\sum_{|\beta|\leq m}(-1)^{|\beta|}t^\beta h_\beta(t)$, where $h_\beta(t)$ is continuous and bounded if $p=1$ and $h_\beta(t)\in L^q$, $\frac{1}{p}+\frac{1}{q}=1$, if $1\leq p<2$. Define the functions $H_\sigma(t)$ and $\alpha_\sigma(t)$ as in the proof of Theorem 5, and let $V_\sigma=H_\sigma(t)V$. As in the proof of Theorem 5, we have

$V = \sum_{\sigma} V_\sigma$ almost everywhere; and $\mathrm{supp}\,(V_\sigma) \subseteq S_\sigma{}^0$. We have proved in Carmichael [13] that for each fixed σ, $f_\sigma(z) = \langle V_\sigma, \alpha_\sigma(t) e^{-2\pi i \langle z, t \rangle} \rangle$ is an element of $G_\sigma{}^0$; and by the same method as we used to prove (13) in Theorem 3, we have $f_\sigma(z) \to \hat{V}_\sigma = W_\sigma$ in $\mathcal{S}'$ as $Im(z) \to 0$, $z \in B_\sigma$. Thus for each $\varphi \in \mathcal{S}$

$$\langle U, \varphi \rangle = \langle \hat{V}, \sigma \rangle = \langle \sum_{\sigma} W_\sigma, \varphi \rangle,$$

where each W_σ is the boundary value of a function $f_\sigma(z) \in G_\sigma{}^0$; and $W_\sigma = \hat{V}_\sigma$, $\mathrm{supp}\,(V_\sigma) \subseteq S_\sigma{}^0$.

We now put

$$K(t, z) = (\pi)^{-n} \prod_{j=1}^{n} \frac{(\mathrm{sgn}\, y_j) y_j}{|t_j - z_j|^2},$$

where $y_j \neq 0$, $j = 1, \ldots, n$. $K(t; z)$ is the n-dimensional Poisson kernel. Now let $U \in \mathcal{D}'_{L^p}$, $1 < p < \infty$; and define $P(U; z \in B_\sigma)$ by

$$P(U; z \in B_\sigma) = \langle U_t, K(t; z) \rangle.$$

We call $P(U; z \in B_\sigma)$ the generalized Poisson integral of U for $z \in B_\sigma$, and the following theorem shows that $P(U; z \in B_\sigma)$ is well defined.

THEOREM 7. *Let $U \in \mathcal{D}'_{L^p}$, $1 < p < \infty$. Then $P(U; z \in B_\sigma)$ exists and is an n-harmonic function.*

PROOF. Since

$$\prod_{j=1}^{n} \frac{y_j}{\pi |t_j - z_j|^2} = \prod_{j=1}^{n} \left(\frac{1}{t_j - z_j} - \frac{1}{t_j - \bar{z}_j} \right),$$

then we have

$$\begin{aligned} (19) \quad P(U; z \in B_\sigma) &= C(U; z \in B_\sigma) - C(U; (\bar{z}_1, z_2, \ldots, z_n), z \in B_\sigma) + \\ &+ \ldots + (-1)^j C(U; (\bar{z}_1, \ldots, \bar{z}_j, z_{j+1}, \ldots, z_n), z \in B_\sigma) + \\ &+ \ldots + (-1)^n C(U; (\bar{z}_1, \ldots, \bar{z}_n), z \in B_\sigma), \end{aligned}$$

where there are 2^n elements in the sum on the right hand side of (19). (i.e. there are 2^n possible ways of taking complex conjugates of the components of z.) In (19) the j in $(-1)^j$ represents the number of complex conjugates taken. But Tillmann [4] has shown that the generalized Cauchy integral for $U\in\mathfrak{D}'_{L^p}$, $1<p<\infty$, exists if $y_j\neq 0$, $j=1, \ldots, n$. Thus the sum in (19) is well-defined; hence so is $P(U;\ z\in B_\sigma)$. We recall that a complex valued function of n complex variables is n-harmonic if it is harmonic in each variable separately. It is straighforward to show that

$$\text{(20)} \qquad \left(\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial y_j^2}\right)K(t;\ z)=0,\ j=1,\ \ldots,\ n,$$

and that

$$\text{(21)} \qquad \left(\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial y_j^2}\right)\langle U,\ K(t;\ z)\rangle=\langle U,\ \left(\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial y_j^2}\right)K(t;\ z)\rangle,$$

$j=1, \ldots, n$. Thus by (20) and (21) we have that $P(U;\ z\in B_\sigma)$ is n-harmonic as desired, and the proof is complete.

In general $P(U;\ z\in B_\sigma)$ will not be analytic because of the complex conjugates in the sum in (19). We note that Bremermann [19] has obtained a result similar to Theorem 7 for the space $\mathcal{O}'_0$. Our result is more general since $\mathcal{O}'_0$ is a proper subspace of $\mathfrak{D}'_{L^p}$. We also note that Beltrami and Wohlers [8] have obtained results similar to some of the theorems in this section. However, their work is for only 1 dimension and only for the space $\mathfrak{D}'_{L^2}$; so that again our results are more general.

The following theorems relate the generalized Poisson integral to the generalized Cauchy integral, the Fourier-Laplace transform, and the boundary value problem.

THEOREM 8. *Let $U\in\mathfrak{D}'_{L^p}$, $1<p\leq 2$. If U is the $\mathcal{S}'$ boundary value of $f(z)\in G_\sigma{}^0$ for fixed σ as $Im(z)\to 0$, then there exists an element $V\in\mathcal{S}'$ with* supp $(V)\subseteq S_\sigma{}^0$ *such that*

$$\text{(22)} \qquad f(z)=\langle V,\ e^{-2\pi i\langle z,\ t\rangle}\rangle=C(U;\ z\in B_\sigma)=P(U;\ z\in B_\sigma),\ z\in B_\sigma\,.$$

PROOF. Let $z\in B_\sigma$ for some fixed σ. $P(U;\ z\in B_\sigma)$ has the representation (19) in terms of generalized Cauchy integrals. But by Theorem 5, all the generalized Cauchy integrals in (19) vanish except $C(U;\ z\in B_\sigma)$. Thus $P(U;\ z\in B_\sigma)=C(U;\ z\in B_\sigma)$. By Corollary 1 we also know that there exists a $V \in \mathcal{S}'$ such that $\operatorname{supp}(V) \subseteq S_\sigma^0$ and $f(z) = \langle V, e^{-2\pi i \langle z, t \rangle} \rangle = C(U; z \in B_\sigma)$. Thus (22) is obtained.

The following theorem shows that the generalized Poisson integral of an element $U \in \mathfrak{D}'_{L^p}$, $1 < p \leq 2$, has a substantially different property than the generalized Cauchy integral. We shall prove that for any fixed octant B_σ, $P(U; z \in B_\sigma) \to U$ in $\mathcal{S}'$ as $Im(z) \to 0$. If no other assumption is made on U other than $U \in \mathfrak{D}'_{L^p}$, $1 < p \leq 2$, then, as we have seen, $C(U; z \in B_\sigma)$ converges in $\mathcal{S}'$ only to one of the elements in the decomposition of U for any fixed B_σ.

THEOREM 9. *Let $U \in \mathfrak{D}'_{L^p}$, $1 < p \leq 2$. Let γ be one of the 2^n values of the n-tuple σ, and let γ be fixed. Then $P(U; z \in B_\gamma) \to U$ in $\mathcal{S}'$ as $Im(z) \to 0$, $z \in B_\gamma$.*

PROOF. For the moment let σ be any of the 2^n possible n-tuples. By Theorem 6, $U = \sum_\sigma W_\sigma$, where for each σ, W_σ is the $\mathcal{S}'$ boundary value of the function

$$f_\sigma(z) = \langle V_\sigma, \alpha_\sigma(t) e^{-2\pi i \langle z, t \rangle} \rangle = C(U; z \in B_\sigma),$$

which is analytic in B_σ. Now let γ be one of the n-tuples σ, and let γ be fixed. Let $z \in B_\gamma$, and suppose $B_{\tilde\sigma}$ denotes any of the other $2^n - 1$ octants. By taking the appropriate complex conjugates in any of the elements in the sum (19), we see that for each $B_{\tilde\sigma}$

$$\text{(23)} \qquad C(U; z \in B_{\tilde\sigma}) = (-1)^j C(U; (\bar{z}_1, \ldots, \bar{z}_j, z_{j+1}, \ldots, z_n), z \in B_\gamma),$$

where the $(-1)^j$ appears because we are changing j of the values in $\prod_{j=1}^{n} \operatorname{sgn} y_j$. Thus if $z \in B_\gamma$, then for each element in the sum in (19) we have by (23) and the above stated consequences from Theorem 6 that

$$\text{(24)} \qquad (-1)^j C(U; (\bar{z}_1, \ldots, \bar{z}_j, z_{j+1}, \ldots, z_n), z \in B_\gamma) \to W_{\tilde\sigma}$$

in $\mathcal{S}'$ as $Im(z) \to 0$, $z \in B_\gamma$. For the particular fixed value γ of σ we have $C(U; z \in B_\gamma) \to W_\gamma$ in $\mathcal{S}'$ as $Im(z) \to 0$, $z \in B_\gamma$. Thus by this fact, (24), and (19) we obtain

$$P(U; z \in B_\gamma) \to W_\gamma + \sum_{\tilde\sigma} W_{\tilde\sigma} = U$$

in $\mathcal{S}'$ as $Im(z) \to 0$, $z \in B_\gamma$, as desired.

THEOREM 10. *Let* $U \in \mathcal{D}'_{L^p}$, $1<p\leq 2$, *be the Fourier transform of* $V \in \mathcal{S}'$, *where* $\operatorname{supp}(V) \subseteq S_\sigma^0$. *Then* U *is the* $\mathcal{S}'$ *boundary value of a function* $f(z) \in G_\sigma^0$ *such that*

$$f(z) = \langle V, e^{-2\pi i\langle z, t\rangle}\rangle = C(U;\ z \in B_\sigma) = P(U;\ z \in B_\sigma),\ z \in B_\sigma. \tag{25}$$

PROOF. Under the stated hypothesis we have by Theorem 3 that U is the $\mathcal{S}'$ boundary value of the function

$$f(z) = \langle V, e^{-2\pi i\langle z, t\rangle}\rangle = C(U;\ z \in B_\sigma), \tag{26}$$

which is analytic in B_σ; and we have proved in Carmichael [13] that $f(z)$ satisties inequality (14) for $A=(0, ..., 0)$. Thus $f(z) \in G_\sigma^0$. But now the assumptions of Theorem 8 are satisfied for $f(z)$ defined in (26). Thus by Theorem 8, we have (25).

Section V. The Fourier-Laplace Transform of Elements in $\mathcal{D}'_{L^p}$.

In this section we wish to obtain conditions on an analytic function $f(z)$ such that $f(z)$ can be represented by the Fourier-Laplace transform of an element in $\mathcal{D}'_{L^p}$ and such that the convergence in $\mathcal{S}'$ of $f(z)$ to a distribution can be proved. This is a different approach to the boundary value problem than that taken in section IV of this paper; for in the theorems of section IV, an assumption is always made either concerning the convergence or the distribution to which an analytic function converges. We make no such assumptions in this section.

Let $g(t) \in L^q$, $1<q<\infty$. If there exists a function $f(x) \in L^p$, $1 \leq p < \infty$, such that

$$f(x) = \underset{N\to\infty}{\text{l.i.m.}} \int_{-N}^{N} \cdots \int_{-N}^{N} g(t) e^{-2\pi i\langle z, t\rangle} dt,$$

where l.i.m. denotes the limit in the L^p norm, then $f(x)$ is said to be the Fourier transform of $g(t)$. *For* $1<q\leq 2$, $f(x) \in L^p$ is known to exist where p and q are related by $\frac{1}{p}+\frac{1}{q}=1$. In this section we shall assume the existence of the Fourier transform $f(x)$ of $g(t)$ for all q, $1<q<\infty$.

Because of this assumption, the proofs of our theorems in this section are fundamentally different from those in section III of Carmichael [1]. There $f(x) \in L^p$, $1 \leq p \leq 2$; thus $f(x)$ possessed a Fourier transform itself. In the present paper $f(x)$ is the Fourier transform of some other function and does not necessarily posses a Fourier transform itself. This change is necessary in order to obtain generalizations of Theorems 1 and 2 in Carmichael [1], where we now let $f(x) \in L^p$, $1 \leq p < \infty$, instead of restricting p to $1 \leq p \leq 2$. In both cases $f(x)$ is the limit function of $f(z)$, which is analytic in a subset of $\boldsymbol{C}^n$, as $Im(z) \rightarrow 0$.

For convenience we shall prove the results of this section for functions analytic in the octant $B_{\bar{1}} = \{z : Im(z_j) > 0,\ j = 1, \ldots, n\}$; but the same proofs apply to any octant B_σ. Let $A = (a_1, \ldots, a_n)$ be a fixed n-tuple of nonnegative real numbers. Let n denote the dimension and ε be any fixed positive real number. Throughout this section by $\lim_{y \to 0} f(z) = f(x)$, we mean that $f(z)$ converges to $f(x)$ for almost all real x as $y_j = Im(z_j) \rightarrow 0+$, $j = 1, \ldots, n$.

THEOREM 11. *Let $f(z)$ be analytic in $B_{\bar{1}}$ and satisfy*

$$(27) \quad |f(z)| \leq C_{\bar{1}}(1 + |z|)^{-n-\varepsilon} \exp [2\pi \langle A, (|y_1|, \ldots, |y_n|) \rangle], \quad z \in B_{\bar{1}}.$$

Let $\lim_{y \to 0} f(z) = f(x)$, where $f(x)$ is continuous and $f(x) \in L^p$, $1 \leq p < \infty$. Let $f(x)$ be the Fourier transform of $g(t) \in L^q$, $1 < q < \infty$. Then $\operatorname{supp}(g) \subseteq S_{\bar{1}}^{A}$ *almost everywhere, and*

$$f(z) = \int_{S_{\bar{1}}^{A}} g(t) e^{-2\pi i \langle z, t \rangle} dt, \quad z \in B_{\bar{1}}.$$

PROOF. Let $z = (z_1, \ldots, z_n)$ be arbitrary but fixed in $B_{\bar{1}}$. Let $v = \delta + i\xi$. In each coordinate plane in $\boldsymbol{C}^n$ construct a contour C_j, $j = 1, \ldots, n$, composed of the semi-circle $|v_j| = r_j$, $Im(v_j) \geq 0$, $r_j > |z_j|$, $j = 1, \ldots, n$, and that part of the real axis δ_j enclosed by this semi-circle. Now consider C_n. $f(z_1, \ldots, z_n, v_n)$ is analytic with respect to v_n in $\{v_n : Im(v_n) > 0\}$ and continuous on $Im(v_n) = 0$. By Cauchy's integral theorem we thus have

$$(28) \qquad f(z) \frac{1}{2\pi i} \int_{C_n} \frac{f(z_1, \ldots, z_{n-1}, v_n)}{v_n - z_n} dv_n,$$

where $\nu_n = \delta_n + i\xi_n$. Consider also

$$(29) \qquad \frac{1}{2\pi i}\int_{C_n} \frac{f(z, \ldots, z_{n-1}, v_n)\{\exp(-2\pi i(v_n - z_n)U_n) - 1\}}{v_n - z_n} dv_n$$

where U_n is any real number. The integrand is analytic in v_n inside C_n and continuous on the boundary of C_n. Thus by Cauchy's theorem, the value of the integral in (29) is zero. Adding (28) and (29) we obtain

$$(30) \qquad f(z) = \frac{1}{2\pi i}\int_{C_n} \frac{f(z_1, \ldots, z_{n-1}, v_n)\exp(-2\pi i(v_n - z_n)U_n)}{v_n - z_n} dv_n.$$

If $U_n < (-a_n)$, a_n being the n th. coordinate of A, then by (27), the part of the integral in (30) over the semi-circle tends to zero as $r_n \to \infty$. Hence letting $r_n \to \infty$ in (30) we obtain

$$(31) \qquad f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(z_1, \ldots, z_{n-1}, \delta_n)\exp(-2\pi i(\delta_n - z_n)U_n)}{\delta_n - z_n} d\delta_n.$$

where $\delta_n = Re(v_n)$ and $U_n < (-a_n)$. We now consider $f(z_1, \ldots, z_{n-1}, \delta_n)$ in the integrand of (31). Using exactly the same argument as above with respect to z_{n-1} we obtain

$$f(z) = \left(\frac{1}{2\pi i}\right)^2 \int_{-\infty}^{\infty} \frac{\exp(-2\pi i(\delta_n - z_n)U_n)}{\delta_n - z_n}$$

$$\int_{-\infty}^{\infty} \frac{f(z_1, \ldots, z_{n-2}, \delta_{n-1}, \delta_n)\exp(-2\pi i(\delta_{n-1} - z_{n-1})U_{n-1})}{\delta_{n-1} - z_{n-1}} d\delta_{n-1} d\delta_n,$$

where $U_{n-1} < (-a_{n-1})$. Continuing in this way for each of the remaining points z_{n-2} down to z_1, we obtain

$$(32) \qquad f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{f(\delta)\exp(-2\pi i\langle(\delta - z), U\rangle}{\prod_{j=1}^{n}(\delta_j - z_j)} d\delta,$$

where $\delta=(\delta_1, \ldots, \delta_n)$, $d\delta=d\delta_1 \ldots d\delta_n$, and $U=(U_1, \ldots, U_n)$ such that $U_j<(-a_j)$, $j=1, \ldots, n$.

We now make the convention that $\int\limits_{[N]}$ will denote $\int\limits_{-N}^{N} \ldots \int\limits_{-N}^{N}$. Since $f \in L^p$, $1 \leq p < \infty$, is the Fourier transform in L^p of $g \in L^q$, $1<q<\infty$, then

$$(33) \qquad \int\limits_{[N]} \frac{f(\delta) \exp(-2\pi i\langle(\delta-z), U\rangle}{\prod_{j=1}^{n}(\delta_j-z_j)} d\delta=$$

$$=\int\limits_{[N]} \frac{\exp(-2\pi i\langle(\delta-z), U\rangle}{\prod_{j=1}^{n}(\delta_j-z_j)} \int\limits_{R_n} g(\eta) \exp(-2\pi i\langle\delta, \eta\rangle) d\eta d\delta=$$

$$=\int\limits_{R_n} g(\eta) \int\limits_{[N]} \frac{\exp(-2\pi i\langle(\delta-z), U\rangle) \exp(-2\pi i\langle\delta, \eta\rangle)}{\prod_{j=1}^{n}(\delta_j-z_j)} d\delta d\eta,$$

where the change of order of integration is easily justified. Since $Im(z_j)>0$, $j=1, \ldots, n$, then

$$\frac{\exp(-2\pi i\langle(\delta-z), U\rangle)}{\prod_{j=1}^{n}(\delta_j-z_j)}=(2\pi i)^n \int\limits_{-\infty}^{-U_n} \ldots \int\limits_{-\infty}^{-U_1} \exp(2\pi i\langle(\delta-z), t\rangle) dt.$$

Thus

$$(34) \qquad \int\limits_{[N]} \frac{\exp(-2\pi i\langle(\delta-z), U\rangle) \exp(-2\pi i\langle\delta, \eta\rangle)}{\prod_{j=1}^{n}(\delta_j-z_j)} d\delta=$$

$$=(2\pi i)^n \int\limits_{[N]} \exp(-2\pi i\langle\delta, \eta\rangle)\{\int\limits_{-\infty}^{-U_n} \ldots \int\limits_{-\infty}^{-U_1} \exp(2\pi i\langle(\delta-z), t\rangle) dt d\delta=$$

$$=(2\pi i)^n \int\limits_{-\infty}^{-U_n} \ldots \int\limits_{-\infty}^{-U_1} \exp(-2\pi i\langle t, z\rangle) \int\limits_{[N]} \exp(2\pi i\langle t-\eta, \delta\rangle) d\delta dt=$$

$$=(2\pi i)^n \int_{-\infty}^{-U_n} \dots \int_{-\infty}^{-U_1} \exp(-2\pi i\langle t, z\rangle) \prod_{j=1}^{n} \frac{\sin N(t_j-\eta_j)}{\pi(t_j-\eta_j)}\, dt,$$

where the change of order of integration is justified. Putting (34) into (33) we have

$$\int_{[N]} \frac{f(\delta)\exp(-2\pi i\langle(\delta-z),\ U\rangle)}{\prod_{j=1}^{n}(\delta_j-z_j)}\, d\delta = \tag{35}$$

$$=(2\pi i)^n \int_{R_n} g(\eta)\{\int_{-\infty}^{-U_n} \dots \int_{-\infty}^{-U_1} \exp(-2\pi i\langle t, z\rangle) \prod_{j=1}^{n} \frac{\sin N(t_j-\eta_j)}{\pi(t_j-\eta_j)}\, dt d\eta$$

$$=(2\pi i)^n \int_{-\infty}^{-U_n} \dots \int_{-\infty}^{-U_1} \exp(-2\pi i\langle t, z\rangle) \int_{R_n} g(\eta) \prod_{j=1}^{n} \frac{\sin N(\eta_j-t_j)}{\pi(\eta_j-t_j)}\, d\eta dt,$$

where again we have changed the order of integration. Now it is well known that if $g(\eta)\in L^q$, $1<q<\infty$, then

$$\lim_{N\to\infty} \left\| \int_{R_n} g(\eta) \prod_{j=1}^{n} \frac{\sin N(\eta_j-t_j)}{\pi(\eta_j-t_j)}\, d\eta - g(t) \right\|_{L_q} = 0.$$

Thus letting $N\to\infty$ on both sides of (35) and recalling (32) we have

$$f(z)=\int_{-\infty}^{-U_n} \dots \int_{-\infty}^{-U_1} g(t)\exp(-2\pi i\langle t, z\rangle)dt. \tag{36}$$

Since $z\in B_{\bar{1}}$ was choosen arbitrarily, then (36) holds for $z\in B_{\bar{1}}$ and all n-tuples of real numbers U such that $U_j<(-a_j)$, $j=1, \dots, n$. But $f(z)$ does not depend on U. Thus $g(t)$ must vanish almost everywhere on $\{t : t_j>a_j$ for at least one j, $j=1, \dots, n\}$, which implies that $\operatorname{supp}(g)\subseteq \subseteq S_{\bar{1}}^{A}=\{t : -\infty<t_j\leq a_j,\ j=1, \dots, n\}$ almost everywhere. From this fact and (36), we obtain immediately the desired representation of $f(z)$; and the proof is complete.

Using Theorems 11 and 2 we can now prove the main result of this section. Let D a constant such that $D\in\boldsymbol{R}^1$ and $D>0$. Choose D large enough so that for $z\in B_{\bar{1}}$

$$|D+\langle z, z\rangle|\geq\tfrac{1}{2}(1+|z|)^2. \tag{37}$$

THEOREM 12. *Let $f(z)$ be analytic in $B_{\bar{1}}$ and satisfy (14) for $\sigma=(1, \ldots, 1)$. Let $\lim_{y\to 0} f(z)=f(x)$, where $f(x)$ is continuous. Let D be a constant such that (37) holds; and suppose $(D+\langle x, x\rangle)^{-R}f(x)\in L^p$, $1\leq p<\infty$, and $(D+\langle x, x\rangle)^{-R}f(x)$ is the Fourier transform in L^p of a function $g(t)\in L^q$, $1<q<\infty$, where the constant R is choosen so large that $N-2R\leq -n-\varepsilon$, $\varepsilon>0$. Then there exists an element $U\in\mathfrak{D}'_{L^q}$, $1<q<\infty$, having support in $S^A_{\bar{1}}$ almost everywhere such that*

$$f(z)=\langle U, e^{-2\pi i\langle z, t\rangle}\rangle, \quad z\in B_{\bar{1}},$$

and $f(z)\to f(x)=\hat{U}\in\mathcal{S}'$ in the topology of $\mathcal{S}'$ as $Im(x)\to 0$.

PROOF. Let D be a constant such that (37) holds. Then $(D+\langle z, z\rangle)^{-R}f(z)$ satisfies the assumptions of Theorem 11. Thus $\operatorname{supp}(g)\subseteq S^A_{\bar{1}}$ almost everywhere and

$$(D+\langle z, z\rangle)^{-R}f(z)=\int_{S^A_{\bar{1}}} g(t)e^{-2\pi i\langle z, t\rangle}dt. \tag{38}$$

Let Δ be a differential operator defined by $\Delta=\frac{1}{4\pi^2}\sum_{j=1}^{n}\frac{\partial^2}{\partial t_j^2}$, and put $U=(D-\Delta)^R g(t)$. By the characterization theorem of Schwartz ([10], Théorème XXV, p. 201), $U\in\mathfrak{D}'_{L^q}$, $1<q<\infty$. Also $\operatorname{supp}(U)=\operatorname{supp}(g)\subseteq\subseteq S^A_{\bar{1}}$ almost everywhere. Let $\alpha(t)\in\mathcal{E}$ such that $\alpha(t)=1$ on $S^A_{\bar{1}}$ and $\operatorname{supp}(\alpha(t))\subseteq S^{A+\eta}_{\bar{1}}$, for some fixed $\eta>0$. Then

$$\langle U, e^{-2\pi i\langle z, t\rangle}\rangle=\langle U, \alpha(t)e^{-2\pi i\langle z, t\rangle}\rangle$$

is well defined since $\mathfrak{D}'_{L^q}\subset\mathcal{S}'$. Thus

$$\begin{aligned}\langle U, e^{-2\pi i\langle z, t\rangle}\rangle&=\langle (D-\Delta)^R g(t), \alpha(t)e^{-2\pi i\langle z, t\rangle}\rangle\\&=(D+\langle z, z\rangle)^R\int_{S^A_{\bar{1}}} g(t)e^{-2\pi i\langle z, t\rangle}dt.\end{aligned} \tag{39}$$

By (38) and (39) we obtain $f(z)=\langle U, e^{-2\pi i\langle z, t\rangle}\rangle$, $z\in B_{\Gamma}^{A}$, as desired. Using Theorem 2 and exactly the same proof used to show (13) in Theorem 3, we obtain $f(z)\to V=\hat{U}\in\mathcal{S}'$ in the topology of $\mathcal{S}'$ as $Im(z)\to 0$. Now

$$\begin{aligned}\langle V, \varphi\rangle&=\langle (D-\Delta)^R g(t), \hat{\varphi}(t)\rangle\\&=\langle (D+\langle x, x\rangle)^R \hat{g}(x), \varphi\rangle.\end{aligned}$$

By hypothesis the Fourier transform of $g(t)$ is $(D+\langle x, x\rangle)^{-R}f(x)$. Thus

$$\begin{aligned}\langle V, \varphi\rangle&=\langle (D+\langle x, x\rangle)^R(D+\langle x, x\rangle)^{-R}f(x), \varphi\rangle\\&=\langle f(x), \varphi\rangle;\end{aligned}$$

and the proof is complete.

Section VI. Tubular Radial Domains and Tubular Cones.

In this section we shall consider functions which are analytic in tubular radial domains and shall generalize some of our results obtained above. We then consider functions analytic in a still more general subset of $\boldsymbol{C}^n$, the tubular cone, and shall obtain an interesting theorem concerning distributional boundary values in $\mathfrak{D}'_{L^p}$ of such a function. For convenience in this section we change $(-2\pi i\langle x, t\rangle)$ to $(2\pi i\langle x, t\rangle)$ in the definition of Fourier transform. We do this so that our notation and calculations in this section will be similar to those of Vladimirov [12]; for we shall use some of Vladimirov's results here. This change of sign in the exponetial term of the Fourier transform does not change the properties of the Fourier transform at all.

Let C be an open connected cone, and let C' be an arbitrary compact subcone of C. Put $T^C=\boldsymbol{R}^n+iC$. Let $f(z)$ satisfy

$$(40)\quad |f(z)|\leq K(C')(1+|z|)^N \exp[2\pi(b+\eta)|y|],\ z\in T^{C'},\ y=Im(z),$$

where $K(C')$ is a constant depending on C', N is a fixed real number, b is a fixed nonnegative real number, and η is any positive real number. We denote the set of all functions which are analytic in T^C and satisfy

(40) by $Q_C{}^b$. In Carmichael [20] we have announced the following generalization of Theorem 4.

THEOREM 13. *Let $f(z)\in Q_C{}^b$, and let $f(z)$ converge in $\mathcal{S}'$ to U as $Im(z)\to 0$, $Im(z)\in C'\subset C$. Then $U\in\mathcal{S}'$; and there exists an element $V\in\mathcal{S}'$ having support in $\{t : u_C(t)\le b\}$ such that $U=\hat{V}$ and $f(z)= =\langle V, e^{2\pi i\langle z,t\rangle}\rangle$, $z\in T^{C'}$.*

A detailed proof of Theorem 13 will appear in Carmichael [21] as will the proof of the following theorem, which we shall need here.

THEOREM 14. *Let $V\in\mathcal{S}'$ and* supp $(V)\subseteq\{t : u_C(t)\le b\}$, *where b is any fixed nonnegative real number. Let $\varphi\in\mathcal{S}$ and C' be an arbitrary compact subcone of $O(C)$. Then*

$$\langle\langle V, e^{2\pi i\langle z,t\rangle}\rangle, \varphi\rangle=\langle V, e^{-2\pi\langle y,t\rangle}\hat{\varphi}\rangle,\ z\in T^{C'}.$$

We now obtain a generalization of Theorem 3.

THEOREM 15. *Let C be an open connected cone. Let $V\in\mathcal{S}'$ such that* supp $(V)\subseteq\{t : u_C(t)\le 0\}=I$ *and $\hat{V}=U\in\mathcal{D}'_{L^p}$, $1<p\le 2$. Then $V=\sum_{|\beta|\le m} t^\beta h_\beta(t)$, $h_\beta(t)\in L^q$, $\frac{1}{p}+\frac{1}{q}=1$; $\langle V, e^{2\pi i\langle z,t\rangle}\rangle$ is analytic in $T^{O(C)}$;*

$$\langle V, e^{2\pi i\langle z,t\rangle}\rangle=\langle U, \int_I e^{2\pi i\langle z-t,\delta\rangle}d\delta\rangle,\ z\in T^{C'}, \tag{41}$$

$C'\subset O(C)$, as elements of $\mathcal{S}'$; and

$$\langle U, \int_I {}^{2\pi i\langle z-t,\delta\rangle}d\delta\rangle\to U$$

in $\mathcal{S}'$ as $Im(z)\to 0$, $Im(z)\in C'\subset O(C)$.

PROOF. By the same calculation as in (9) in the proof of Theorem 3, we have the desired representation of V. (Because of our change in the Fourier transform in this section, the term $(-1)^{|\beta|}$ does not appear in this representation of V as it did in (9).) Let $\beta(t)\in\mathcal{E}$ such that $\beta(t)=1$ for $t\ge 0$, $\beta(t)=0$ for $t\le(-\varepsilon)$, $\varepsilon>0$, and $0\le\beta(t)\le 1$; and put $\alpha(t)=\beta(\langle t, y\rangle)$, $y\in O(C)$. Vladimirov ([12], Theorem 2, p. 239) has shown that $\langle V, e^{2\pi i\langle z,t\rangle}\rangle = \langle V, \alpha(t)e^{2\pi i\langle z,t\rangle}\rangle$ is analytic for $z \in T^{0(C)}$. Now put $H(t)=1$ on supp (V) and $H(t)=0$ elsewhere. By Theorems 14 and 1 we have for $z \in T^{C'}$, $C' \subset 0(C)$, that

$$\begin{aligned}\langle\langle V, e^{2\pi i\langle z,t\rangle}\rangle, \varphi\rangle &= \langle V, \alpha(t)e^{-2\pi\langle y,t\rangle}\hat{\varphi}(t)\rangle \\ &= \langle \mathfrak{F}[H(t)\alpha(t)e^{-2\pi\langle y,t\rangle}V], \varphi(x)\rangle \\ &= \langle U * \int_I e^{2\pi i\langle z,t\rangle}dt, \varphi(x)\rangle \\ &= \langle\langle U, \int_I e^{2\pi i\langle z-t,\delta\rangle}d\delta\rangle, \varphi(x)\rangle,\end{aligned}$$

where $\varphi \in \mathfrak{S}$. Thus (41) is obtained. By exactly the same method as in proving (13) in Theorem 3, we have $\langle V, e^{2\pi i\langle z,t\rangle}\rangle \to U$ in $\mathfrak{S}'$ as $Im(z) \to 0$, $Im(z) \in C' \subset 0(C)$. By this fact and (41), we thus obtain

$$\langle U, \int_I e^{2\pi i\langle z-t,\delta\rangle}d\delta\rangle \to U$$

in $\mathfrak{S}'$ as $Im(z) \to 0$, $Im(z) \in C' \subset 0(C)$, as desired; and the proof is complete.

We note that if T^C is just an octant B_σ, than $\int_I e^{2\pi i\langle z-t,\delta\rangle}d\delta$ is the usual Cauchy kernel. The following corollary is a generalization of Corollary 1.

COROLLARY 2. *Let $U \in \mathfrak{D}'_{L^p}$, $1 < p \leq 2$, be the boundary value in $\mathfrak{S}'$ of $f(z) \in Q_C{}^0$. Let C' be an arbitrary compact subcone of C. Then*

$$f(z) = \langle U, \int_I e^{2\pi i\langle z-t,\delta\rangle}d\delta\rangle, \quad z \in T^{C'}, \tag{42}$$

where U is the Fourier transform of an element $V \in \mathfrak{S}'$ which has support in $I = \{t : u_C(t) \leq 0\}$.

PROOF. By Theorem 13, there exists an element $V \in \mathfrak{S}'$ with

$\text{supp}(V)\subseteq I$ such that $U=\hat{V}$ and

(43) $$f(z)=\langle V,\ e^{2\pi i\langle z,t\rangle}\rangle,\quad z\in T^{C'}.$$

But for such a V, (41) holds. Thus by (41) and (43), we have (42).

We now let C be an open cone which is not necessarily connected. For such a cone, $T^C=\boldsymbol{R}^n+iC$ is a tubular cone.

THEOREM 16. *Let C be an open, disconnected cone such that $C=\bigcup_{j=1}^{r} C_j$, C_j being an open connected cone, $j=1, \ldots, r$. Let $f(z)$ be analytic in the tubular cone T^C and satisfy* (40) *for $z\in T^{C'}$, C' being an arbitrary compact subcone of C. For each C_j, $j=1, \ldots, r$, let C'_j be an arbitrary compact subcone of C_j; and suppose*

$$\lim_{\substack{y\to 0\\ y\in C'_j\subset C_j}} f(z)=U_j\in\mathfrak{D}'_{L^p},\quad 1\leq p\leq\infty,$$

in the topology of $\mathcal{S}'$ for each $j=1, \ldots, r$ such that $U_1=U_2=\ldots=U_r$. *Then*

(44) *$f(z)$ is analytic in $\bigcup_{j=1}^{r} T^{0(C_j)}$;*

(45) *there exist an element $V\in\mathcal{S}'$ with* $\text{supp}(V)\subseteq\{t: u_{0(C)}(t)\leq b\rho_C\}$ *such that if $1\leq p\leq 2$, then $V=\sum_{|\beta|\leq m} t^\beta h_\beta(t)$, $h_\beta(t)$ being continuous and bounded if $p=1$, and $h_\beta(t)\in L^q$, $\dfrac{1}{p}+\dfrac{1}{q}=1$, if $1<p\leq 2$;*

(46) *$f(z)=\langle V,\ e^{2\pi i\langle z,t\rangle}\rangle$, $z\in\bigcup_{j=1}^{r} T^{0(C_j)}$;*

(47) *$|f(z)|\leq M(C')(1+|z|^K(1+|y|^{-R}) \exp[2\pi b\rho_C|y|]$, $z\in T^{C'}$, $C'\subset\bigcup_{j=1}^{r} 0(C_j)$, $y=Im(z)$, where K and R are nonnegative real numbers which do not depend on C'.*

PROOF. By hypothesis $f(z)$ is analytic in T^C and satisfies (40) for $z\in T^{C'}$, $C'\subset C$. Thus $f(z)$ is analytic in T^{C_j} for each $j=1, \ldots, r$; and $f(z)$ satisfies (40) for $z\in T^{C'_j}$, $C'_j\subset S_j$, $j=1, \ldots, r$. By hypothesis $f(z)\to U_j$, $j=1, \ldots, r$, as $Im(z)\to 0$, $Im(z)\in C'_j\subset C_j$; and $U_j\in\mathfrak{D}'_{L^p}$, $1\leq p\leq\infty$. Thus by Theorem 13, for each U_j there exists a $V_j \in \mathcal{S}'$ such that $\operatorname{supp}(V_j) \subseteq \subseteq \{t : u_{C_j}(t) \leq b\}$, $\hat{V}_j = U_j$, and

$$(48) \qquad f(z) = \langle V_j, e^{2\pi i \langle z, t\rangle}\rangle, \; z \in T^{C'_j}, \; C'_j \subset C_j, \; j = 1, \ldots, r.$$

By hypothesis $U_1 = U_2 = \ldots = U_r$, and we call this common value U. Then $\mathfrak{F}^{-1}(U_1) = \ldots = \mathfrak{F}^{-1}(U_r) = \mathfrak{F}^{-1}(U)$. But $V_j = \mathfrak{F}^{-1}(U_j)$, $j = 1, \ldots, r$. Thus $V_1 = V_2 = \ldots = V_r$, and we call this common value V. We then have $\hat{V} = U$; and if $1 \leq p \leq 2$, then by a calculation as in (9) in the proof of Theorem 3, we have $V = \sum_{|\beta| \leq m} t^\beta h_\beta(t)$, where $h_\beta(t)$ is continuous and bounded if $p = 1$ and $h_\beta(t) \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$, if $1 < p \leq 2$.

Now $\operatorname{supp}(V_j) \subseteq \{t : u_{C_j}(t) \leq b\}$, $j = 1, \ldots, r$. Thus V vanishes on $\bigcup_{j=1}^{r} \{t : u_{C_j}(t) > b\}$. From the definition of ρ_C we have
$u_{0(C)}(t) \leq \rho_C u_C(t)$. Since

$$u_C(t) = \max_{j=1, \ldots, r} u_{C_j}(t),$$

we then have

$$(49) \qquad u_{0(C)}(t) \leq \rho_C \max_{j=1, \ldots, r} u_{C_j}(t);$$

and by a lemma of Vladimirov ([12], Lemma, p. 200), $\rho_C < \infty$. Now suppose that $u_{0(C)}(t) > b\rho_C$, and consider the set $J = \{t : u_{0(C)}(t) > b\rho_C\}$. If $t \in J$, then by (49), $t \in \{t : \max_{j=1, \ldots, r} u_{C_j}(t) > b\}$; hence $t \in \bigcup_{j=1}^{r} \{t : u_{C_j}(t) > b\}$; and on this set V vanishes. Thus V vanishes if $t \in J$ which implies that $\operatorname{supp}(V) \subseteq \{t : u_{0(C)}(t) \leq b\rho_C\}$ as desired. Since $V \in \mathcal{S}'$ and $\operatorname{supp}(V) \subseteq \subseteq \{t : u_{0(C)}(t) \leq b\rho_C\}$, we have by a theorem of Vladimirov ([12], Theorem 2, p. 239) that $\langle V, e^{2\pi i \langle z, t\rangle}\rangle$ is analytic in $T^{0(C)}$ and satisfies

$$(50) \qquad |\langle V, e^{2\pi i \langle z, t\rangle}\rangle| \leq M(C')(1 + |z|)^K (1 + |y|^{-R}) \exp[2\pi b \rho_C |y|],$$

$y = Im(z)$, $z \in T^{C'}$, $C' \subset 0(C)$. By hypothesis $f(z)$ is analytic in T^C; *thus* $f(z)$ is analytic in T^{C_j} for each $j = 1, \ldots, r$. It is then immediate by Bochner's theorem on analytic extension of tubular domains (Bochner and Martin [22], Chapter V) that $f(z)$ is analytic in $T^{0(C_j)}$ for each $j = 1, \ldots, r$; and

(44) is obtained. By (48), $f(z)$ and $\langle V, e^{2\pi i\langle z,t\rangle}\rangle$ are identical in a subset of $T^{0(C_j)}$ for each $j=1, ..., r$. Thus by the identity theorem for analytic functions, $f(z)=\langle V, {}^{2\pi i\langle z,t\rangle}\rangle$, $z\in T^{0(C_j)}$, for each $j=1, ..., r$; and (46) is obtained. But then by (46) and (50), we have (47); and the proof is complete.

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