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DISTRIBUTIONAL BOUNDARY VALUES IN $\mathfrak{D}'_{L^p}$

RICHARD D. CARMICHAEL *)

Section I. Introduction.

In papers by De Jager [1], Lauwerier [2], and Beltrami and Wohlers [3] the following problem has been considered. Let $f(z)$ be analytic in the half plane $Im(z)>0$ and be bounded by a polynomial in $Im(z)>\delta>0$. Under what conditions does f(z) converge distributionally to an element of a distribution space, and what are the properties of this element? Lauwerier showed that the analytic function $f(z)$ converged in $\mathfrak{Z}'$ to an element $U\in\mathfrak{Z}'$ which was the Fourier transform of an element $V\in\mathfrak{D}'$ which vanished for $Re(z)<0$. The problem is not as straightforward in $\mathfrak{S}'$ as was shown first by De Jager and then by Beltrami and Wohlers. With the convergence in $\mathfrak{S}'$ being assumed it was then shown that the limit distribution was an element of $\mathfrak{S}'$ and was the Fourier transform of an element in $\mathfrak{S}$ which vanished for $Re(z)<0$. In [4] we have extended the results of the above authors to n dimensions. n being an arbitrary positive integer and have given conditions under which the convergence in $\mathfrak{S}'$ of [1] and [3] is proved. Throughout [4] we used a boundedness condition of $f(z)$ which resulted in a more general concept of support than that in [1], [2] or [3].

In [3] Beltrami and Wohlers have stated a boundary value theorem concerning the space $\mathfrak{D}'_{L^2}$, which is a subspace of $\mathfrak{S}'$, with the convergence being that of $\mathfrak{S}'$. The result is that a necessary and sufficient condition that $U\in\mathfrak{D}'_{L^2}$ be the boundary value in the $\mathfrak{S}'$ topology of a

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specified function $f(z)$ is that U be the Fourier transform of the product of a polynomial and a L^2 function having support in $Re(z)<0$. Thus as before the convergence must be assumed; and if the limit distribution is to be an element of $\mathfrak{D}'_{L^2}$, then this must be assumed also. Beltrami and Wohlers note the reason for the convergence being that of $\mathcal{S}'$. Their result is a distribution version of a result of Titchmarsh ([9], Theorem 95, p. 128) which replaces L^2 functions by L^2 functions multiplied by a polynomial and replaces the Hardy space H^2 by a less restrictive class of functions. The sacrifice for this more general setting is that the topology of $\mathcal{S}'$ replaces that of the L^2 norm.

In this paper we shall consider the boundary value problem with respect to the space $\mathfrak{D}'_{L^p}$, $1\leq p\leq 2$. In particular we shall be concerned with obtaining conditions under which the convergence in $\mathcal{S}'$ of a certain analytic function $f(z)$, which is assumed to be an element of L^p, $1\leq p\leq 2$, on $Im(z)=0$, to a distribution is proved and shall determine the properties of this distribution. As will be seen, the limit distribution will not necessarily be an element of $\mathfrak{D}'_{L^p}$, $\frac{1}{q}+\frac{1}{p}=1$, but will be the Fourier transform of such an element. The limit distribution will be characterized in terms of functions in L^2 if $p=2$. These results will appear in section 3. Converse results to those of section 3 will appear in section 4. A characterization theorem for $\mathfrak{D}'_{L^p}$ will be represented in section 5.

Throughout this paper by $f(x)\in L^p$, $1\leq p\leq 2$, or $U\in\mathfrak{D}'_{L^p}$, $1\leq p\leq 2$, we mean that $f(x)\in L^p$ for some p where $1\leq p\leq 2$ or $U\in\mathfrak{D}'_{L^p}$ for some p where $1\leq p\leq 2$.

Section II. Definitions and Notation.

The n dimensional notation used here will be that of Schwartz [5]. Thus by the point $x\in\mathbb{R}^n$ we mean $x=(x_1, \ldots, x_n)$, and by $z\in\mathbb{C}^n$ we mean $z=(z_1, \ldots, z_n)=(x_1+iy_1, \ldots, x_n+iy_n)$. The product of two vectors x and y in $\mathbb{R}^n$ will be denoted by $\langle x, y\rangle=x_1y_1+\ldots+x_ny_n$. Similarly we define $\langle x, z\rangle$ where $x\in\mathbb{R}^n$ and $z\in\mathbb{C}^n$. The absolute value of z is defined by $|z|=\max_{1\leq j\leq n}|z_j|$. By D^α, α being an n-tuple of nonnegative integers we mean $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ where $D_j = \frac{1}{2\pi i}\frac{\partial}{\partial x_j}$. Similarly we write $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. All integrals will be n-fold over $\mathbb{R}^n$ unless otherwise specified; that is by $\int_{-\infty}^{\infty}$ we mean $\int_{\mathbb{R}^n} = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty}$, where there are n integrals on the right hand side of the last equality. In this paper we shall be concerned with the octant $Im(z_j) > 0$, $1 \le j \le n$, which from now will be described by $Im(z) > 0$. It will frequently be necessary to delete the j th. component of an n-fold expression. We shall denote this by a $\sim$ over the concerned expression. Thus by $\tilde{x}$ we mean $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$, or by $\tilde{dx}$ we mean $dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n$.

The Fourier transform of a function $f(t)$ will be denoted by $\widehat{f}$ or $\mathfrak{F}[f(t); x]$. By the Fourier transform of an element $f(t) \in L^1(\mathbb{R}^n)$ we mean the function $\widehat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \langle x, t \rangle} dt$; while that of an element $f(t) \in L^p(\mathbb{R}^n)$, $1 < p \le 2$, is defined as $\widehat{f}(x) = \underset{N \to \infty}{\text{l.i.m.}} \int_{-N}^{N} \dots \int_{-N}^{N} f(t) e^{-2\pi i \langle x, t \rangle} dt$, where l.i.m. denotes the limit in the L^q norm, $\frac{1}{p} + \frac{1}{q} = 1$. In the case $f(t) \in L^1$, then $\widehat{f}$ is continuos and bounded on $\mathbb{R}^n$; while if $f(t) \in L^p$, $1 < p \le 2$, then $\widehat{f} \in L^q, \frac{1}{p} + \frac{1}{q} = 1$. In case the integral $\int_{-\infty}^{\infty} f(t) e^{-\pi 2i \langle z, t \rangle} dt$ exists for z in some region of $\mathbb{C}^n$ then it will be called the Fourier transform of $f(t)$ in that region and will be denoted by $\widehat{f}(z)$.

$\mathcal{S}$ will denote the space of functions having derivatives of all order and satisfying $\sup_x |x^\beta D^\alpha \varphi(x)| < \infty$ for all multi-indicies α and β of non negative integers. By $\varphi_\lambda \to \varphi$ in $\mathcal{S}$ as $\lambda \to \lambda_0$, where $\varphi_\lambda \in \mathcal{S}$ for each λ and $\varphi \in \mathcal{S}$, we mean that $\lim_{\lambda \to \lambda_0} \sup_x | x^\beta D^\alpha [\varphi_\lambda(x) - \varphi(x)] | = 0$, where α and $\beta$are as above. $\mathcal{S}'$ is the space of continuous linear functionals on $\mathcal{S}$, where continuity means that $\varphi_\lambda \to \varphi$ in $\mathcal{S}$ as $\lambda \to \lambda_0$ implies $\langle T, \varphi_\lambda \rangle \to \langle T, \varphi \rangle$ as $\lambda \to \lambda_0$, $T \in \mathcal{S}'$. It is well known ([5], pp. 249, 251) that the Fourier transfor is a continuous isomorphism of $\mathcal{S}$ onto

$\mathcal{S}$ with the same being true of $\mathcal{S}'$ under the definition

$$\langle \widehat{U}, \varphi \rangle = \langle U, \widehat{\varphi} \rangle,$$

where $\varphi \in \mathcal{S}$ and $U \in \mathcal{S}'$.

The space $\mathcal{D}_{L^p}$ is the vector space of functions which are indefinitely differentiable and whose derivatives are elements of $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$. We say that $\varphi_\lambda \in \mathcal{D}_{L^p}$ converges to 0 in $\mathcal{D}_{L^p}$ as $\lambda \to \lambda_0$ if $\lim_{\lambda \to \lambda_0} \left(\int_{-\infty}^{\infty} |D^\alpha \varphi_\lambda(x)|^p \, dx \right)^{1/p} = 0$ for every multi-inidex α of nonnegative integers. The space $\mathcal{D}'_{L^p}$, $1 < p \le \infty$, is the space of continuous linear functionals on $\mathcal{D}_{L^q}$, $\frac{1}{p} + \frac{1}{q} = 1$, with continuity being defined as in the case of $\mathcal{S}'$. The space $\mathcal{D}'_{L^1}$ is the dual of the space $\dot{\mathcal{B}}$, which is a subspace of $\mathcal{B} \equiv \mathcal{D}_{L^\infty}$, which are functions which vanish at ∞ together with each of its derivatives. Schwartz ([5], Theorem 15, p. 201) has obtained a characterization of distributions in $\mathcal{D}'_{L^p}$; for $U \in \mathcal{D}'_{L^p}$ it is necessary and sufficient that U be a finite sum of derivatives of functions in L^p. For a complete discussion of $\mathcal{D}_{L^p}$ and $\mathcal{D}'_{L^p}$ the reader is referred to [5], p. 199.

The support of a function $g(x)$ is the closure in $\mathbb{R}^n$ of $\{x : x \in \mathbb{R}^n, g(x) \ne 0\}$ and is denoted by supp (g). Let U be an element of one of the above distribution spaces. The support of U is the complement in $\mathbb{R}^n$ of the union of all open sets in which $U = 0$. U is equal to 0 in an open set $\mathcal{O}$ if $\langle U, \varphi \rangle = 0$ for every φ in the appropriate function space such that supp $(\varphi) \subset \mathcal{O}$. We denote the support of U by supp (U). Let φ be an element of one of the above function spaces, and let $f(x)$ be a suitable function such that

$$\langle f, \varphi \rangle \overset{\text{def}}{\equiv} \int_{-\infty}^{\infty} f(x)\varphi(x)dx$$

exists and is finite. Then f is a regular distribution, and we note that supp (f) is the same as a function and a distribution.

Let $f(z)$ be a function of n complex variables. We say that $f(z)$ has a distributional boundary value U in $\mathcal{S}'$ if

$$\langle f(x+iy), \varphi \rangle \to \langle U, \varphi \rangle$$

in $\mathcal{S}'$ as $Im(z)=y\to 0+$ for $\varphi\in\mathcal{S}$. Here by $y\to 0+$, we mean $y_j\to 0+$, $1\leq j\leq n$.

Section III. Distributional Boundary Values in $\mathfrak{D}'_{L^p}$.

Mikusinski [6] has shown that if $f(z)$ is analytic in the half plane $Re(z)>0$, $e^{-k|z|}f(z)$ is bounded in $Re(z)>0$, and $f(iy)\in L^p(\mathbb{R}^1)$, $1\leq p\leq 2$, then $f(z)$ can be represented as a Laplace type integral of a continuous and bounded function with support in $(-k, \infty)$ in the case of L^1 or of a function in $L^q(\mathbb{R}^1)$, $\frac{1}{p}+\frac{1}{q}=1$, which vanishes a.e. outside $(-k, \infty)$ in the case of L^p, $1<p\leq 2$. We shall now extend this result to n-dimensions. This extension will be used in the proof of the main result of this paper.

THEOREM. 1 *Let $f(z)$ be analytic in $Im(z)>0$, continuous on $Im(z)=0$, and*

$$(1)\qquad |f(z)|\leq Ce^{2\pi\langle A, (|y_1|, \dots, |y_n|)\rangle}, Im(z)>0,$$

for some constant C and n-tuple A of real numbers. Let $f(x)\in L^p(\mathbb{R}^n)$, $1\leq p\leq 2$. Then $f(z)$ can be represented for $Im(z)>0$ as an absolutely convergent integral

$$(2)\qquad f(z)=\int_{-\infty}^{\infty} g(t)e^{-2\pi i\langle z, t\rangle}dt.$$

If $p=1$, $g(t)$ is continuous and bounded;

$$\text{supp}\,(g)\subseteq S_A{}^+=\{x: -\infty<x_j\leq A_j,\ 1\leq j\leq n\};$$

and $g(t)$ is given by the absolutely convergent integral

$$(3)\qquad g(t)=\int_{-\infty}^{\infty} f(x)e^{2\pi i\langle x, t\rangle}dx.$$

If $1<p\leq 2$, *the function* $g(t)$ *belongs to* L^q, $\frac{1}{p}+\frac{1}{q}=1$; $g(t)$ *vanishes a.e. outside* $S_A{}^+$; *and* $g(t)$ *is given as the limit in the mean with exponent* q,

$$(4)\qquad g(t)=\underset{N\to\infty}{\text{l.i.m.}}\int_{-N}^{N}\dots\int_{-N}^{N}f(x)e^{2\pi i\langle x,\,t\rangle}dx.$$

PROOF. Let p be fixed, $1\leq p\leq 2$. Let α be any n-tuple of real numbers such that $\alpha_j>0$, $1\leq j\leq n$. Consider α to be fixed for the present. Let $B\in\mathbb{R}^1$, $B>0$, and let it be large enough so that

$$(5)\qquad |B+\langle\alpha,\ z\rangle|\geq\tfrac{1}{2}(1+|z|)^2,\ Im(z)\geq 0,$$

for the fixed α. Consider the function

$$(6)\qquad g_\alpha(t)=B^{\frac{n+1}{2}}\int_{-\infty}^{\infty}\frac{f(x)}{(B+\langle\alpha,\ x\rangle)^{\frac{n+1}{2}}}e^{2\pi i\langle x,\,t\rangle}dx,$$

which exists. It is immediate that $g_\alpha(t)$ is continuous and bounded in $\mathbb{R}^n$. We show that supp $(g_\alpha)\subseteq S_A{}^+$. Let $(\beta_1,\ \dots,\ \beta_n)$ be a fixed vector such that $\beta_j>0$, $1\leq j\leq n$. For a fixed j construct a rectangle with vertices at $(-\eta,\ 0)$, $(\eta,\ 0)$, $(\eta,\ \beta_j)$, and $(-\eta,\ \beta_j)$ in the j th. Coordinate plane of $\mathbb{C}^n$. Integrating the function

$$\frac{f(z)\exp(2\pi i\langle z,\ t\rangle}{(B+\langle\alpha,\ z\rangle)^{\frac{n+1}{2}}}$$

around the boundary of this rectangle and adding on the other $n-1$ integrals we obtain by Cauchy's theorem for $\mathbb{C}^1$ that

$$\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\int_{-\eta}^{\eta}\frac{f(x)}{]B+\langle\alpha,\ x\rangle)^{\frac{n+1}{2}}}e^{2\pi i\langle x,\,t\rangle}dx_j d\widehat{x}=$$

$$= -i\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{0}^{\beta_j}\frac{f(x_1+i\beta_1, \ldots, \eta+i\sigma, \ldots, x_n+i\beta_n)}{(B+\langle\alpha, (x_1+i\beta_1, \ldots, \eta+\sigma, \ldots, x_n+i\beta_n)\rangle)^{\frac{n+1}{2}}}$$

$$\exp[2\pi i\langle(x_1+i\beta_1, \ldots, \eta+i\sigma, \ldots, x_n+i\beta_n), t\rangle]d\sigma d\tilde{x}$$

(7)
$$+\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\eta}^{\eta}\frac{f(x+i\beta)}{(B+\langle\alpha, x+i\beta\rangle^{\frac{n+1}{2}}}e^{2\pi i\langle x+i\beta, t\rangle}dx_j d\tilde{x}+$$

$$+i\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{0}^{\beta_j}\frac{f(x_1+i\beta_1, \ldots, -\eta+i\sigma, \ldots, x_n+i\beta_n)}{(B+\langle\alpha, (x_1+i\beta_1, \ldots, -\eta+i\sigma, \ldots, x_n+i\beta_n)\rangle)^{\frac{n+1}{2}}}$$

$$\exp[2\pi i\langle(x_1+i\beta_1, \ldots, -\eta+i\sigma, \ldots, x_n+i\beta_n), t\rangle]d\sigma d\tilde{x}=$$

$$=I_1+I_2+I_3,$$

say.

With B_j fixed we estimate I_1 using (1).

(8)
$$|I_1|\leq 2CKQ_{A\beta t}\beta_j e^{2\pi(A_j-t_j)\beta_j}(1+|\eta|)^{-\left(\frac{3}{2}\right)},$$

where

$$Q_{A\beta t}=\exp[2\pi\langle\widetilde{A-t}, \tilde{\beta}\rangle]$$

and K is a constant such that

$$K\geq\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(1+|\widetilde{x+i\beta}|)^{-n+\frac{1}{2}}d\tilde{x}.$$

Hence from the inequality (8) we see that $\lim_{\eta\to\infty} I_1=0$. A similar argument can be used to show $\lim_{\eta\to\infty} I_3=0$. Thus changing the order of integration which clearly can be done, and letting $\eta\to\infty$ on the left hand side of (7) and in I_2 we have from (6) that

(9)
$$g_\alpha(t)=B^{\frac{n+1}{2}}\int_{-\infty}^{\infty}\frac{f(x+i\beta)}{(B+\langle\alpha, x+i\beta\rangle)^{\frac{n+1}{2}}}e^{2\pi i\langle x+i\beta, t\rangle}dx,$$

Estimating (9) we have

$$|g_\alpha(t)| \leq 2B^{\frac{n+1}{2}} CDQ_{A\beta t} e^{2\pi(A_j - t_j)\beta_j} \tag{10}$$

where $Q_{A\beta t}$ is as before and D is a constant such that

$$D \geq \int_{-\infty}^{\infty} (1+|x+i\beta|)^{-n-1} dx.$$

Assuming $t_j > A_j$ and letting $\beta_j \to \infty$ in (10) we see that $g_\alpha(t)=0$. Since j was arbitrary $1 \leq j \leq n$, then $\operatorname{supp}(g_\alpha) \subseteq S_A^+$ as desired. From (9) we have

$$e^{2\pi\langle y, t\rangle} g_\alpha(t) = B^{\frac{n+1}{2}} \int_{-\infty} \frac{f(z)}{(B+\langle\alpha, z\rangle)^{\frac{n+1}{2}}} e^{2\pi i\langle x, t\rangle} dx,$$

where have replaced β with y. From inequalities (1) and (5) we see that as a function of x alone

$$\frac{B^{\frac{n+1}{2}} f(z)}{(B+\langle\alpha, z\rangle)^{\frac{n+1}{2}}} \in L^1 \cap L^2.$$

Hence by the Plancherel theory $e^{2\pi\langle y, t\rangle}{}_\alpha g_\alpha(t) \in L^2$ and

$$\frac{B^{\frac{n+1}{2}} f(z)}{(B+\langle\alpha, z\rangle)^{\frac{n+1}{2}}} = \int_{-\infty}^{\infty} g_\alpha(t) e^{-2\pi i\langle z, t\rangle} dt.$$

Here we have used the fact that $e^{2\pi\langle y, t\rangle} g_\alpha(t) \in L^1$ for $Im(z) > 0$ since $\operatorname{supp}(g_\alpha) \subseteq S_A^+$ and $g_\alpha(t)$ is bounded. Hence we have replaced the limit in the L^2 norm integral by an ordinary one. We now wish to let $\alpha \to 0$. This can be done as in [6], pp. 293-294, and the desired results can be obtained; for the methods hold equally well for n-dimensions with obvious modifications. This completes the proof.

We shall also need the following.

LEMMA 1. *Let* $U \in \mathcal{S}'$ *and* supp $(U) \subseteq S_A{}^+$. *Then*

$$(11) \qquad \int_{-\infty}^{\infty} \langle U, e^{-2\pi i \langle z, t \rangle} \rangle \varphi(x) dx = \langle U, e^{2\pi \langle y, t \rangle} \widehat{\varphi} \rangle$$

for $\varphi \in \mathcal{S}$.

PROOF. Let $\alpha(t) \in \mathcal{E}$, the space of functions having derivatives of all order, such that $\alpha(t) = 1$ on $S_A{}^+$ and supp $(\alpha) \subseteq S^+_{A+\varepsilon}$ for any $\varepsilon > 0$. Then $\alpha(t) e^{-2\pi i \langle z, t \rangle} \in \mathcal{S}$ independently of $\alpha(t)$ for $Im(z) > 0$. Writting $\langle U, e^{-2\pi i \langle z, t \rangle} \rangle = \langle U, \alpha(t) e^{-2\pi i \langle z, t \rangle} \rangle$ we have that the integral in (11) is a Riemann integral which can be approximated by Riemann sums. Consider the Cartesian product on n intervals $[-Y, Y] \otimes \ldots \otimes [-Y, Y]$. Divide the coordinate axes from $-Y$ to Y into partitions by planes which are parallel to the coordinate planes. Let $\Delta V j$ be the volumn and $(x_{1_j}, \ldots, x_{n_j})$ be a point of the j th. small parallelpiped formed by the above division. Let $\mathfrak{R}$ denote the appropriate mesh. Then.

$$(12) \qquad \int_{-\infty}^{\infty} \langle U, e^{-2\pi i \langle z, t \rangle} \rangle \varphi(x) dx = \lim_{Y \to \infty} \int_{-Y}^{Y} \cdots \int_{-Y}^{Y} \langle U, \alpha(t) e^{-2\pi i \langle z, t \rangle} \rangle \varphi(x) dx =$$

$$= \langle U, \lim_{Y \to \infty} \lim_{\substack{N \to \infty \\ |\mathfrak{R}| \to 0}} \sum_{j=1}^{N^n} \alpha(t) \exp[-2\pi i \langle x_{1_j} + iy_1, \ldots, x_{n_j} + iy_n), t \rangle]$$

$$\varphi(x_{1_j}, \ldots, x_{n_j}) \Delta V_j \rangle.$$

It can be shown without difficulty that the limit in the right hand side of the last equality in (12) converges in $\mathcal{S}$ to

$$\int_{-\infty}^{\infty} \alpha(t) e^{-2\pi i \langle z, t \rangle} \varphi(x) dx = \alpha(t) e^{2\pi \langle y, t \rangle} \widehat{\varphi}(t).$$

Thus by (12) and the fact that $U \in \mathcal{S}'$ we have (11), and the proof is complete.

We shall now give the main result of this paper, which gives conditions under which the convergence is proved instead of assumed as was the case in the result of Beltrami and Wohlers.

THEOREM. 2. *Let $f(z)$ be analytic in $Im(z)>0$, continuous on $Im(z)=0$, and let $f(x)\in L^p$ for some p, $1\le p\le 2$. Let*

$$|f(z)|\le C(1+|z|)^N e^{2\pi\langle A,(|y_1|,\dots,|y_n|)\rangle},\ Im(z)>0,$$

for some constants C, N, and n-tuple A of real numbers. Then there exists an element $U\in\mathfrak{D}'_{L^\infty}$ if $p=1$ or $U\in\mathfrak{D}'_{L^q}$, $\frac{1}{q}+\frac{1}{p}=1$, if $1<p\le 2$ such that supp $(U)\subseteq S_A{}^+$ *if $p=1$ or* supp $(U)\subseteq S_A{}^+$ a.e. *if $1<p\le 2$. Furthermore*

$$f(z)=\langle U,\ e^{-2\pi i\langle z,t\rangle}\rangle,\ Im(z)>0,$$

and $f(z)$ converges in $\mathcal{S}'$ to an element $V\in\mathcal{S}'$ as $Im(z)\to 0+$ where V is the product of a polynomial and L^2 function if $p=2$.

PROOF. Let p be fixed $1\le p\le 2$. Let $B\in\mathbb{R}^1$ and $B>0$. Let B be large enough so that

$$(13)\qquad |B+\langle z,\ z\rangle|\ge\tfrac{1}{2}(1+|z|)^2.$$

Let R be a constant which is large enough so that $N-2R\le -n-\varepsilon$, $\varepsilon>0$. Then $f(z)\ (B+\langle z,z\rangle)^{-R}$ satisfies the conditions of Theorem 1. Thus there exists a function $g(t)$, which belongs to L^q if $1<p\le 2$ or is continuous and bounded if $p=1$ and is given by (4) or (3) respectively, such that

$$(14)\qquad f(z)(B+\langle z,\ z\rangle)^{-R}=\int_{-\infty}^{\infty} g(t)e^{-2\pi i\langle z,t\rangle}dt.$$

Let Δ be a differential operator which is defined by

$$\Delta=\frac{1}{4\pi^2}\sum_{n}^{j=1}\frac{\partial^2}{\partial t_j^2}.$$

Then by the characterization theorem of Schwartz ([5], Theorem 15, p. 201) the distribution $U=(B-\Delta)^R g$ is an element of $\mathfrak{D}'_{L^\infty}$ if $p=1$ or of $\mathfrak{D}'_{L^q}$ if $1<p\le 2$. Also supp $(U)\equiv$ supp (g). Let $\alpha(t)$ be defined as

in the proof of Lemma 1. Then

$$\langle U, e^{-2\pi i\langle z, t\rangle}\rangle=\langle (B-\Delta)^R g, \alpha(t)e^{-2\pi i\langle z, t\rangle}\rangle=$$

(15) $$=(B+\langle z, z\rangle)^R \int_{-\infty}^{A_n}\cdots\int_{-\infty}^{A_1} g(t)e^{-2\pi i\langle z, t\rangle}dt.$$

The integration here is over $S_A{}^+$ since at worst $\mathrm{supp}\,(g)\subseteq S_A{}^+$ a.e., and integration over sets of measure zero gives no information. Thus from (14) and (15) we see that

$$f(z)=\langle U, e^{-2\pi i\langle z, t\rangle}\rangle,\ Im(z)>0.$$

It remains to show the boundary value results. Let $\varphi\in\mathcal{S}$. Since $U\in\mathfrak{D}'_{L^q}$ or $U\in\mathfrak{D}'_{L^\infty}$ and both space are in $\mathcal{S}'$ then by Lemma 1 we have that

(16) $$\langle f(z), \varphi\rangle=\langle U, \alpha(t)e^{2\pi\langle y, t\rangle}\widehat{\varphi}(t)\rangle.$$

It is easy to show that $\alpha(t)e^{2\pi\langle y, t\rangle}\widehat{\varphi}(t)\to\alpha(t)\widehat{\varphi}$ in $\mathcal{S}$ as $Im(z)\to 0+$. Thus by (16) and the fact that $U\in\mathfrak{D}'_{L^\infty}$ or $\mathfrak{D}'_{L^q}$ we see that

$$\langle f(z), \varphi\rangle\to\langle U, \alpha(t)\widehat{\varphi}(t)\rangle=\langle V, \varphi\rangle$$

in $\mathcal{S}'$ as $Im(z)\to 0+$ where $V=\widehat{U}$ and $V\in\mathcal{S}'$. Again let $\varphi\in\mathcal{S}$. Then for $p=2$ we have

$$\langle V, \varphi\rangle=\langle\widehat{U}, \varphi\rangle=\langle\mathfrak{F}[(B-\Delta)^R g], \varphi\rangle=$$

$$=\int_{-\infty}^{\infty} g(t)[(B-\Delta)^R\widehat{\varphi}(t)]dt$$

$$=\int_{-\infty}^{\infty} g(t)\mathfrak{F}[(B+\langle x, x\rangle)^R\varphi(x);\ t]dt$$

$$=\langle (B+\langle x, x\rangle)^R\widehat{g}, \varphi\rangle.$$

Thus $V=(B+\langle x, x\rangle)^R\widehat{g}$. Since $g\in L^2$, then $\widehat{g}\in L^2$. The proof is complete.

If we knew that $\widehat{g}$ existed for $g \in L^q$, $\frac{1}{p}+\frac{1}{q}=1$, $1 \leq p<2$, then the calculation of the element V in the last theorem would have been valid for $1 \leq p \leq 2$. However $\widehat{g}$ does not necessarily exist for $g \in L^q$, $q>2$. In a future paper we shall show that under the assumption of the existence of the Fourier transform of $g \in L^q$, $q \geq 1$, results may be obtained for $\mathcal{D}'_{L^q}$, $q \geq 1$.

Lions ([5], Proposition 8, p. 397, [7]) has obtained results similar to Theorem 2 for the space $\mathcal{S}'$ but without consideration of the boundary value problem. We have shown that under the assumption of integrability of $f(x)$ on the boundary of the octant one can make the stronger statement that the distribution U, whose existence is proved, is in $\mathcal{D}'_{L^\infty}$ or $\mathcal{D}'_{L^q}$, $1<p \leq 2$, $\frac{1}{p}+\frac{1}{q}=1$, which are proper subspaces of $\mathcal{S}'$. We note that if the integrability on the boundary had not been assumed and if the boundedness condition of $f(z)$ had been assume valid on the boundary then the distribution U can be shown to be in $\mathcal{D}'_{L^\infty}$ and $\mathcal{D}'_{L^q}$, $\frac{1}{p}+\frac{1}{q}=1$, $1<p \leq 2$ for all q. To see this we need only look at $f(z)(B+\langle z, z\rangle)^{-R}$. Then for $Im(z)=0$ this product belongs to L^p for any p. Applying Theorem 1 and the noting the proof of Theorem 2 this result is obtained.

In [4] we have considered function $f(z)$ of n complex variables which are analytic in $Im(z)>0$ and satisfy

$$|f(z)| \leq C_\delta(1+|z|)^N e^{2\pi\langle A, (|y_1|, \ldots, |y_n|)\rangle}$$

in any octant $Im(z) \geq \delta>0$, $\delta=(\delta_1, \ldots, \delta_n)$, $\delta_j>0$ for $1 \leq j \leq n$, where N is a constant, C_δ is a constant which may depend on δ, and A is as before.Denote these functions by $G_A{}^+$. Similarly we could define the set $G_A{}^-$ of functions analytic in $Im(z)<0$ and satisfying the above inequality in any octant $Im(z) \leq \delta<0$. We shall obtain a boundary value result concerning $G_A{}^+$.

Let $U \in \mathcal{D}'_{L^q}$, $1 \leq p \leq 2$. Then by the characterization theorem of Schwartz, $U=\sum_{|\alpha| \leq m} D^\alpha g_\alpha$, where $g_\alpha \in L^p$ for all α under consideration. There exists a $V \in \mathcal{S}'$ such that $U=\widehat{V}$. Let $\varphi \in \mathcal{S}$. Then

$$\langle V, \varphi\rangle=\langle \mathfrak{F}^{-1}U, \varphi\rangle=\langle U, \mathfrak{F}^{-1}\varphi\rangle$$

$$=\sum_{|\alpha|\leq m}(-1)^{|\alpha|}\int_{-\infty}^{\infty} g_\alpha(t)D^\alpha[\mathfrak{F}^{-1}(\varphi(x);\ t)]dt$$

$$=\sum_{|\alpha|\leq m}(-1)^{|\alpha|}\int_{-\infty}^{\infty} g_\alpha(t)\mathfrak{F}^{-1}[x^\alpha\varphi(x);\ t]dt$$

$$=\langle \sum_{|\alpha|\leq m}(-1)^{|\alpha|}t^\alpha\mathfrak{F}^{-1}g_\alpha,\ \varphi\rangle.$$

Hence $V=\sum_{|\alpha|\leq m}(-1)^{|\alpha|}t^\alpha\mathfrak{F}^{-1}g_\alpha$ in $\mathcal{S}'$, and we note that supp $(\mathfrak{F}^{-1}g_\alpha)\equiv$ $\equiv$ supp (V) for each α. Since $g_\alpha\in L^p$, $1\leq p\leq 2$, then $h_\alpha\equiv\mathfrak{F}^{-1}g_\alpha\in L^q$, $1<p\leq 2$, $\frac{1}{p}+\frac{1}{q}=1$, or h_α is continuous and bounded if $p=1$. By $\mathfrak{F}^{-1}g_\alpha$ here we mean

$$\int_{-\infty}^{\infty} g_\alpha(t)e^{2\pi i\langle x,t\rangle}dt$$

if $p=1$ and

$$\operatorname*{l.im.}_{N\to\infty}\int_{-N}^{N}\dots\int_{-N}^{N} g_\alpha(t)e^{2\pi i\langle x,t\rangle}dt$$

if $1<p\leq 2$. Of course the reciprocal relation $\widehat{h_\alpha}=g_\alpha$ holds only if $p=2$.

In [4] we have proved that if $f(z)\in G_A{}^+$ and converges in $\mathcal{S}'$ to an element U, then $U\in\mathcal{S}'$ and there exists an element $V\in\mathcal{S}'$ having support in $S_A{}^+$ such that $U=\widehat{V}$ and $f(z)=\langle V, e^{-2\pi i\langle z,t\rangle}\rangle$, $Im(z)>0$. As an immediate consequence of this result and the above calculation we have the following.

THEOREM 3. *Let $f(z)\in G_A{}^+$ and let $f(z)\to U\in\mathfrak{D}'_{L^p}$, $1\leq p\leq 2$, in the $\mathcal{S}'$ topology. Then there is an element $V\in\mathcal{S}'$ having support in $S_A{}^+$ such that $U=\widehat{V}$ and $f(z)=\langle V, e^{-2\pi i\langle z,t\rangle}\rangle$. Furthermore $V=\sum_{|\alpha|\leq m}(-1)^{|\alpha|}t^\alpha h_\alpha$, where $h_\alpha\in L^q$, $\frac{1}{p}+\frac{1}{q}=1$, if $1<p\leq 2$ or h_α is continuous and bounded if $p=1$; and* supp $(h_\alpha)\subseteq S_A{}^+$ *for each* α.

Section IV. Converse Results.

In this section we shall present converse results to those of Section III. The following is a converse to Theorem 2.

THEOREM 4. *Let* $U \in \mathfrak{D}'_{L^p}$, $1 \leq p \leq 2$, *and* $\operatorname{supp}(U) \subseteq S_A{}^+$. *Then there is a function* $f(z)$ *which is analytic in* $Im(z) > 0$, *satisfies*

$$(17) \qquad |f(z)| \leq C(1+|z|)^N e^{2\pi\langle A, (|y_1|, \ldots, |y_n|)\rangle}, \quad Im(z) \geq \delta > 0,$$

and $f(z) \to V \in \mathcal{S}'$ *in the topology of* $\mathcal{S}'$. $V = \sum_{|\beta| \leq m} t^\beta h_\beta$, *where* $h_\beta \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$, *if* $1 < p \leq 2$, *and* h_β *is continuous and bounded if* $p = 1$.

PROOF. Let $\alpha(t)$ be as defined in Lemma 1. Consider the function

$$f(z) = \langle U, \alpha(t) e^{-2\pi i\langle z, t\rangle}\rangle, \quad Im(z) > 0.$$

This function is analytic in $Im(z) > 0$ since it is analytic in each variable separately. To show the boundedness condition we use the Schwartz characterization theorem and obtain $U = \sum_{|\beta| \leq m} D^\beta g_\beta$ where $g_\beta \in L^p$ if $U \in \mathfrak{D}'_{L^p}$. We note that $\operatorname{supp}(g_\beta) \subseteq S_A{}^+$ for each β. Using Leibnitz's rule we have

$$|f(z)| = \left| \sum_{|\beta| \leq m} (-1)^{|\beta|} \int_{-\infty}^{\infty} g_\beta(t) \sum_{\gamma+\eta=\beta} \frac{\beta!}{\gamma!\eta!} D^\gamma \alpha(t) (-z_1)^{\eta_1} \ldots (-z_n)^{\eta_n} e^{-2\pi i\langle z, t\rangle} dt \right|$$

$$\leq \sum_{|\beta| \leq m} \sum_{\gamma+\eta=\beta} \frac{\beta!}{\gamma!\eta!} |z_1|^{\eta_1} \ldots |z_n|^{\eta_n} \int_{-\infty}^{A_n} \ldots \int_{-\infty}^{A_1} |g_\beta(t) D^\gamma \alpha(t) e^{-2\pi i\langle z, t\rangle}| \, dt.$$

For $t \in \mathbb{R}^n$ such that $t_j < 0$ for some j, $1 \leq j \leq n$, we have

$$(18) \qquad |f(z)| \leq \sum_{|\beta| \leq m} \sum_{\gamma+\eta=\beta} \frac{\beta!}{\gamma!\eta!} |z_1|^{\eta_1} \ldots |z_n|^{\eta_n} K_{\beta\gamma}$$

where

$$K_{\beta\gamma} \geq \int_{-\infty}^{A_n} \ldots \int_{-\infty}^{A_1} | g_\beta(t) D^\gamma \alpha(t) e^{-2\pi i \langle z, t \rangle} | \, dt.$$

Hence

$$| f(z) | \leq Q e^{2\pi \langle A, (|y_1|, \ldots, |y_n|) \rangle} \sum_{|\beta| \leq m} \sum_{\gamma+\eta=\beta} \frac{\beta!}{\gamma!\eta!} | z_1 |^{\eta_1} \ldots | z_n |^{\eta_n} K_{\beta\gamma} ,$$

where Q is a constant and A is an n-tuple of real numbers. For

$$t \in \{t : t_j \geq 0, \ 1 \leq j \leq n\} \cap S_A^+$$

we have

$$(19 \qquad | f(z) | \leq e^{2\pi \langle A, (|y_1|, \ldots, |y_n|) \rangle} \sum_{|\beta| \leq m} \sum_{\gamma+\eta=\beta} \frac{\beta!}{\gamma!\eta!} | z_1 |^{\eta} \ldots | z_n |^{\eta_n} D_{\beta\gamma}$$

where

$$D_{\beta\gamma} \geq \int_{-\infty}^{A_n} \ldots \int_{-\infty}^{A_1} | g_\beta(t) D^\gamma \alpha(t) | \, dt.$$

Letting $R_{\beta\gamma} = \max (K_{\beta\gamma} , D_{\beta\gamma})$, $\eta_j = \max\limits_{1 \leq i \leq n} \eta_i$, and $| z | = \max\limits_{1 \leq i \leq n} | z_i |$ we get from (18) and (19) that

$$f(z) | \leq Q e^{2\pi \langle A, (|y_1|, \ldots, |y_u|) \rangle} \sum_{|\beta| \leq m} \sum_{\gamma+\eta=\beta} \frac{\beta!}{\gamma!\eta!} | z |^{n\eta_j} R_{\beta\gamma} .$$

Since both sums on the right are finite then there exists a constant M and a nonnegative integer N such that

$$\sum_{|\beta| \leq m} \sum_{\gamma+\eta=\beta} \frac{\beta!}{\gamma!\eta!} | z |^{n\eta_j} R_{\beta\gamma} \leq M(1 + | z |)^N, \ Im(z) \geq \delta > 0.$$

Thus (17) is obtained with $C = QM$. It follows as in the proof of Theorem 2 that $f(z)$ converges in the $\mathcal{S}'$ topology to an element $V = \widehat{U} \in \mathcal{S}'$

as $Im(z)\to 0+$. Let $\varphi\in\mathcal{S}$. Then

$$\begin{aligned}\langle V,\ \varphi\rangle=\langle\widehat{U},\ \varphi\rangle&=\langle U,\ \widehat{\varphi}\rangle=\\ &=\sum_{|\beta|\leq m}\langle D^{\beta}g_{\beta},\ \widehat{\varphi}\rangle=\\ &=\sum_{|\beta|\leq m}(-1)^{|\beta|}\int_{-\infty}^{\infty}g_{\beta}(t)D^{\beta}\widehat{\varphi}(t)dt=\\ &=\sum_{|\beta|\leq m}(-1)^{2|\beta|}\int_{-\infty}^{\infty}g_{\beta}(t)\mathfrak{F}[x^{\beta}\varphi(x);\ t]dt=\\ &=\langle\sum_{|\beta|\leq m}t^{\beta}\widehat{g_{\beta}},\ \varphi\rangle.\end{aligned}$$

If $g_{\beta}\in L^{p}$, $1<p\leq 2$, then $h_{\beta}=\widehat{g_{\beta}}\in L^{q}$, $\frac{1}{p}+\frac{1}{q}=1$.
If $g_{\beta}\in L^{1}$, *then* h_{β} *is continuous and bounded. The proof is complete.*

We note that Schwartz ([5], p. 256) first recognized the form of the Fourier transform of an element in $\mathfrak{D}'_{L^p}$, $1\leq p\leq 2$.

We now present another converse result.

THEOREM 5. *Let* $U=\sum_{|\beta|\leq m}t^{\beta}\widehat{\Psi}$, $\Psi\in L^{p}$, $1\leq p\leq 2$, *and* supp $(\Psi)\subseteq\subseteq S_{A}{}^{+}$. *Then U is the Fourier transform of an element in* $\mathfrak{D}'_{L^p}$, $1\leq p\leq 2$; *and there exists a function $f(z)$ wich is analytic in $Im(z)>0$, satisfies*

$$|f(z)|\leq C(1+|z|)^{N}e^{2\pi\langle A,(|y_1|,\ldots,|y_n|)\rangle},\ Im(z)\geq\delta>0,$$

and $f(z)\to U$ in $\mathcal{S}'$ as $Im(z)\to 0+$.

We shall present a sketch of the proof and shall leave the details to the interested reader. By a calculation similar to that of Theorem 4 we obtain $U=\mathfrak{F}[\sum_{|\beta|\leq m}D^{\beta}\Psi]$. Defining the function

$$f(z)=\langle\sum_{|\beta|\leq m}D^{\beta}\Psi,\ \alpha(t)e^{-2\pi i\langle z,t\rangle}\rangle$$

the boundedness and convergence results follow as in Theorems 4 and 2, respectively.

Section V. Characterization Theorem for $\mathfrak{D}'_{L^p}$.

Hörmander ([8], p. 4) has defined the space $\mathfrak{D}'$ to be the space of all linear forms U on $\mathfrak{D}$ such that to every compact set $K \subset \mathbb{R}^n$ there exist constants C and K such that

$$(20) \qquad |\langle U, \varphi\rangle| \leq C \sum_{|\alpha|\leq m} \sup |D^\alpha \varphi|, \quad \varphi \in \mathfrak{D}(K).$$

With this definition one can prove as a theorem ([8], p. 5, Theorem 1.3.) the usual definition of element of $\mathfrak{D}'$. The reverse can also be shown. That is if one is given the usual definition of $\mathfrak{D}'$ that $U \in \mathfrak{D}'$ if U is a continuous linear functional on $\mathfrak{D}$, then (20) can be shown to hold for every $\varphi \in \mathfrak{D}$. The inequality (20) thus serves as a characterization of elements of $\mathfrak{D}'$.

Throughout this paper we have used the characterization theorem of Schwartz for the spaces of distributions $\mathfrak{D}'_{L^p}$. We shall now give another characterization theorem for $\mathfrak{D}'_{L^p}$similar to that described above for $\mathfrak{D}'$.

THEOREM. 6 *A linear functional U on $\mathfrak{D}_{L^q}$ belongs to $\mathfrak{D}'_{L^p}$, $\frac{1}{p}+\frac{1}{q}=1$, $1<p\leq\infty$, if and only if there exist constants C and m depending only on U such that*

$$(21) \qquad |\langle U, \varphi\rangle| \leq C \sum_{|\alpha|\leq m} \| D^\alpha \varphi \|_{L^q}$$

for every $\varphi \in \mathfrak{D}_{L^q}$.

PROOF. The sufficiency is an immediate consequence of the definition of convergence in $\mathfrak{D}_{L^p}$. To prove the necessity suppose the inequality does not hold for any C and m. Then for every λ we can find a $\varphi_\lambda \in \mathfrak{D}_{L^q}$ such that

$$(22) \qquad |\langle U, \varphi\rangle| > \lambda \sum_{|\alpha|\leq \lambda} \| D^\alpha \varphi_\lambda \|_{L^q}$$

Then $\| D^\alpha \varphi_\lambda \|_{L^q} \neq 0$ for if it were (22) would be contradicted. Put

$$\| \varphi \|_\lambda = \sum_{|\alpha|\leq \lambda} \| D^\alpha \varphi \|_{L^q}.$$

Then the function

$$\Psi_\lambda = \frac{\varphi_\lambda}{\lambda \| \varphi_\lambda \|_\lambda}$$

is an element of $\mathfrak{D}_{L^q}$ such that $\| \Psi_\lambda \|_\lambda = 1/\lambda$ and

$$|\langle U, \Psi_\lambda \rangle| > 1 \tag{23}$$

for every λ. For any η such that $0 \leq \eta \leq \lambda$ we have

$$\| \Psi_\lambda \|_\eta = \frac{1}{\lambda} \frac{\| \varphi_\lambda \|_\eta}{\| \varphi_\lambda \|_\lambda} \leq \frac{1}{\lambda}. \tag{24}$$

From (24) we conclude that $\Psi_\lambda \to 0$ in $\mathfrak{D}_{L^q}$ as $\lambda \to \infty$. Then since $U \in \mathfrak{D}'_{L^q}$ by assumption, we must have $\langle U, \Psi_\lambda \rangle \to 0$ as $\lambda \to \infty$ which contradicts (23) Hence (21) must hold, and the proof is complete.

A similar characterization holds for $\mathfrak{D}'_{L^1}$ which is the dual space of $\mathfrak{B}$, the subspace of $\mathfrak{D}_{L^\infty}$ whose elements converge to zero at ∞ along with each of their derivatives. As an application of Theorem 6 one can use it to obtain the boundedness condition (17) of the function $f(z)$ in Theorem 4.

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