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The Fourier transform on L^1_{loc} and its application to Cauchy problems for linear partial differential equations

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THE FOURIER TRANSFORM ON L^1_{loc} AND ITS APPLICATION TO CAUCHY PROBLEMS FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction.

In [5] Treves introduces the Banach spaces K^s. Here s is a real parameter. Loosely speaking an element in K^s is something whose Fourier transform is square integrable with respect to the measure $e^{2s|\xi|}d\xi$. Treves uses these spaces in connection with applications of the Ovsjannikov theorem [3]. Let s be a real number and let $g(t)$ be an increasing function for $t \geq 0$. We define other Banach spaces K_g^s. Again loosely speaking an element in K_g^s is something whose Fourier transform is integrable with respect to the measure $\exp(sg(|\xi|))d\xi$. By varying s and g we can define a linear space $\mathfrak{J}$, the union of the spaces K_g^s, and the Fourier transform on $\mathfrak{J}$ such that $\mathfrak{F}(\mathfrak{J}) = L^1_{loc}$, the set of all locally integrable functions in R^n.

The Ovsjannikov theorem is a main tool in [5]. In Sect. 2 we prove a higher order Carathéodory version of that theorem. The spaces K_g^s and $\mathfrak{J}$ are defined in Sect. 3. There we also define the function classes $B(m, g, s, \rho)$, $C(m, g, s, \rho)$, $MB(g, \rho)$ and $MC(g, \rho)$. An element $f(t)$ in $B(m, g, s, \rho)$ is a function in $|t| \leq \rho$ with values in K_g^s that is $m-1$ times continuously differentiable with a Bochner integrable m : th derivative. If the m : th derivative is continuous the f is in $C(m, g, s, \rho)$. The sets $MB(g, \rho)$ and $MC(g, \rho)$ are two different sets of multipliers.

In Sect. 4 we state and prove a theorem for a Cauchy problem for

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a linear partial differential equation with solutions in $B(m, g, s, \rho)$. It is a rather generally applicable theorem. There are many questions that arise about the nature of the spaces K_g^s and $\mathfrak{J}$. In Sect. 5 we shall only point out one property that shows the difference between distributions and elements in K_g^s.

In [6] Treves defines the Banach spaces K^{sp}. They are identical with the spaces K_g^s. They are also used in two simple examples of Cauchy problems for the heat equation in two dimensions. So the idea with application of the Ovsjannikov theorem to the spaces K_g^s can be found in [6]. Therefore a part of the intermediate results in Sect. 3 are also found in [6].

It is obvious that the higher order Ovsjannikov theorem can be used to simplify old proofs with the original theorem. The Carathéodory version has also many interesting implications.

We shall use the following notation. The n-dimensional Euclidian space is called R^n and its dual R_n. We write $(t, x)=(t, x_1, \dots, x_n)\in R^{n+1}$, $\xi=(\xi_1, \dots, \xi_n)\in R_n$, $\langle x, \xi\rangle=x_1\xi_1+\dots+x_n\xi_n$, $D_t=-i\partial/\partial t$, $D_j=-i\partial/\partial x_j$, $1\leq j\leq n$, $i=\sqrt{-1}$. By $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ we denote the Fourier transform and its inverse.

$$\mathfrak{F}u(\xi)=\widehat{u}(\xi)=\int_{R^n} e^{-i\langle x, \xi\rangle}u(x)dx.$$

If $\alpha=(\alpha_1, \dots, \alpha_n)$ is a multi-index with non-negative integers as components then $|\alpha|=\alpha_1+\dots+\alpha_n$ and $D_x^\alpha=D_1^{\alpha_1}\dots D_n^{\alpha_n}$. L^p is the set of all functions whose p: th power is integrable in R^n (or R_n). $L^1_{\mathrm{loc}}(R_n)$ is the set of all locally integrable functions in R_n.

2. A higher order Carathéodory version of the Ovsjannikov theorem.

Let $I=\{s;\ -2<s<-1\}$. We assume that we have a one-parameter family of Banach spaces E_s, $s\in I$. We define $E=\cup E_s$, $s\in I$. The spaces E_s have the following property

(2.1) *If* $-2<s'<s<-1$ *then* $E_s\subset E_{s'}$ *and the natural injection* $E_s\to E_{s'}$ *has a norm* ≤ 1.

There exists an integer $m\geq 1$, a constant $C>1$, and linear operators

$\Lambda_j : E \to E$, $1 \le j \le m$, such that

(2.2) *the restriction of* Λ_j *to* E_s *is a bounded linear operator* $E_s \to E_{s'}$ *with norm less than* $C(s-s')^{-j}$ *if* $s > s'$, $1 \le j \le m$.

We now consider functions depending on t having their values in E. As to the theory of Bochner integrable functions see Hille-Phillips [1], Ch. 3. There exist operator valued functions $B_j(t)$, $1 \le j \le m$, such that

(2.3) *for some number* ρ *and for any* $s \in I$, *if* $v(t)$ *is a continuous function* $\{t;\ |t| \le \rho\} \to E_s$ *then* $B_j(t)v(t)$ *is a Bochner integrable function in* $\{t;\ |t| \le \rho\}$ *with values in* E_s, $1 \le j \le m$.

and

(2.4) *for all* s *and, almost all* t, $|t| \le \rho$, $B_j(t)$ *is a bounded linear map of* E_s *into itself. There further exists an integrable function* $L_1(r) \ge 1$ *in* $0 \le r \le \rho$, *such that* $\| B_j(t) \| \le L_1(|t|)$, $|t| \le \rho$, $1 \le j \le m$, $L_1(r)$ *being independent of* s.

Now we state our Carathéodory version of the Ovsjannikov theorem.

THEOREM 1. *We assume that* $B_j(t)$ *and* Λ_j, $1 \le j \le m$, *are defined as above satisfying* (2.1)-(2.4). *For a certain fixed* $s' \in I$ *there exists a function* $f(t)$ *such that*

(2.5) $f(t)$ *is a Bochner integrable function* $\{t;\ |t| \le \rho\} \to E_{s'}$.

Let D_t^{-j} *denote iterated integration* j *times from the origin. Then it follows that for every fixed* $s \in I$, $s < s'$, *there exist a constant* ε, $0 < \varepsilon \le \rho$, *and a unique Bochner integrable function* $v(t)$ *with values in* E_s *such that*

$$v(t) = \sum_{j=1}^{m} B_j(t)\Lambda_j D_t^{-j} v + f(t), \quad |t| \le \varepsilon. \tag{2.6}$$

Let

$$u(t) = D_t^{-m} v.$$

Then $u(t)$ *is* $m-1$ *times continuously differentiable. Also* $d^m u/dt^m = u^{(m)}$

exists and is Bochner integrable. From (2.6) *it follows that u is the unique solution of the Cauchy problem*

$$u^{(m)}=\sum_{j=1}^{m} B_j(t)\Lambda_j u^{(m-j)}+f(t),\ u^{(j)}(0)=0,\ 0\leq j<m,\ |t|\leq\varepsilon, \tag{2.7}$$

that is valued in E_s and has these differentiability properties.

Proof. We define

$$v^{-1}=v^0=0,\ v^{k+1}=\sum_{j=1}^{m} B_j(t)\Lambda_j D_t^{-j}v^k+f(t),\ k=0,\ 1,\ 2,\ \ldots . \tag{2.8}$$

It is clear that all v^k are Bochner integrable for $|t|\leq\rho$. We further define

$$w^k=v^k-v^{k-1},\ k=0,\ 1,\ 2,\ \ldots . \tag{2.9}$$

Then we have

$$w^{k+1}=\sum_{j=1}^{m} B_j(t)\Lambda_j D_t^{-j}w^k,\ k=1,\ 2,\ \ldots . \tag{2.10}$$

We now define, see (2.4), and (2.2),

$$L(r)=Cem^{m+1}(L_1(r)+\|f(-r)\|_{s'}+\|f(r)\|_{s'}). \tag{2.11}$$

We assume that

$$L(r)\geq 1,\ \text{and that}\ K(r)=\int_0^r L(s)ds. \tag{2.12}$$

Then $K'(r)$ exists a.e. and $K'(r)=L(r)$ a.e. Let $d=s'-s$, $s<s'$. We may assume that $K(\rho)\leq 1$. If this is not satisfied we choose a new smaller ρ so it is satisfied. With the new ρ we assert that for $|t|\leq\rho$

$$\|w^{k+1}(t)\|_s\leq L(|t|)(K(|t|))^k d^{-mk},\ k=0,\ 1,\ 2,\ \ldots,\ s<s'. \tag{2.13}$$

It follows from (2.8), (2.9), (2.11) and (2.1) that (2.13) is true for $k=0$. Now we note that

$$|D_t^{-1}(K(|t|))^k|\leq|D_t^{-1}L(|t|)(K(|t|))^k|\leq(K(|t|))^{k+1}(k+1)^{-1},$$

since

$$d((K(t))^{k+1})/dt=(k+1)L(t)(K(t))^{k} \text{ a.e.}$$

and

$$L(t)\geq 1.$$

If (2.13) is true for $k=l$ then (2.10), (2.13), (2.2), (2.3) and (2.4) give

$$(2.14) \qquad \| w^{l+2}(t) \|_s \leq$$

$$\leq CL_1(t) \sum_{j=1}^{m} (d-\delta)^{-ml}\delta^{-j}(K(t))^{l+j}(l+1)^{-1} \dots (l+j)^{-1}.$$

Here we have chosen

$$\delta=(s'-s)(ml+1)^{-1}.$$

Noting that

$$(d-\delta)^{-ml}\delta^{-j}=d^{-ml-j}(1+1/ml)^{ml}(ml+1)^{j},\ d<1, \text{ and } K(|\,t\,|)\leq 1,$$

(2.14) now gives

$$\| w^{l+2}(t) \|_s \leq meCL_1(|\,t\,|)d^{-m(l+1)}m^m(K(|\,t\,|))^{l+1}\leq$$

$$\leq L(|\,t\,|)d^{-m(l+1)}(K(|\,t\,|))^{l+1}.$$

By that we have proved (2.13).

We now choose ε from $K(2\varepsilon)=(s'-s)^m$ if $(s'-s)^m\leq K(\rho)$. Otherwise we choose $\varepsilon=\rho$. It follows that

$$K(|\,t\,|)(s'-s)^{-m}\leq K(\varepsilon)(s'-s)^{-m}<1, \ |\,t\,|<\varepsilon.$$

Then

$$v_k(t)=\sum_{j=1}^{k} w_j(t), \ k=1,\ 2,\ \dots,$$

converges in E_s for $|\,t\,|\leq\varepsilon$. The limit function $v(t)$ is Bochner integrable in $|\,t\,|\leq\varepsilon$, sinces $\| v^k(t) \|_s\leq L(|\,t\,|)(1-K(|\,t\,|)d^{-m})^{-1}$. See [1], Th. 3.7.9. Depending on our choice of ε it is clear that for some $s''>s$ v^k also converges towards v in $E_{s''}$ for $|t|\leq\varepsilon$. From (2.8) we get

$$\| v-\sum_{j=1}^{m} B_j\Lambda_j D_t^{-j} v-f \|_s=$$

$$=\| (v-v^{k+1})-\sum_{j=1}^{m} B_j\Lambda_j D_t^{-j}(v-v^{k}) \|_s\leq$$

$$\leq\| v-v^{k+1} \|_s+\sum_{j=1}^{m} \| B_j\Lambda_j \|_{s''s} | D_t^{-j} \| v-v^{k} \|_{s''} |.$$

It is now clear that v solves (2.6). Let now $w=v$ be a solution of (2.6) when $f=0$. Let $d=s-s''$. Then for some $w^k=w$ one proves as above that for some $\varepsilon'\leq\varepsilon$, $K(|t|)d^{-m}<1$, $|t|\leq\varepsilon'$ and

$$\| w \|_{s''}\leq L(|t|)(K(|t|)^k d^{-mk},\ k=1,\ 2,\ \ldots,\ |t|\leq\varepsilon'.$$

That implies $w(t)=0$ in $|t|\leq\varepsilon'$ when the values are in $E_{s''}$. But then $w(t)=0$ also when the values are taken in E_s. It is clear that after an adjustment of L and K we can translate the procedure to start in $t=t_0\neq 0$, $|t_0|\leq\varepsilon$. The uniqueness is proved. We have now proved Theorem 1.

3. The inverse Fourier transform of L^1_{loc}.

We shall now construct a linear space $\mathfrak{J}$ such that the Fourier transform of $\mathfrak{J}$ is L^1_{loc}. We start by constructing concrete examples of Banach spaces satisfying (2.1).

As before we let $I=\{s;\ -2<s<-1\}$. For $b>0$ we define

$$h_j(b,\ \xi)=1+|\xi_j|^b,\ 1\leq j\leq n,$$

and

$$h(b,\ \xi)=h_1(b,\ \xi)+\ \ldots\ +h_n(b,\ \xi).$$

If a realvalued continuous function $g(\xi)$ is such that

$$g(\xi)\geq h(b,\ \xi),\ \xi\in R_n,$$

then g is said to belong to the class $\mathfrak{G}(b)$. Now let

$$A=\mathfrak{F}^{-1}(C_0^\infty(R_n)).$$

Let further $s \in I$ and $g \in \mathfrak{G}(b)$. On A we define the norm $\| \ \|_{g,s}$ by

$$\| u \|_{g,s} = \int | \widehat{u}(\xi) | e^{sg(\xi)} d\xi. \tag{3.1}$$

DEFINITION 3.1. *The completion of A under the norm* (3.1) *is called K_g^s.*

A Cauchy sequence from A under $\| \ \|_{g,s}$ defines an element u in K_g^s. If $s' < s$ then it also defines an element $\bar{u}$ in $K_g^{s'}$. We identify u with $\bar{u}$. The proof of the following proposition is now trivial.

PROPOSITION 3.1. *The spaces $K_g^s = E_s$ satisfy* (2.1).

For an arbitrary K_g^s and for an arbitrary element u in K_g^s we shall now define the Fourier transform $\widehat{u}$ of u. This will give a natural justification for the identification of elements in K_g^s and $K_g^{s'}$ above.

Take a Cauchy sequence (u_j) from A that defines an element u in K_g^s. Then $(\widehat{u}_j)$ defines a Cauchy sequence in L^1 with respect to the measure $e^{sg(\xi)} d\xi$. We call the limit function $\widehat{u}$. We note that $\widehat{u}$ is independent of the choice of the Cauchy sequence (u_j) that defines u. We also note that $\widehat{u}$ is in $L^1_{\text{loc}}(R_n)$ and that a subsequence of $\widehat{u}_j e^{sg(\xi)}$ tends to $\widehat{u} e^{sg(\xi)}$ a.e. If (u_j) also defines an element u' in some $K_{g'}^{s'}$ then for a suitably chosen subsequence one has

$$\widehat{u}_{j_k} e^{sg(\xi)} \to \widehat{u} e^{sg(\xi)} \quad \text{and} \quad \widehat{u}_{j_k} e^{s'g'(\xi)} \to \widehat{u}' e^{s'g'(\xi)} \text{ a.e., } k \to \infty.$$

Since g and g' are continuous we have $\widehat{u} = \widehat{u}'$ a.e.

On the other hand take an arbitrary element $\bar{u} \in L^1_{\text{loc}}(R_n)$. It is then obvious that we can choose a $g \in \mathfrak{G}(1)$ such that for this fixed $\bar{u}$, $\bar{u} e^{sg(\xi)}$ is in $L^1(R_n)$ for all $s \in I$. Then we can choose a sequence of functions $\bar{u}_j$ with compact support such that $\bar{u}_j e^{sg(\xi)} \to \bar{u} e^{sg(\xi)}$ when $j \to \infty$. To every j we can choose a sequence $(\bar{u}_{jk})$, $\bar{u}_{jk} \in C_0^\infty(R_n)$, $k = 1, 2, \ldots$, such that $\bar{u}_{jk} \to \bar{u}_j$ in $L^1(R_n)$, when $k \to \infty$. It is now easy to find a sequence (u_j) from A such that $\widehat{u}_j e^{sg(\xi)}$ tends to $\bar{u} e^{sg(\xi)}$ in $L^1(R_n)$. That shows that (u_j) defines an element u in K_g^s with $\widehat{u} = \bar{u}$.

Using the discussion above we now define the linear space $\mathfrak{F}$.

DEFINITION 3.2. *If* $u \in K_g^s$, $u' \in K_{g'}^{s'}$ *and if* $\hat{u} = \hat{u}'$ *a.e. then we let* $u = u'$. *No we define the set* $\mathfrak{I}$ *as*

$$\mathfrak{I} = \cup K_g^s,\ s \in I,\ g \in \mathfrak{G}(b),\ b > 0.$$

The set $\mathfrak{I}$ is a linear space since $u \in K_g^s$ and $u' \in K_{g'}^{s'}$ implies that both u and u' are in $K_{4(g+g')}^{-1/2}$. We now define two important linear mappings on $\mathfrak{I}$ into itself.

DEFINITION 3.3. *Let* $u \in \mathfrak{I}$ *and let* s'' *be a real number. We define*

$$H_b^{s''} : u \to \mathfrak{F}^{-1}((h(b,\ \xi))^{s''} u).$$

DEFINITION 3.4. *Let* $u \in \mathfrak{I}$. *We define*

$$D_j : u \to \mathfrak{F}^{-1}(\xi_j u),\ 1 \leq j \leq n.$$

The definition of D_j agrees with the usual definition of differentiation on A and on other spaces on which the Fourier transform is already defined. The following propositions are used when we convert a partial differential equation into a form on which we can use Theorem 1.

PROPOSITION 3.2. *Let* $P(\xi)$ *be an arbitrary polynomial of degree* $\leq m$ *and let* $b > 0$ *and* $g \in \mathfrak{G}(b)$. *Then* $P(D_x) H_b^{-m/b}$ *is a continuous map from* K_g^s *into itself.*

PROOF. Evident.

PROPOSITION 3.3. *Let* $g \in \mathfrak{G}(b)$ *and let* $m > 0$. *Then the operator* H_b^m *is a bounded linear operator from* K_g^s *into* $K_g^{s'}$ *for* $s' < s$. *There exists a constant* $C > 0$ *such that the norm of the operator is equal or less than* $C(s-s')^{-m}$.

PROOF. Let $u \in K_g^s$. Then we have

$$\| H_b^m u \|_{g,s'} = \int | u(\xi) | (h(b,\ \xi))^m e^{s' g(\xi)} d\xi \leq$$
$$\leq \| u \|_{g,s} \sup_{\xi} (h(b,\ \xi))^m e^{(s'-s) h(b,\ \xi)}.$$

Since $\lambda^m e^{-\lambda}$ is bounded for $\lambda \geq 0$ we have the wanted estimate above.

DEFINITION 3.5. *The function $f(t)$ depends on the real variable t and has its values in the space K_g^s. Let m be a non-negative integer. If for some $\rho>0$ $f(t)$ is an m times continuously differentiable function for $|t|\leq\rho$ then we say that f belongs to the function $C(m, g, s, \rho)$. If $f(t)$ is Bochner integrable for $|t|\leq\rho$ then we say that $f(t)$ is in the class $B(0, g, s, \rho)$. If for some integer $m>0$ $f(t)\in C(m-1, g, s, \rho)$ and if also $d^m f/dt^m$ exists a.e. and is in $B(0, g, s, \rho)$ then we say that $f(t)$ is in the function class $B(m, g, s, \rho)$.*

We shall now define two kinds of multipliers connected with the spaces defined above. Here $*$ denotes convolution in the distribution sense and $\mathcal{S}'$ the Schwartz space $\mathcal{S}'$.

DEFINITION 3.6. *The function $a(t)$ is valued in $\mathcal{S}'$ for $|t|\leq\rho$. We further suppose that for all $f(t)$ in $C(0, g, s, \rho)$ $\widehat{a}(t)*\widehat{f}(t)$ exists with values in $L^1_{\text{loc}}(R_n)$ for almost all t in $|t|\leq\rho$, and that*

$$\widehat{a}(t) * \widehat{f}(t)e^{sg(\xi)}\in L^1(R_n), \quad |t|\leq\rho. \text{ a.e.}$$

We define

$$a(t)f(t)=\mathfrak{F}^{-1}(a(t) * f(t)).$$

If

$$a(t)f(t)\in B(0, g, s, \rho), \quad f\in C(0, g, s, \rho)$$

and if there exists an integrable function $L(r)$, $0\leq r\leq\rho$, such that L is independent of f and s and such that

$$\| a(t)f(t) \|_{g,s}\leq L(|t|) \, \| f(t) \|_{g,s}, \quad f\in C(0, g, s, \rho), \; s\in I, \; |t|\leq\rho,$$

then we say that $a(t)$ belongs to the function class $MB(g, \rho)$. If in the last inequality we can choose $L(r)$ as a constant and if

$$a(t)f(t)\in C(0, g, s, \rho), \quad f(t)\in C(0, g, s, \rho)$$

then we say that $a(t)$ is in the function class $MC(g, \rho)$.

We shall now give a sufficient condition for a complex valued function $a(t, x)$ to be in $MC(g, \rho)$.

Lemma 1. *We assume that the complex valued function* $a(t, x)$ *is such that* $\hat{a}(t, \xi)$ *exists and is continuous in* $\{(t, \xi);\ |t|\leq\rho,\ \xi\in R_n\}$. *Let* $g(\xi)=h(b, \xi)$ *for some* b, $0<b\leq 1$. *Further there exist constants* $L'>0$ *and* $r<-2$ *such that*

$$|\hat{a}(t, \xi)|\leq L'e^{rg(\xi)},\quad |t|\leq\rho,\quad \xi\in R_n. \tag{3.3}$$

Then it follows that $a(t, x)$ *belongs to* $MB(g, \rho)$.

Proof. Let $f(t)\in C(0, g, s, \rho)$. Then

$$\|a(t, x)f(t)\|_{g,s}\leq \int\Big(\int|\hat{a}(t, \xi)|\,|\hat{f}(t, \eta-\xi)|\,d\xi\Big)e^{sg(\eta)}d\eta\leq$$
$$\leq\int\int L'e^{(r-s)g(\xi)+k(\xi,\eta)}e^{sg(\eta-\xi)}|\hat{f}(t, \eta-\xi)|\,d\xi d\eta.$$

Here

$$k(\xi, \eta)=s(h(b, \eta)+h(b, \xi)-h(b, \eta-\xi)).$$

For $a'\geq 0$, $a''\geq 0$ $(a'+a'')^b\leq a'^b+a''^b$ when $0\leq b\leq 1$. This gives $k(\xi, \eta)\leq 0$ since $s\in I$. It follows that

$$\|a(t, x)f(t)\|_{g,s}\leq\|f(t)\|_{g,s}L'\int e^{(r+2)h(b,\xi)}d\xi\leq L\,\|f(t)\|_{g,s}.$$

Here L is independent of s, t, $|t|\leq\rho$. It is now easy to verify that $a(t, x)f(t)$ is continuous. Let $t_j\to t$ when $j\to\infty$. We get

$$\|a(t_j, x)f(t_j)-a(t, x)f(t)\|_{g,s}\leq\|(a(t_j, x)-a(t, x))f(t)\|_{g,s}+$$
$$+\|a(t_j, x)(f(t_j))-f(t))\|_{g,s}=E+F.$$

We can use the computations above to prove that $\lim_{j\to\infty}F=0$. The continuity of $\hat{a}(t, \xi)$, (3.3), the same computations and the theorem of dominated convergence gives $\lim_{j\to\infty}E=0$. The lemma is proved.

Even for $g\in\mathcal{G}(b)$, $b>1$, $MC(g, \rho)$ is not empty since all constants are in $MC(g, \rho)$. As an example of functions in $MC(g, \rho)$ when $g=h(b, \xi)$, $b\leq 1$, we can take the set A which is independent of t. As another example of functions independent of t which are in $MC(h(b, \xi), \rho)$ $0<b<1$, we take the functions in γ_0^δ, $\delta=1/b$, satisfying the estimate in Lemma 5.7.2, [2], p. 147.

We shall end this section by giving some examples in the form of propositions. These examples give some relations between the spaces above and other well-known spaces.

PROPOSITION 3.4. $C(0, g, s, \rho) \subset B(0, g, s, \rho)$, $g \in \mathcal{G}(b)$, $b>0$, $s \in I$, $\rho>0$.

PROOF. Evident.

PROPOSITION 3.5. *Let* $F=\{t;\ |t| \leq \rho\} \times R_n$. The function $\bar{f}(t, \xi)$ *is measurable in* F. *For a fixed* $g \in \mathcal{G}(b)$, $b>0$, *and a fixed* $s \in I$ $\bar{f}(t, \xi) e^{sg(\xi)}$ *is integrable in* F. *Then for almost all* t *in* $|t| \leq \rho$, $\bar{f}(t, \xi)$ *is the Fourier transform of an element* $f(t)$ *in* K_g^s. *The function* $f(t)$ *is Bochner integrable in* $|t| \leq \rho$.

PROOF. That $f(t)$ is in K_g^s for almost all t follows from the Fubini theorem. It also follows from that theorem that $\| f(t) \|_{g,s}$ is integrable. The dual of K_g^s is isomorphic to $L^\infty(R_n)$. The duality is given by

$$\langle u, v \rangle = \int \tilde{u}(\xi) v(\xi) e^{sg(\xi)} d\xi, \quad u \in K_g^s, \quad v \in L^\infty(R_n).$$

Since K_g^s is separable we only have to show that for all $v \in L^\infty(R_n)$ $\langle f(t), v \rangle$ is measurable. Then f is Bochner integrable. But also this follows from the Fubini theorem. See [1], Th. 3.7.4 and Th. 3.5.3. The proof is completed.

We give another important example of functions in $B(0, g, s, \rho)$.

PROPOSITION 3.6. *Let* $f(t, x)$ *be in* $L^1(\{t;\ |t| \leq \rho\} \times R^n)$. *Then it follows that for any* $g \in \mathcal{G}(b)$, $b>0$, *and for any* $s \in I$ $f(t, x)$ *can be indentified as an element in* $B(0, g, s, \rho)$.

PROOF. We know that $\tilde{f}(t, \xi) \in L^\infty(R_n)$ for almost all t. We first show that for every $v \in L^\infty(R_n)$

$$h(t) = \int \tilde{f}(t, \xi) v(\xi) e^{sg(\xi)} d\xi = \int \left(\int f(t, x) e^{i\langle x, \xi \rangle} v(\xi) e^{sg(\xi)} dx \right) d\xi,$$

is measurable. It is obvious that the last integrand is measurable and also integrable in $\{t;\ |t| \leq \rho\} \times R^n \times R_n$. Then the Fubini theorem says that $h(t)$ is integrable in $|t| \leq \rho$. So $f(t, x)$ is weakly measurable as a function of t with values in K_g^s. That $\| f(t) \|_{g,s}$ is integrable is shown by an analogous procedure. The proof is completed.

We end this section by the following proposition.

PROPOSITION 3.7. *Let $\mathcal{S}$ and $\mathcal{E}'$ denote the corresponding Schwartz spaces. Then for every $b>0$, $s\in I$, and $g\in\mathcal{G}(b)$ the spaces $\mathcal{S}$, $\mathcal{E}'$, $L^p(R^n)$, $1\leq p\leq 2$, $\mathfrak{F}^{-1}(L^q)$, $1\leq q\leq\infty$, are all contained in K_g^s.*

PROOF. The Paley-Wiener theorem, [2], p. 21, for the case $\mathcal{E}'$ and the fact that $\mathfrak{F}(L^p(R^n))\subset L^q(R_n)$, $1\leq p\leq 2$, $1/q+1/p=1$. If $\widehat{u}\in L^q(R_n)$, $1<q\leq\infty$ then

$$\| u \|_{g,s}=\int |\widehat{u}| e^{sg(\xi)}d\xi\leq \|\widehat{u}\|_{L^q}(\int e^{psg(\xi)}d\xi)^{1/p}.$$

If $\widehat{u}\in L^1(R_n)$ then $\| u\|_{g,s}\leq \|\widehat{u}\|_{L^1}$. The proposition is proved.

4. A Cauchy problem with solutions in $B(m, g, s, \rho)$.

We shall now formulate a theorem for a Cauchy problem. Note that we have a higher order Ovsjannikov theorem to our disposal. So in the proof we need not rewrite the original equation as a system of order one in the time variable.

THEOREM 2. *Let $b>0$ and $\rho>0$ be fixed real numbers and let $s'\in I=\{s;\ -2<s<-1\}$. We assume that $g\in\mathcal{G}(b)$ and that $m>0$ is a fixed integer. There exist functions $a_{j\alpha}$ belonging to $MB(g, \rho)$ for*

$$|\alpha|\leq jb,\ 1\leq j\leq m. \tag{4.1}$$

The given elements u_{0j} in $K_g^{s'}$ are such that

$$H_b^m u_{0j}\in K_g^{s'}\ \ 0\leq j<m. \tag{4.2}$$

The function $f(t)$ belongs to $B(0, g, s', \rho)$. It follows that to every $s<s'$, $s\in I$, there exist an ε, $0<\varepsilon\leq\delta$ and a unique function $u\in B(m, g, s, \varepsilon)$ such that

$$D_t^m u=\sum_{j=1}^{m}\sum_{|\alpha|\leq jb} a_{j\alpha}D_x^\alpha D_t^{m-j}u+f(t); \tag{4.3}$$

and

$$D_t^k u(0)=u_{0k}\ ,\ 0\leq k<m. \tag{4.4}$$

PROOF. We want to solve the Cauchy problem above using Theorem 1. We have defined $D_t = -i\partial/\partial t$. But after a multiplication by appropriate powers of i in (4.3) and (4.4) we may let $D_t = \partial/\partial t$ for the moment. Then we let

$$\bar{u} = \sum_{k=0}^{m-1} t^k (k!)^{-1} u_{0k} .$$

We note that

$$D_x^\alpha u_{0k} = D_x^\alpha H_b^{-m} H_b^m u_{0k} .$$

Then (4.1) and Prop. 3.2 says that $D_x^\alpha u_{0k} \in K_g^{s'}$, $|\alpha| \leq mb$, and

$$\bar{f}(t) = D_t^m \bar{u} - \sum_{j=1}^{m} \sum_{|\alpha| \leq jb} a_{j\alpha} D_x^\alpha D_t^{m-j} \bar{u} \in B(0,\ g,\ s',\ \rho).$$

So we assume in the following that $u_{0k}=0$, $0 \leq k < m$, in (4.4) and that we have a new f in (4.3) equal to $f - \bar{f}$ in the original notation. We now define

$$\Lambda_j = H_b{}^j, \quad B_j(t) = \sum_{|\alpha| \leq jb} a_{j\alpha} D_x^\alpha H_b^{-j}, \quad 1 \leq j \leq m.$$

It follows from Prop. 3.3 and 3.2 and from Def. 3.7 that the Λ_j and B_j satisfy the hypothesis of Theorem 1 with $E_s = K_g^s$. Then (4.3) and (4.4) are equivalent to (2.7). We apply Theorem 1. Theorem 2 is proved.

It is obvious that we can modify Theorem 1 to prove the following version of Theorem 2.

THEOREM 2′. *If in the hypothesis of Theorem* 2 $MB(g, \rho)$ *is replaced by* $MC(g, \rho)$ *and* $B(0, g, s', \rho)$ *by* $C(0, g, s', \rho)$ *then the conclusion of Theorem* 2 *is still true if* $B(m, g, s, \varepsilon)$ *is replaced by* $C(m, g, s, \varepsilon)$.

It may be noted that in the constant coefficient case we can change the interval I to $\{s;\ s < -1\}$. Then we can choose s so small that we at the same time can choose $\varepsilon = \rho$. In this case we get globality also in the time direction. The same is true when the coefficients depend on time only. But it is also clear that we change the class of multipliers when we change I.

5. Comments.

We start by a set of infinitely differentiable functions $C_0^\infty(B)$, B being a compact set in R^n. The completition of this set under the K_g^s-norm then may lead to elements that cannot be identified as distributions with compact support. We exemplify this by using Theorem 2 and the fundamental solution of the heat equation. Take an arbitrary function $u_0 \in C_0^\infty(R^1)$, $u_0 \geq 0$, $u_0 \neq 0$. The solution of

$$iD_t u = -D_x^2 u, \; u(0, x) = u_0(x), \tag{5.1}$$

is

$$u(t, x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4t} u_0(y) dy.$$

It is then clear that $u(t, x) > 0$ for all (t, x) with $t > 0$. So $u(t, x)$ does not have compact support in x for fixed t. On the other hand for $g = h(2, \xi)$, u_0 belongs to K_g^s together with all its derivatives. According to Theorem 2 (5.1) has a unique solution $u(t) \in C(1, g, s, \rho)$. We may go outside I here and choose $s' = -1$. It is not hard to verify that $u(t, x) \in C(1, g, s, \rho)$. So $u(t, x) = u(t)$ by the uniqueness of $u(t)$. The proof of Theorem 1 then gives the proof of the assertion above.

Note that the constant b in Theorem 2, in a certain sense, plays the same role as the constant $d = 1/b$ in the formulation of theorems for Gevrey classes $G(d)$ or $\gamma(d)$. See for instance [4] Theorem 2, and the remark after Lemma 1 above. Se also the introduction in [5].

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