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Abstract boundary-value operators and their adjoints

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ABSTRACT BOUNDARY-VALUE OPERATORS AND THEIR ADJOINTS

JEAN-PIERRE AUBIN *)

Introduction.

Let U be a Hilbert space, U_0 be a closed subspace of U such that there exists a Hilbert space K satisfying

(*) $\qquad U \subset K$ with a stronger topology; U_0 is dense in K.

An *abstract boundary value operator* $\mathfrak{F} = \wedge \times \beta_0$ will be in this paper the product of two operators Λ and β_0 defined on the space U such that

(**) $$U_0 \subset D = \ker \beta_0 .$$

The maps Λ and β_0 play respectively the roles of a differential operator and of a differential boundary operator when U is a space of functions or distributions.

On the other hand, if D is a closed subspace of U *containing* U_0, we shall consider the pair (D, Λ) like an *unbounded* operator of domain D dense in K (by (*)).

Then, with each boundary operator $\mathfrak{F}$ we may associate an unbounded operator $(D, \wedge)$ with domain $D = \ker \beta_0$ and consersely, with each $(D, \wedge)$ we may associate a monempty class of equivalent boundary value operators.

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The main aim of this paper is to construct an « *adjoint boundary value operator* » $\mathfrak{F}^* = \wedge^* \times \beta_1^*$ of $\mathfrak{F}$, product of two operators defined on a suitable space U^* (depending on U_0, K and Λ), which is associated with the adjoint $(D^*, \wedge^*)$ of the unbounded operator $(D, \wedge)$ by

$$(***) \qquad D^* = \ker \beta_1^*.$$

The main tool used for the construction of the adjoint $\mathfrak{F}^*$ is the *Green's formula.* This is why the first section of this paper is devoted to the construction of an abstract Green's formula associated with U, U_0, K and Λ (Theorem 1.1). In Section 2, we assiate with each $(D, \wedge)$ a Green's formula: Namely, we will construct operatos β_0, β_1, β^*_0 and β_1^* such that

$$(****) \qquad (u, \wedge v) - (\wedge^* u, v) = \langle \beta_1^* u, \beta_1 v \rangle - \langle \beta_0^* u, \beta_0 v \rangle$$

(where $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ denote duality pairings on suitable pairs of spaces (Theorem 2.1)) and point out some consequences (results of density and connections between closed operators and a priori estimates).

We define and study the boundary value operators and their adjoint in the third section (Green's formula associated with a boundary operator, relations between the transpose and the adjoint, characterization of the ranges and characterization of the well posed boundary value operators). Naturally, these abstract results are already known for concrete differential boundary value operators (see for instance Chapeter 2 and the appendix of the book by J. L. Lions and E. Magenes [2]). We give a brief example in Section 4.

The bibliography related to these problems is so abundant that we will refer only to the book of J. L. Lions and E. Magenes, where further information can be found.

Let us only mention that another study of « abstract boundary values » related to unbonuded operators lies in Chapter XII, § 4, Sections 20 to 31 of Part 2 of the book *Linear Operators* by N. Dunford and J. T. Schwartz [1].

National Conventions.

All the operators used in this paper are assumed (once and for all) linear and continuous. In particular, a right inverse will mean a continuous linear right inverse and a projector will mean a continuous projector.

We will denote by $E \times F$ the product of two spaces E and F, by $u \times v$ ($u \in E$, $v \in F$) an element of $E \times F$ and by $A \times B$ (where A maps a space G into E, B maps G into F) the operator mapping $u \in G$ onto $Au \times Bu \in E \times F$.

The topological dual of a space E will always be denoted by E', the transpose of an operator by A'.

The duality pairings between a space and its dual will be denoted by $(\cdot, \cdot)$ if the spaces are denoted by Latin letters and by $\langle \cdot, \cdot \rangle$ if the spaces are denoted Greek letters.

Contents.

1. Green' Formula.

1-1. *The Framework.*

We denote by U and E two Hilbert spaces and by Λ an operator mapping U into E. We also introduce a surjective operator β from U onto a Hilbert space Φ and set

(1.1) $$U_0 = \ker \beta$$

DIAGRAM 1.

$$\begin{array}{ccc} U & \xrightarrow{\ \Lambda\ } & E \\ \Big\downarrow \beta & & \\ \Phi & & \end{array}$$

Finally, let us suppose that there exists a Hilbert space K satisfying:

(1.2) $$\begin{cases} i) \ U \text{ is contained in } K \text{ with a stronger topology} \\ ii) \ U_0 \text{ (and thus } U) \text{ is dense in } K. \end{cases}$$

We will assume throughout this paper that the choice of Φ, β, K is done once for all.

By analogy with applications, we will term:

Φ the « *boundary space* »

β the « *boundary operator* »

U_0 the « *minimal domain* »

K a « *normal space* ».

NOTE 1.1. We always may consider a closed space U_0 as the kernel of an operator β. We may choose for instance Φ a topological complement of U_0 and β the projector onto Φ with kernel U_0.

NOTE 1.2. In the applications, the above items will respectively be:

U and E spaces *of* distributions on a manifold

Λ a differential operator

U_0 the closure of the space of indefinitely diffrentiable functions with compact support

K a normal space of distributions on the manifold which contains U

Φ a convenient space of distributions on the boundary of the manifold and β a convenient boundary differential operator.

NOTE 1.3.

LEMMA 1.1. The assumption (1.2) is equivalent to:

$$(1.3) \qquad \begin{cases} i) \ K' \text{ is a dense subspace of } U' \\ ii) \ K' \text{ is a dense subspace of } U'_0 . \end{cases}$$

Indeed, if i denotes the injection from U into K, its transpose i' is injective since $i(U)$ is dense in K and $i'(K')$ is dense in U' since i is injective. In the same way, if j the injection from U_0 into K, j' is the injection from K' into U'_0 and $j'(K')$ is dense in U'_0.

1.2. *The Adjoint Framework.*

Let us introduce the

DEFINITION 1.1. The adjoint Λ^*, mapping E' into U'_0 is the operator defined by:

$$(1.4) \qquad (\Lambda^* u, v) = (u, \Lambda v) \text{ for all } u \in E', \text{ for all } v \text{ in } U_0 .$$

Let us recall that the transpose Λ' of Λ, mapping E' into U' is defined by:

$$(1.5) \qquad (\Lambda' u, v) = (u, \Lambda v) \text{ for all } u \in E', \text{ for all } v \text{ in } U$$

(Thus Λ^* is the transpose of the restriction of Λ to U_0).

In order to compare these two operators, it is natural, by Lemma 1.1, to introduce

DEFINITION 1.2. The domain U^* of Λ^* will be the subspace of the elements u of E' such that $\Lambda^* u$ belongs to K' equipped the following norm:

$$\| u \|_{U^*} = (\| u \|_{E'}^2 + \| \Lambda^* u \|_{K'}^2)^{1/2}. \tag{1.6}$$

The domain U^* is a Hilbert space (see note 1.7).

NOTE 1.4. We will see that we may consider the pair (U_0, Λ) as an unbounded operator (depending on the choice of the normal space K).

We now will prove

THEOREM 1.1 (*Green's Formula*). There exists a continuous linear operator β^* from U^* into Φ' related to β, Λ and K by the following Green's formula:

$$(u, \Lambda v) - (\Lambda^* u, v) = \langle \beta^* u, \beta v \rangle \tag{1.7}$$

for all u in U^* and for all v in U.

In order to keep the analogy, we may call β^* the « *adjoint boundary operator* ». We then have associated with Diagram 1 the adjoint

DIAGRAM 2.

$$\begin{array}{ccc} U^* & \xrightarrow{\Lambda^*} & K' \\ \Big\downarrow{\scriptstyle \beta^*} & & \\ \Phi' & & \end{array}$$

PROOF. Let us choose u in U^*, v in U. Then $\Lambda' u$ belongs to U' (since $U^* \subset E'$) and $\Lambda^* u$ belongs to K' and thus, to U'.

(We identify Λ^* to $i' \Lambda^*$ where i is the injection from U into K).

Then:

(1.8) $\Lambda' u - \Lambda^* u$ belongs to the orthogonal $U_0^\perp$ of U_0

since, by definition of these operators

$$(\Lambda' u - \Lambda^* u, v) = 0 \text{ for all } v \text{ in } U_0 .$$

On the other hand, let σ be a right inverse of β. Since U_0 is the kernel of β, $\sigma\beta$ is a projector whose kernel is U_0 and, *by* transposition,

(1.9) $\beta'\sigma'$ is a projector from U' onto U_0.

Then (1.8) and (1.9) implies that

$$\Lambda' u - \Lambda^* u = \beta'\sigma'(\Lambda' - \Lambda^*) = \beta'\beta^* u \tag{1.10}$$

where

(1.11) $\beta^* = \sigma'(\Lambda' - \Lambda^*)$ maps U^* into Φ'.

We finally notice that (1.7) is equivalent to (1.10).

NOTE 1.5. *The operator* β^* *does not depend on the choice of the right inverse* σ *of* β.

Indeed, if Green's formula holds, we have $\Lambda' - \Lambda^* = \beta'\beta^*$. If σ is any right inverse of β, we have $\sigma'\beta' = 1$ and thus, (1.11) holds.

NOTE 1.6. The operator β^* depends on the choice of the operator β for which $U_0 = \ker \beta$. Nevertheless, we may state the

PROPOSITION 1.1. Let *us* assume that U_0 is also the kernel of a surjective operator γ from U onto Ψ. Then there exists an isomorphism θ from Ψ onto Φ such that

$$\beta = \theta\gamma; \quad \gamma^* = \theta'\beta^* \tag{1.12}$$

and Green's formula may be written

$$(u, \Lambda v) - (\Lambda^* u, v) = \langle \gamma^* u, \gamma v \rangle . \tag{1.13}$$

NOTE 1.7. *The domain U^* is a Hilbert space.* If u_n is a Cauchy sequence in U^*, then, by (1.6), we have

(1.14) u_n converges to u in E'; $\wedge^* u_n$ converges to f in K'.

If v belongs to U_0, we have $(\wedge^* u_n, v)=(u_n, \wedge v)$ and, taking the limit, we obtain

(1.15) $$(f, v)=(u, \wedge v) \text{ for all } v \text{ in } U_0 .$$

Then $f= \wedge^* u$ and since f belongs to K', u belongs to U^*. The Cauchy sequence converges to u in U^*.

NOTE 1.8. The operator β^* depends on the choice of K in the following way. Let $K_1 \subset K_2$ be two normal spaces satisfying (1.2). Let U_1^*, U_2^*, β_1^* and β_2^* be the associated domains and adjoint boundary operators. Then:

$$\beta_2^* = \text{ the restriction of } \beta_1^* \text{ to } U_2^* \subset U_1^*$$

(« Smaller » is the space K_1, « larger » is the space U_1^*).

Indeed, if i_1 and i_2 denote the inijection from U into K_1 and K_2 respectively, we will have

$$\beta_1^* = \sigma'(\wedge' - i'_1 \wedge^*); \quad \beta_2^* = \sigma'(\wedge' - i'_2 \wedge^*).$$

Since K_1 is dense in K_2 (by (1.2)), i'_2 is the restriction of i'_1 to K'_2.

1.3 *Two Lemmas About Projectors.*

We will need below the two following lemmas. Let E be a topological vector space, μ and ν two continuous projectors. Let *us* recall that:

(1.16) $$\ker \mu \subset \ker \nu \text{ iff } \nu\mu = \nu$$

(1.17) $$\nu(E) \subset \mu(E) \text{ iff } \mu\nu = \nu.$$

LEMMA 1.2. Let μ and ν two projectors satisfying (1.16). Then $\mu_1 = \mu + (1+\mu)\nu$ is a projector satisfying

$$(1.18)\qquad \begin{cases} i)\ \ker\mu = \ker\mu_1 \\ ii)\ \nu\mu_1 = \mu_1\nu = \nu. \end{cases}$$

The geometrical form of this lemma is the following: If two closed subspaces M and N possess topological complements, if $M \subset N$ and if P is a topological complement of N, then there exists a topological complement Q of M containing P.

PROOF. Using (1.16), we may check that $\mu_1^2 = \mu$ and that $\nu\mu_1 = = \mu_1\nu = \nu$. Since $\mu\mu_1 = \mu$, we obtained $\ker\mu_1 \subset \ker\mu$. If $\mu(x) = 0$, then $\nu(x) = 0$ by (1.16), then $\mu_1(x) = 0$ and $\ker\mu \subset \ker\mu_1$.

LEMMA 1.3. Let μ_0 be a projector onto a closed subspace P. The following operators

$$(1.19)\qquad \mu = \rho + (1-\rho)\mu_0$$

where ρ is an operator mapping E into P are projectors onto P and all the projectors onto P can be written in the form (1.19).

Indeed, the range of a projector μ is P iff, by (1.17):

$$(1.20)\qquad \mu\mu_0 = \mu_0 \text{ and } \mu_0\mu = \mu.$$

Then, if μ is a projector onto P, we have (taking $\rho = \mu$)

$$(1.21)\qquad \mu = \mu + (1-\mu)\mu_0 .$$

Conversely, let ρ map E into $P = \mu_0(E)$. Then:

$$(1.22)\qquad \mu_0\rho = \rho.$$

If μ is defined by (1.19), (1.22) implies that $\mu^2 = \mu$ and that:

$$\mu\mu_0 = \rho\mu_0 + (1-\rho)\mu_0 = \mu_0$$

$$\mu_0\mu = \mu_0\rho + \mu_0 - \mu_0\rho\mu_0 = \rho + (1-\rho)\mu_0 = \mu.$$

Then μ is a projector onto P.

NOTE 1.9. The results of this section hold when the spaces are locally convex if we assume that U_0 possesses a topological complement (or that β possesses a continuos right inverse). We only have to modify (1.6): We equip U^* with the weakest topology for which $\wedge^*$ and the injection from U^* into K' are continuous.

2. K-Unbounded Operators and Their Adjoints.

2.1. *Definitions.*

Let *us* consider a space D such that

(2.1) D is a closed subspace of U containing U_0.

Let *us* associate with D the subspace D^* of U^* defined by

$$\text{(2.2)} \quad \begin{cases} D^* \text{ is the subspace of } u \in U^* \text{ such that:} \\ (\wedge^* u,\ v) = (u,\ \wedge v) \text{ for all } v \in D. \end{cases}$$

We first notice that

LEMMA 2.1. The subspace D^* of U^* is the space of elements u of E' such that there eixsts a (unique) solution f in K' of

(2.3) $(f,\ v) = (u,\ \wedge v)$ for all $v \in D$.

PROOF. Since U_0 is dense in K, then D is dense in K and there exists at most a solution of (2.3). Let us denote by $\tilde{D}$ the space defined in Lemma 2.1. Is is clear that D^* is contained in $\tilde{D}$. On the other hand, let $u \in \tilde{D}$ be a solution of (2.3). Since this equation holds for all v in $U_0 \subset D$, we deduce that $f = \wedge^* u$ and since $f \in K$, that u belongs to U^*. Then $\tilde{D} \subset D^*$ and the lemma is proved. This lemma suggests to slightly extend the usual definition of the unbounded operators .

DEFINITION 2.1. Let D be a closed subspace of U which contains U_0. We will call the pair $(D, \wedge)$ the K-unbounded operator associated with $\wedge$ of domain D (dense in K).

If D^* is defined by (2.2) (or (2.3)), we will say that $(D^*, \wedge^*)$ is the adjoint of $(D, \wedge)$.

NOTE 2.1. In the case where $K=E$, this is the usual definition of unbounded operators (with dense domain containing U_0) and their adjoints (see Lemma 2.1).

The choice of D^*, like the coice of β^*, depends on the choice of the space K'.

By Definition 1.1, we see that $(U^*, \wedge^*)$ is the adjoint of $(U_0, \wedge)$ and we will denote by $(U_0^*, \wedge^*)$ the adjoint of (U, Λ).

2.2 *Green's Formula Associated with an Unbounded Operator.*

Let π be a continuous projector of Φ. Let us associate with π:

$$(2.4)\qquad \begin{cases} i)\ \Phi_0=\pi\Phi;\ \Phi'_1=(1-\pi')\Phi'=\Phi_0{}^\perp \\ ii)\ \Phi'_0=\pi'\Phi'=\Phi_1{}^\perp;\ \Phi_1=(1-\pi)\Phi. \end{cases}$$

Indeed, we may identify Φ'_0 with the dual of Φ_0 and Φ'_1 with the dual of Φ_1 in the following: If $\langle \psi, \varphi \rangle$ denotes the duality pairing on $\Phi' \times \Phi$, the duality pairings on $\Phi'_0 \times \Phi_0$ and $\Phi'_1 \times \Phi_1$ will be respectively:

$$\langle \pi'\psi, \pi\psi \rangle \text{ and } \langle (1-\pi')\psi, (1-\pi)\varphi \rangle.$$

We also associate with π and β the following boundary operators:

$$(2.5)\qquad \begin{cases} i)\ \beta_0=\pi\beta \in \mathcal{L}(U;\ \Phi_0);\ \beta_1^*=(1-\pi')\beta^* \in \mathcal{L}(U^*,\ \Phi'_1) \\ ii)\ \beta_0^*=-\pi'\beta^* \in \mathcal{L}(U^*;\ \Phi'_0);\ \beta_1=(1-\pi)\beta \in \mathcal{L}(U;\ \Phi_1). \end{cases}$$

Finally, with the projector π *we may associate the following unbounded operator* (D, Λ) *where* $D=\ker \pi\beta=\ker \beta_0 \supset U_0$. Conversely, we will prove the:

THEOREM 2.1 Let $(D, \wedge)$ be an unbounded operator, $(D^*, \wedge^*)$ be its adjoint. Then there exists a projector π of Φ such that

$$(2.6)\qquad D=\ker \pi\beta=\ker \beta_0\,;\ D^*=\ker(1-\pi')\beta^*=\ker \beta_1$$

and we may write Green's formula in the following form:

$$(2.7)\qquad (u, \wedge v)-(\wedge^* u, v)=\langle \beta_1^* u, \beta_1 v\rangle-\langle \beta_0^* u, \beta_0 v\rangle$$

for all $u \in U^*$, for all $v \in U$.

PROOF. By Lemma 1.2, there exests projectors μ and ν such that

$$\begin{cases} i) \; U_0 = \ker \mu; \; D = \ker \nu \\ ii) \; \mu\nu = \nu\mu = \nu. \end{cases}$$

On the other hand, there exists a right inverse σ of β such that

$$\mu = \sigma\beta \; (\beta\sigma = 1).$$

Let *us* associate with D (and thus, with ν) the following operator

$$\pi = \beta\nu\sigma.$$

Then we can check that π is a projector and that $D = \ker \pi\beta$. Indeed, $\pi^2 = \beta\nu\sigma\beta\nu\sigma = \beta\nu\mu\nu\sigma = \beta\nu\sigma = \pi$ and $\pi\beta = \beta\nu$.
Since σ is one to one, we obtain:

$$D = \ker \nu = \ker \mu\nu = \ker \sigma\beta\nu = \ker \beta\nu = \ker \pi\beta.$$

We now have to prove that $D^* = (1-\pi')\beta^*$. We first notice that Green's formula (2.7) associated with π is a direct consequence of Green's formula of Theorem 1.1 and of (2.5). Then if $u \in \ker (1-\pi')\beta^*$ and $v \in D = \ker \pi\beta$, we will have $(\wedge^* u, v) = (u, \wedge v)$. Then $\ker (1-\pi')\beta^* \subset D^*$. On the other hand, let u belong to D^*, v belong to D. Then, by Green's formula, we deduce that

$$\langle \beta_1{}^* u, \beta_1 v \rangle = 0 \text{ for all } v \text{ in } D. \tag{2.8}$$

But:

$$\beta_1 \in \mathcal{L}(D, \Phi_1) \tag{2.9}$$

is surjective.

Indeed, if φ belongs to Φ_1 (i.e., if $\pi\varphi = 0$) and if σ is a right inverse of β, then $u = \sigma\varphi$ belongs to D since

$$\beta_0 u = \pi\beta u = \pi\beta\sigma\varphi = \pi\varphi = 0.$$

Then (2.8) and (2.9) imply that $\beta_1^*=(1-\pi')\beta^* u=0$ and *we have proved that* $D^*=\ker\beta_1^*=\ker(1-\pi')\beta^*$ *when* $D=\ker\beta_0=\ker\pi\beta$.

NOTE 2.2. Since $(U_0^*, \wedge^*)$ is, by definition, the adjoint of $(U, \wedge)$, we deduce from Theorem 2.1 that

(2.10) $$U_0^*=\ker\beta^*.$$

2.3. *Closed K-unbounded Operators.*

Let $(D, \wedge)$ be a K-unbounded operator, $(D^*, \wedge^*)$ be its adjoint. We associate with them their graphs $\Gamma(D)$ and $\Gamma^*(D^*)$ defined by:

(2.11)
- *i)* $\Gamma(D)$ is the subspace of $K\times E$ of the $v\times\wedge v$ when v ranges D
- *ii)* $\Gamma^*(D^*)$ is the subspace of $K'\times E'$ of the $-\wedge^* u\times u$ when u ranges D^*.

We thus introduce

DEFINITION 2.2. We will say that $(D, \wedge)$ is a closed K-unbounded operator iff its graph $\Gamma(D)$ is closed in $K\times E$. This amounts to saying that if a sequence $u_n\in D$ satisfies:

$$u_n \text{ converges to } u \text{ in } K \text{ and } \wedge u_n \text{ converges to } f \text{ in } E$$

then u belongs to D and $f=\wedge u$.

The classical properties of the closed operators remain. Namely:

PROPOSITION 2.1. The graph $\Gamma^*(D^*)$ is the orthogonal in $K'\times E'$ of the graph $\Gamma(D)$ and thus, is closed.

If $(D, \wedge)$ is closed, then:

(2.12)
- *i)* D^* is dense in E^*
- *ii)* D is the space of v of K such that:
$$(u, \wedge v)-(\wedge^* u, v)=0 \text{ for all } u \text{ of } D^*$$
- *iii)* $\beta_0^*(D^*)$ is dense in Φ'_0.

In particular, if (U_0, Λ) is closed, then U^* is dense in E' and $\beta^*(U^*)$ is dense in Φ'.

We have only to prove the last statement, the proof of the others being analogous to the classical one.

By the Hahn-Banach Theorem, we have to prove that if $\psi \in \Phi_0$ satisfies

$$\langle \psi, \beta_0^* u \rangle = 0 \text{ for all } u \text{ in } D^*,$$

then $\psi = 0$.

Let $v = \sigma\psi$. Then $\beta_0 v = \pi\beta v = \pi\beta\sigma\psi = \pi\psi = \psi$ since ψ belongs to Ψ_0 and $\beta_1 v = (1-\pi)\beta v = (1-\pi)\psi = 0$. Then

$$\langle \psi, \beta_0^* u \rangle = \langle \beta_0 v, \beta_0^* u \rangle = (\Lambda^* u, v) - (u, \Lambda v) = 0 \text{ for all } u \text{ in } D^*.$$

This implies, by (2.12) *ii*) that u belongs to D and thus, that $\psi = \beta_0 v = \pi\beta v = 0$.

If we choose $D = U_0$, then $D^* = U^*$ is dense in E'. The projector associated with U_0 is the identity and $\beta_0 = \beta$, $\beta_0^* = -\beta^*$. Then $\beta^*(U^*)$ is dense in Φ'.

The next proposition gives an analytic definition of the closed K-unbounded operators, where appears the « *apriori estimates* ».

PROPOSITION 2.2. The following statements are equivalent:

(2.13) $\quad (D, \Lambda)$ is a closed K-unbounded operator

(2.14) $\quad \| u \|_U \leq c(\| \Lambda u \|_E^2 + \| u \|_K^2)^{1/2}$ for all $u \in D$

(2.15) $\quad \| u \|_U \leq c(\| \Lambda u \|_E^2 + \| \beta_0 u \|_{\Phi_0}^2 + \| u \|_K^2)^{1/2}$ for all $u \in U$.

First of all, we always have this inequality:

(2.16) $\quad (\| \Lambda u \|_E^2 + \| u \|_K^2)^{1/2} \leq c \| u \|_U$ for all $u \in U$

since Λ is continuous from U into E and since U is contained in K with a stronger topology. The two first statements are equivalent. Indeed, to say that $\Gamma(D)$ is closed amounts to saying that D equipped with the graph norm $(\| \Lambda u \|_E^2 + \| u \|_K^2)^{1/2}$ is complete. Since D is also complete for the norm $\| u \|_U$ by assumption, these two norms are equivalent when $\Gamma(D)$ is closed and conversely, if these two norms are equivalent, D is complete for the graph norm and $\Gamma(D)$ is closed in $K \times E$. The statement (2.15) implies obviously (2.14). The converse is true. If σ_0 is a right inverse of $\beta_0 = \pi\beta$, then $u = \sigma_0\beta_0 u + (1-\sigma_0\beta_0)u$ and $(1-\sigma_0\beta_0)u$ belongs to D. Then:

$$\| u \|_U^2 \leq 2(\| \sigma_0\beta_0 u \|_U^2 + c \| \wedge u - \wedge \sigma_0\beta_0 u \|_E^2 + \| u - \sigma_0\beta_0 u \|_E^2)$$
$$\leq c'(\| \wedge u \|_E^2 + \| \beta_0 u \|_{\Phi_0}^2 + \| u \|_K^2).$$

Let Z be the kernel of Λ.

THEOREM 2.2. Let *us* assume that D is not contained in Z. If the injection from D into K is compact, then the injection from D^* into E' is comparct.

Let B^* be a bounded set of U^*. Then B^* is bounded in E' and $\wedge^* B^*$ is bounded in K'. We may extract from a sequence of B^* a subsequence u_n such that:

(2.17) u_n converges weakly to 0 in E';

$\wedge^* u_n$ converges weakly to 0 in K'

and we have to prove that u_n converges strongly in E'. Let B be a bounded set of D. Then B is compact in K by assumption and thus, by the Banach-Steinhauss Theorem:

(2.18) $(\wedge^* u_n, v)$ converges to 0 uniformly on the compact B of K.

Then $(u_n, \wedge v) = (\wedge^* u_n, v)$ converges uniformly to 0 on the bounded set $\wedge(B)$ of E, i.e., u_n converges strongly to 0 in E'. (Since D is not contained in Z, $\wedge(B)$ is not empty if B is not contained in Z).

COROLLARY 2.1. Let *us* assume that $(D, \wedge)$ is closed and that the injection from D into K is compact.

Let $Z = \ker \wedge$ and $Z^* = \ker \wedge^*$ be the kernels of $\wedge$ and $\wedge^*$, $N = D \cap Z$ and $N^* = D^* \cap Z^*$ the kernels of $(D, \wedge)$ and $(D^*, \wedge)$. Then N and N^* are finite dimensional spaces and the ranges $\wedge(D)$ and $\wedge^*(D^*)$ are closed in E and K' respectively.

By theorem 2.2, it is sufficient to prove the theorem only for $(D, \wedge)$. But this result for $(D, \wedge)$ is classical. Let *us* nevertheless sketch the proof. In u belongs to $N=Z\cap D$, then the norms $\| u \|_U$ and $\| u \|_K$ are equivalent and since the injection is compact, the unit ball of N is compact and thus, N is a finite dimensional space. If M is a topological complement of N in D, Λ is a one-to-one map from M onto $\wedge(D)$. Then $\wedge(D)$ will be closed iff

$$\| u \|_U \leq c \| \wedge u \|_E \text{ for all } u \text{ in } M. \tag{2.19}$$

But (2.19) holds. If not, there would be a sequence u_n of M such that $\| u_n \|_U=1$ and such that $\wedge u_n$ converges to 0 in K. Since the injection from D into K is compact, (a subsequence of) u_n converges to an element u of K. Since $(D, \wedge)$ is closed, we deduce that $u=0$. Since the norm $\| u \|_U$ is equivalent to the graph norm, we deduce that $\| u \|_U$ converges to 0. We have obtained a contradiction.

NOTE 2.3. If we assume that the domains D posses a topological complement (and not only that they are closed), then Lemma 2.1, Theorem 2.1 and Proposition 2.1 hold again when the spaces are locally convex and Proposition 2.2, Theorem 2.2, and Corollary 2.1 hold again when the spaces are Banach spaces.

3. Boundary Value Operators and their Adjoints.

3.1. *Definitions.*

DEFINITION 3.1. Let γ_0 be a surjective operator from U onto a Hilbert space Ψ_0 such that:

$$D= \ker \gamma_0 \supset U_0 = \ker \beta. \tag{3.1}$$

We will say that $\mathcal{S}=\mathcal{S}(\Psi_0, \gamma_0)= \wedge \times \gamma_0$, mapping U into $E \times \Psi_0$, is a boundary value operator associated with the K-unbounded operator $(D, \wedge)$. Two boundary value operators associated with the same unbounded operator $(D, \wedge)$ will be said to be equivalent.

First of all, Theorem 2.1 implies that there exists always at least one boundary value operator associated with (D, Λ) namely, the operator

$\mathcal{S} = \mathcal{S}(\Phi_0, \pi\beta)$ where π is a projector from Φ onto Φ_0 constructed in the proof of Theorem 2.1.

Another way to define the equivalence between two boundary value operators is given by

PROPOSITION 3.1. Two boundary value operators $\mathcal{S}(\Phi_0, \beta_0)$ and $\mathcal{S}(\Psi_0, \gamma_0)$ are equivalent iff there exists an isomorphism θ from Ψ_0 onto Φ_0 such that $\beta_0 = \theta\gamma_0$.

We then may identify an unbounded operator $(D, \wedge)$ with an equivalence class of boundary value operators and *say that* $(D, \wedge)$ *is associated with* $\mathcal{S}(\Psi_0, \gamma_0)$ *if* $D = \ker \gamma_0$. In particular, each boundary value operator is equivalent to a boundary value operator $\mathcal{S}(\Phi_0, \pi\beta)$ where π is a projector from Φ onto Φ_0.

We now will define the adjoint of a boundary value operator.

DEFINITION 3.2. Let γ_1^* map U^* into a Hilbert space Ψ'_1. Let *us* denote by $\mathcal{S}^* = \mathcal{S}^*(\Psi'_1, \gamma_1^*)$ the operator $\wedge^* \times \gamma_1^*$ mapping U^* into $K' \times \Psi'_1$.

We will say that $\mathcal{S}^*$ is an adjoint of an operator $\mathcal{S} = \mathcal{S}(\Psi_0, \gamma_1)$ iff there exists

$$(3.2) \quad \begin{cases} i) \text{ a projector } \pi \text{ from } \Phi \text{ onto a closed subspace } \Phi_0 \\ ii) \text{ an isomorphism } \theta_0 \text{ from } \Psi_0 \text{ onto } \Phi_0 \\ iii) \text{ an isomorphism } \theta_1 \text{ from } \Psi'_1 \text{ onto } \Phi'_1 = (1-\pi')\Phi' \end{cases}$$

such that

$$(3.3) \qquad \pi\beta = \theta_0\gamma_0\,; \quad (1-\pi')\beta^* = \theta_1\gamma_1^*.$$

By this definitions, if $\mathcal{S}^*$ is an adjoint of a boundary value operator $\mathcal{S}$, then $\mathcal{S}^*$ is an adjoint of all the operators equivalent to $\mathcal{S}$.

Then Theorem 2.1 implies that there exists always en adjoint operator of $\mathcal{S}$. Namely, if $\mathcal{S}$ is equivalent to $\mathcal{S}(\Phi_0, \pi\beta)$ the operator $\mathcal{S}^*(\Phi'_1, (1-\pi')\beta^*)$ is an adjoint of $\mathcal{S}$.

If $(D, \wedge)$ is associated with $\mathcal{S}(\Psi_0, \gamma_0)$ $(D = \ker \gamma_0)$, then $(D^*, \wedge^*)$ (where $D^* = \ker \gamma_1^*$), the unbounded operator associated with an adjoint $\mathcal{S}^*(\Psi'_1, \gamma_1^*)$, is the adjoint of $(D, \wedge)$.

PROPOSITION 3.2. Let $\mathcal{B}^*(\Psi'_1, \gamma_1^*)$ be an adjoint of $\mathcal{B}(\Psi_0, \gamma_0)$. Let us set

(3.4) $$\gamma_1 = \theta'_1(1-\pi)\beta \text{ and } \gamma_0^* = -\theta'_0\pi'\beta^*,$$

(where π is defined by (3.2) and (3.3)). Then

(3.5) $$\gamma_0 \text{ maps } \ker\gamma_1 \text{ onto } \Psi_0, \ \gamma_1 \text{ maps } \ker\gamma_0 \text{ onto } \Psi_1$$

(3.6) $$\gamma_0 u = \gamma_1 u = 0 \text{ for all } u \in U_0$$

and the following Green's formula holds:

(3.7)) $$(u, \wedge v) - (\wedge^* u, v) = \langle \gamma_1^* u, \gamma_1 v\rangle - \langle \gamma_0^* u, \gamma_0 v\rangle$$

for all u in U^*, for all v in U.

These results are direct consequences of the definition and Theorem 2.1. If we set

(3.8) $$Z = \ker\wedge \text{ and } Z^* = \ker\wedge^*$$

then

(3.9) $$N = D \cap Z = \ker\mathcal{B}; \ N^* = D^* \cap Z^* = \ker\mathcal{B}^*$$

are the kernels of $\mathcal{B}$ (or $(D, \wedge)$) and $\mathcal{B}^*$ (or $(D^*, \wedge^*)$) respectively.

3.2. *Relations Between the Transpose and the Adjoint of a Boundary Value Operator.*

Since the boundary operators $\mathcal{B} = \mathcal{B}(\Psi_0, \gamma_0)$ maps U into $E \times \Psi_0$, its transpose $\mathcal{B}'$ maps $E' \times \Psi'_0$ into U'. Let $\mathcal{B}^* = \mathcal{B}^*(\Psi'_1, \gamma_1^*)$ be an adjoint of $\mathcal{B}$. Then the spaces K' and $\gamma'_1\Psi'_1$ are contained in U'. Let *us* consider

Problem P' (transposed boundary value operator).

Let $f \times \psi_1$ be given in $K' \times \Psi'_1$. Find $u \times \psi_0$ in $E' \times \Psi'_0$ such that

(3.14) $$\mathcal{B}'(u \times \psi_0) = f + \gamma'_1\psi_1.$$

Problem P^* (adjoint boundary value problem).

Let $f \times \psi_1$ be given in $K' \times \Psi'_1$. Find u in U^* such that

$$\Lambda^* u = f \text{ and } \gamma_1 u = \psi_1 . \tag{3.11}$$

We then will prove

THEOREM 3.1. When $f \times \psi_1$ ranges $K' \times \Psi'_1$, the problems P' and P^* are equivalent: If $u \times \psi_0$ is a solution of the problem P', then u belongs to U^*, $\psi_0 = \gamma_0^* u$ and u is the solution of the problem P^*. Conversely, if u is a solution of the problem P^*, then $u \times \gamma_0^* u$ is a solution of the problem P'.

PROOF. Let $u \times \psi_0$ be a solution of the problem P'. Then

$$(u \times \psi_0 , \mathfrak{F} v) = (f + \gamma_1^* \psi_1 , v) \text{ for all } v \text{ in } U \tag{3.12}$$

and this equations may be written explicitly in the following form:

$$(u, \Lambda v) + \langle \psi_0 , \gamma_0 v \rangle = (f, v) + \langle \psi_1 , \gamma_1 v \rangle \text{ for all } v \text{ in } U. \tag{3.13}$$

Since U_0 is contained in U, we obtain in particular by (3.6):

$$(u, \Lambda v) = (\Lambda^* u, v) = (f, v) \text{ for all } v \text{ in } U_0$$

and this implies that:

(3.14) $\Lambda^* u = f$ and that u belongs to U^* since f belongs to K'.

We then can use the Green's formula (3.7) and (3.13) together:

$$\begin{aligned}(u, \Lambda v) &= (\Lambda^* u, v) + \langle \gamma_1^* u, \gamma_1 v \rangle - \langle \gamma_0^* u, \gamma_0 v \rangle \\ &= (f, v) + \langle \psi_1 , \gamma_1 v \rangle - \langle \psi_0 , \gamma_0 v \rangle .\end{aligned}$$

We deduce that:

$$\langle \gamma_1^* - \psi_1 , \gamma_1 v \rangle - \langle \gamma_0^* u - \psi_0 , \gamma_0 v \rangle = 0 \text{ for all } v \text{ in } U. \tag{3.15}$$

Using (3.5), we finally get

$$\gamma_0^* u = \psi_0 ; \quad \gamma_1^* u = \psi_1 . \tag{3.16}$$

Conversely, let u be a solution of the problem P^* and let *us* set $\psi_0 = \gamma_0^* u$. Using Green's formula (3.7), we have

$$(u \times \psi_0, \mathcal{S} v) = (u, \wedge v) + \langle \psi_0, \gamma_0 v \rangle$$
$$= (\wedge^* u, v) + \langle \gamma_1^* u, \gamma_1 v \rangle + \langle \psi_0 - \gamma_0^* u, \gamma_0 v \rangle$$
$$= (f, v) + \langle \psi_1, \gamma_1 v \rangle = (f + \gamma'_1 \psi_1, v).$$

Therefore $u \times \psi_0$ is a solution of the problem P'.

COROLLARY 3.1. The kernel of the transpose $\mathcal{S}'$ is the subspace $N^* \times \gamma_0^* N^*$ of the elements $u \times \psi_0$ of $E' \times \Psi'_0$ satisfying.

(3.17) $$u \in U^*; \ \psi_0 = \gamma_0^* u; \ \wedge^* u = 0 \text{ and } \gamma_1^* u = 0.$$

Thus the kernel N^* of $\mathcal{S}^*$ does not depend on the choice of the normal space K. If we denote by $D^*(K)$ the domain of $\wedge^*$ associated with K, then $N^* \subset \bigcap_K D^*(K)$ for all the spaces K satisfying (1.2).

3.3. *Characterization of the Ranges of $\mathcal{S}$ and $\mathcal{S}^*$.*

THEOREM 3.2. Let *us* assume that the range of $\mathcal{S}$ is closed. Then the range of the adjoint $\mathcal{S}^*$ is also closed. Moreover, the range $\mathcal{S}U$ is the space $\{E \times \Psi_0, N^*, \gamma_0^*\}$ of the $f \times \psi_0$ belonging to $E \times \Psi_0$ such that

(3.18) $$(f, v) + \langle \psi_0, \gamma_0^* v \rangle = 0 \text{ for all } v \in N^* = Z^* \cap D^*$$

and the range $\mathcal{S}^* U^*$ is the space $\{K' \times \Psi'_1, N, \gamma_1\}$ of the $f \times \psi_1$ belonging to $K' \times \Psi'_1$ such that:

(3.19) $$(f, v) + \langle \psi_1, \gamma_1 v \rangle = 0 \text{ for all } v \in N = Z \cap D.$$

The conditions (3.18) and (3.19) are usually called the « *compatibility conditions* ». Let *us* notice that the assumption that $\mathcal{S}U$ is closed *does not depend on the choice of K satisfying* (1.2).

PROOF. The range of $\mathcal{S}$ is the orthogonal of the kernel of $\mathcal{S}'$ (when $\mathcal{S}U$ is closed). But $\ker \mathcal{S}' = N^* \times \gamma_0^* N^*$ by the Corollary 3.1. To say that $f \times \psi_0 \in E \times \Psi_0$ is orthogonal to $N^* \times \gamma_0^* N^*$ amounts to saying that $f \times \psi_0$ satisfies (3.18).

If the range of $\mathcal{S}$ is closed, the range of $\mathcal{S}'$ is also closed, and is equal to the orthogonal of N in U'.

Let $f \times \psi_1$ belong to $K' \times \Psi'_1$. Then $f+\gamma'_1\psi_1$ is orthogonal to N iff (3.19) is fulfilled. Therefore $\{K' \times \Psi'_1, N, \gamma_1\}$ is contained in $\mathcal{S}^*U^*$. On the other hand, $\mathcal{S}^*U^*$ is contained in $\{K' \times \Psi'_1, N, \gamma_1\}$ by Green's formula:

$$(\Lambda^* u, v)+\langle \gamma_1^* u, v \rangle = (u, \Lambda v)+\langle \gamma_0^* u, \gamma_0 v \rangle = 0$$

whenever v belongs to $N=D \cap Z$.

The Banach Theorem implies the:

COROLLARY 3.2. If the range of $\mathcal{S}$ is closed, $\mathcal{S}$ defines an isomorphism mapping U/N onto $\{E \times \Psi_0, N^*, \gamma_0^*\}$ and $\mathcal{S}^*$ defines an isomorphism mapping U^*/N^* onto $\{K' \times \Psi'_1, N, \gamma_1\}$.

We notice

LEMMA 3.1. The range of $\mathcal{S}$ is closed iff the range $\Lambda(D)$ of the associated unbounded operator $\Lambda(D)$ is closed.

This is the consequence of the following fact. Let σ_0 be a right inverese of γ_0. Then, if u is a solution of

$$\Lambda u = f; \quad \gamma_0 u = \psi_0 \tag{3.20}$$

then $v=u-\sigma_0\psi_0$ is a solution of

(3.21)) $$u \in D = \ker \gamma_0 ; \quad \Lambda u = f + \Lambda \sigma_0 \psi_0 .$$

COROLLARY 3.3. Let *us* assume that the following inequality holds:

$$\| u \|_U \leq c(\| \Lambda u \|_E^2 + \| \gamma_0 u \|_{\Psi_0}^2 + \| u \|_K^2)^{1/2} \tag{3.22}$$

and that the injection from U into K is compact.

Then the kernels $N=D \cap Z$ and $N^*=D^* \cap Z^*$ of $\mathcal{S}$ and $\mathcal{S}^*$ are finite dimeinsional spaces and the ranges of $\mathcal{S}$ and $\mathcal{S}^*$ are closed.

By Proposition 2.2, the assumption (3.22) implies that (D, Λ) is closed. Corollary results from Corollary 2.2, Lemma 3 and the equalities $\ker(D, \Lambda) = \ker \mathcal{S}$ and $\ker(D^*, \Lambda^*) = \ker \mathcal{S}^*$.

NOTE 3.1. *If the assumption holds for at least one normal space K, then the conclusions hold for all the normal spaces K.*

3.4. *Well-posed Boundary Value Operators.*

Let $(D, \wedge)$ be a K-unbounded operator, $(D^*, \wedge^*)$ its adjoint, $\mathfrak{F}=\mathfrak{F}(\Psi_0, \gamma_0)$ a boundary value operator, $\mathfrak{F}^*=\mathfrak{F}^*(\Psi'_1, \gamma_1^*)$ an adjoint of $\mathfrak{F}$. We set:

$$Z= \ker \wedge;\ Z^*= \ker \wedge^*;\ N=Z \cap D= \ker \mathfrak{F};\ N^*=Z^* \cap D^*= \ker \mathfrak{F}^*.$$

By Theorem 3.2, we notice that:

PROPOSITION 3.3. The following statements are equivalent

$$(3.23)\quad \begin{cases} i)\ U=Z+D \text{ has a direct sum decomposition into } Z \text{ and } D \\ ii)\ N=N^*=0 \text{ and the range of } \mathfrak{F} \text{ (or } \wedge(D) \text{ is closed)} \\ iii)\ \mathfrak{F} \text{ is an isomorphism from } U \text{ onto } E \times \Psi_0 \\ iv)\ \wedge \text{ is an isomorphism from } D \text{ onto } E. \end{cases}$$

This suggests

DEFINITION 3.3. We will say that the boundary value operator $\mathfrak{F}$ is well posed (or that $(D, \wedge)$ is well posed) if one of the equivalent statements of (3.23) holds.

Theorem 3.1 implies that if $\mathfrak{F}$ is well posed, then their adjoints $\mathfrak{F}^*$ are also well posed.

We shall give a characterization of the well posed boundary value operators.

THEOREM 3.3. If there exists one well posed boundary value operator $\mathfrak{F}$, then β defines an isomorphism from Z onto βZ and all the well posed boundary value operators are equivalent to $\mathfrak{F}(\beta Z, \pi\beta)$ where π is a projector onto βZ. Conversely, if β defines an isomorphism from Z onto βZ, the operators $\mathfrak{F}(\beta Z, \pi\beta)$ where π is a projector onto βZ are well posed. Finally, the adjoint of a well boundary value operator $\mathfrak{F}(\beta Z, \beta\pi)$ is $\mathfrak{F}^*(\beta^* Z^*, (1-\pi')\beta^*)$.

PROOF. If there exists a well posed boundary value operator, then there exists a topological complement D of Z containing U_0 and, by Lemma 1.2, a topological complement Y of U_0 containing Z. If k_0 is a projector onto Y with kernel U_0 and k the projector onto Z with kernel D, we have:

$$k_0k=kk_0=k. \tag{3.24}$$

On the other hand, there exists a right inverse σ of β such that

$$k_0=\sigma\beta. \tag{3.25}$$

Then β is an isomorphism from Y onto Φ and, in particular, between $Z\subset Y$ and its closed range βZ. Moreover, we may associate with k the following operator:

$$\pi=\beta k\sigma \tag{3.26}$$

which is, by (3.24) and (3.25), a projector onto βZ such that

$$\sigma\pi\beta=k. \tag{3.27}$$

Then

$$D=\ker k=\ker\sigma\pi\beta=\ker\pi\beta \tag{3.28}$$

and $\mathfrak{S}$ is equivalent to $\mathfrak{S}(\beta Z, \pi\beta)$. Conversely, let us assume that the restriction β_Z of β to Z is an isomorphism from Z onto βZ. Let β_Z^{-1} be its inverse. Let now π be any projector onto βZ. Then

$$k=\beta_Z^{-1}\pi\beta \tag{3.29}$$

is a proejctor from Z onto K and $U_0\subset D=\ker k=\ker\pi\beta$ since $ku=0$ if u belongs to $U_0=\ker\beta$. Then $\mathfrak{S}(\beta Z, \pi\beta)$ is well posed. We know that $\mathfrak{S}^*((\beta Z)^\perp, (1-\pi')\beta^*)$ is an adjoint of $\mathfrak{S}(\beta Z, \pi\beta)$. The end of the proof of the Theorem will be the consequence of

Proposition 3.4. If $\mathfrak{S}$ is well posed, then β^*Z^* is the orthogonal of βZ.

We always have $\beta^*Z^*\subset(\beta Z)^\perp$ by Green's formula:

$$\langle\beta^*u, \beta v\rangle=(\Lambda^*u, v)-(u, \Lambda v)=0 \text{ if } u\in Z^*, v\in Z.$$

Let now φ belong to $(\beta Z)^{\perp}$ and π a projector onto $\Phi_0=\beta Z$. Then $\varphi=\varphi_1-\varphi_0$ where $\varphi_1=(1-\pi')\varphi$ and $\varphi_0=-\pi'\varphi$ and φ_1 belongs to $(\beta Z)^{\perp}$.

Since $\mathcal{B}^*$ is well posed, there exists a unique solution u satisfying $\Lambda^* u=0$ and $\beta_1^* u=\varphi_1$. Then, if $\beta_1=(1-\pi)\beta$ and $\beta_0^*=-\pi'\beta^*$

$$\langle \varphi, \beta v\rangle=\langle \varphi_1, \beta_1 v\rangle-\langle \varphi_0, \beta_0 v\rangle$$
$$=(u, v)+\langle \beta_0^* u-\varphi_0, \beta_0 v\rangle.$$

This implies that

$$\langle \beta_0^* u-\varphi_0, \beta_0 v\rangle=0 \text{ for all } v\in Z. \tag{3.30}$$

Since $\beta_0=\pi\beta$ maps Z onto $\pi\beta Z=\beta Z=\Phi_0$, we get

$$\varphi_0=\beta_0^* u \text{ and } \varphi=\varphi_1-\varphi_0=\beta_1^* u-\beta_0^* u=\beta^* u. \tag{3.31}$$

Then $(\beta Z)^{\perp}$ is contained in $\beta^* Z^*$ and $\beta^* Z^*$ is the orthogonal of βZ. Theorem 3.3 and Lemma 1.3 imply a theorem of characterization of the well posed boundary value operators proved by J. L. Lions and E. Magenes.

COROLLARY 3.4. Let $\mathcal{B}(\Psi_0, \gamma_0)$ be a well posed boundary value operator. The boundary value operators $\mathcal{B}(\Psi_0, \gamma)$ where

$$\gamma=k\beta+(1-k\beta\sigma_0)\gamma_0;\ k \text{ is an operator from } \Phi \text{ into } \Psi_0 \tag{3.32}$$

and σ_0 is the right inverse of γ_0 mapping Ψ_0 onto Z are well posed. Conversely, every well posed boundary value operator is equivalent to a boundary value operator $\mathcal{B}(\Psi_0, \gamma)$ where γ is defined by (3.32) for a convenient operator k mapping Φ into Ψ_0.

PROOF. Let $\mathcal{B}(\Psi_0, \gamma_0)$ be a well posed boundary value operator. By Theorem 3.3, it is equivalent to $\mathcal{B}(\beta Z, \pi_0\beta)$ where π_0 is a projector onto βZ. If σ_0 is the inverese of γ_0 mapping Ψ_0 onto Z; $\theta_0=\beta\sigma_0$ is the isomorphism from Ψ_0 onto βZ such that

$$\theta_0\gamma_0=\pi_0\beta;\ \theta_0=\beta\sigma_0. \tag{3.33}$$

Then, if γ is defined by (3.32),

$$\theta_0\gamma=\theta_0 k\beta+\theta_0\gamma_0-\theta_0 k\theta_0\gamma_0=(l+(1-l)\pi_0)\beta \tag{3.34}$$

where $l=\theta_0 k$ is an operator mapping Φ into βZ. By Lemma 1.3, $\pi=l+(1-l)\pi_0$ is a projector onto βZ and (3.34) implies that $\mathcal{S}(\Psi_0, \gamma)$ is equivalent to $\mathcal{S}(\beta Z, \pi)$, which is well posed by Theorem 3.3.

Conversely, let $\mathcal{S}$ be a well posed boundary value operator. It is equivalent to $\mathcal{S}(\beta Z, \pi\beta)$ for a convenient projector π onto βZ, which is equal to $l+(1-l)\pi_0$ where l maps Φ into βZ. Let *us* set $\gamma=\theta_0^{-1}\pi\beta$. Then:

$$\begin{aligned}\gamma&=\theta_0^{-1}l\beta+\theta_0^{-1}\pi_0\beta-\theta_0^{-1}l\theta_0\theta_0^{-1}\pi_0\beta=k\beta+(1-k\theta_0)\beta\\&=k\beta+(1-k\beta\sigma_0)\beta\end{aligned} \tag{3.35}$$

where $k=\theta_0^{-1}l$ maps Φ into Ψ_0. This implies that $\mathcal{S}$ is equivalent to $\mathcal{S}(\Psi_0, \gamma)$.

NOTE 3.2 We deduce from Corollary 3.4 that a domain D is a topological complement of Z (a well posed domain in the terminology of J. L. Lions - E. Magenes) iff D is the space of elements u of U such that

$$\gamma_0 u=k\beta(u-\sigma_0\gamma_0 u) \tag{3.36}$$

for a suitable operator k mapping Φ into Ψ_0.

NOTE 3.3. If we assume in Definition 3.1 that γ_0 posses a continuous right inverese (or that D possesses a topological complement), we may assume that the spaces are Banach spaces in Theorem 3.2, Corollaries 3.2 and 3.3 and are locally convex in the other results of this section.

4. Example: Elliptic Differential Boundary Value Operators.

4.1. *The Framework* (*Example*).

Let Ω be a « regular » bounded open set of R^n, with boundary Γ.

Let us choose:

(4.1) $$U=H^{2m}(\Omega);\ E=K=L^2(\Omega)$$

where $L^2(\Omega)$ denotes the space of square integrale functions on Ω and $H^{2m}(\Omega)$ denotes the Sobolev space of u of $L^2(\Omega)$ such that the weak derivatives $D^p u$ of order $|p|=p_1+ \dots +p_n \leq 2m$ belong to $L^2(\Omega)$.

$$D^p u=\left(\frac{\partial^{|p|}u}{\partial x_1^{p_1} \dots \partial x_n^{p_n}}\right)$$

We shall choose

(4.2) $$U_0=H_0^{2m}(\Omega)= \text{closure of } D(\Omega) \text{ into } H^{2m}(\Omega)$$

where $D(\Omega)$ is the space of indefinitely differentiable functions with compact support in Ω. Our main assumption (1.2) is then fulfilled.

Let us set:

(4.3) $$\Phi=\prod_{j=0}^{2m-1} H^{2m-j-1/2}(\Gamma).$$

(See for instance J. L. Lions - E. Magenes [2], Chapter 1, for the precise definitions).

Let us recall that $H_0^{2m}(\Omega)$ is the kernel of the operator $\beta=\gamma_0\times \times\gamma_1\times \dots \times\gamma_{2m-1}$ where γ_j is the operator which associates with u the restriction to Ω of its normal derivatives of order j. We now introduce the following differential operator:

(4.4) $$\Lambda u=\sum_{|p|,\,|q|\leq m}(-1)^{|q|}D^q(a_{pq}(x)D^p u)$$

where the coefficients $a_{pq}(x)$ are $2m$ times continuously differentiable on Ω.

4.2. *The Adjoint Framework and Green's Formula.*

Since $H_0^{2m}(\Omega)$ is the closure of $D(\Omega)$, we deduce that the adjoint operator Λ^* is equal to

(4.5) $$\wedge^* u = \sum_{|p|,|q| \leq m} (-1)^{|p|} D^p(a_{pq}(x) D^q u).$$

Then:

(4.6) $$U^* = \{u \in L^2(\Omega) \text{ such that } \wedge^* u \text{ belongs to } L^2(\Omega)\}$$

and we notice that

(4.7) $$H^{2m}(\Omega) \subset U^* \subset L^2(\Omega).$$

We then deduce from Theorem 1.1 that there exists adjoint boundary operators γ_j^* into $(H^{2m-j-1/2}(\Gamma))' = H^{j+1/2-2m}(\Gamma)$ such that the following Green's formula holds:

(4.8) $$\int_\Omega u \cdot \wedge v dx - \int_\Omega \wedge^* u \cdot v dx = \sum_{j=0}^{2m-1} \int_\Gamma \gamma_j^* u \cdot \gamma_j v \cdot d\sigma$$

where we denote by $\int u \cdot v \cdot d\sigma$ the duality pairing on

$$H^{2m-j-1/2}(\Gamma) \times H^{j+1/2-2m}(\Gamma).$$

4.3. *Examples of Boundary Value Operators.*

We have seen (Definition 3.1 and Theorem 2.1) that a boundary value operator is associated with a projector of the space Φ. We will choose a simple example.

Let k_j $(0 \leq j \leq m-1)$ be m different integers between 0 and $2m-1$ and l_j $(0 \leq j \leq m-1)$ be the integers between 0 and $2m-1$ such that the set $k_0, \ldots, k_{m-1}, l_0, \ldots, l_{m-1}$ is the set $0, \ldots, 2m-1$ of the $2m$ first integers.

We will choose the projector π associated with the following direct sum decomposition $\Phi = \Phi_0 + \Phi_1$ where

(4.9) $$\begin{cases} i) \ \Phi_0 = \prod_{j=0}^{2m-1} H^{2m-k_j-1/2}(\Gamma); \ \Phi'_0 = \prod_{j=0}^{m-1} H^{k_j+1/2-2m}(\Gamma) \\ ii) \ \Phi_1 = \prod_{j=0}^{m-1} H^{2m-l_j-1/2}(\Gamma); \ \Phi'_1 = \prod_{j=0}^{m-1} H^{l_j+1/2-2m}(\Gamma). \end{cases}$$

Then the operators β_0, β_1, β_0^* and β_1^* defined by (2.5) are:

$$(4.10)\qquad \begin{cases} \beta_0=\gamma_{k_0}\times \dots \times \gamma_{k_{m-1}};\ \beta_1^*=\gamma^*_{l_0}\times \dots \times \gamma^*_{l_{m-1}} \\ \beta_1=\gamma_{l_0}\times \dots \times \gamma_{m-1};\ \beta_0^*=-\gamma^*_{k_0}\times \dots \times \gamma^*_{lk_{m-1}} \end{cases}.$$

The following boundary value operators

$$(4.11)\qquad \begin{cases} i)\ \mathcal{B}=\Lambda\times\gamma_{k_0}\times \dots \times \gamma_{k_{m-1}} \\ ii)\ \mathcal{B}^*=\Lambda^*\times\gamma^*_{l_0}\times \dots \times \gamma^*_{l_{m-1}} \end{cases}$$

are associated with the unbounded operator (D, Λ) and its adjoint (D^*, Λ^*) where:

$$(4.12)\qquad \begin{cases} i)\ D=\{u\in H^{2m}(\Omega) \text{ such that } \gamma_{k_0}u= \dots =\gamma_{k_{m-1}}u=0\} \\ ii)\ D^*=\{u\in U^* \text{ such that } \gamma^*_{l_0}u= \dots =\gamma^*_{l_{m-1}}u=0\}. \end{cases}$$

4.4. *Examples of equivalent Boundary Value Operators.*

Let θ_i^j be operators mapping $H^{2m-k_j-1/2}(\Gamma)$ into $H^{2m-k_i-1/2}(\Gamma)$ such that the matrix of operators $\theta=(\theta_i^j)_{0\leq i,\ j\leq m-1}$ is an automorphism of Φ. Then the operator

$$(4.13)\qquad Q=\Lambda\times\delta_{k_0}\times \dots \times\delta_{k_{m-1}}$$

where

$$(4.14)\qquad \delta_{k_i}u=\sum_{j=0}^{m-1}\theta_i^j(\gamma_{k_j}u)$$

is equivalent to $\mathcal{B}$. In particular, a « normal system » of boundary operators, i.e., differential boundary operators δ_{kj}

$$(4.15)\qquad \delta_{k_j}u=\sum_{|l|=k_j} b^l_{k_j}(x)D^l u \qquad (0\leq j\leq m-1)$$

satisfying:

$$(4.16)\qquad \sum_{|l|=m_j} b^l_{k_j}(x)\xi^l\neq 0 \text{ for all } x\in\Gamma, \text{ for all vector } \xi \text{ normal to } \Gamma \text{ at } x$$

is equivalent to the system γ_k , ..., γ_k . (See J. L. Lions - E. Magenes [2], Chapter 2, § 2).

4.5. *Ranges of $\mathcal{B}$ and $\mathcal{B}^*$.*

Let $\mathcal{B}$ and $\mathcal{B}^*$ be two boundary value operators (equivalent to the ones) defined by (4.11). Let us apply Theorem 2.2.

Let *us* assume that the range of $\mathcal{B}$ is closed. The kernels N and N^* of $\mathcal{B}$ and $\mathcal{B}^*$ are:

(4.17) $N=\{u \in H^m(\Omega)$ such that $\wedge u=0$ and $\gamma_{k_0}u= \ldots =\gamma_{k_{m-1}}u=0\}$

(4-18) $N^*=\{u \in H^m(\Omega)$ such that $\wedge^* u=0$ and $\gamma^*_{l_0}u= \ldots =\gamma^*_{l_{m-1}}u=0\}$.

We deduce that:

$$(4.19)\quad \begin{cases} \mathcal{B}U=\{f \times \varphi_{k_0} \times \ldots \times \varphi_{k_{m-1}} \in L^2(\Omega) \times \prod_{j=0}^{m-1} H^{2m-k\ -1/2}(\Gamma) \\ \qquad\qquad \text{such that:} \\ \int_\Omega f(x)v(x)dx + \sum_{i=0}^{m-1} \int_\Gamma \varphi_{k_j}(x)\gamma^*_{k_j}v(x)d\sigma(x)=0 \text{ for all } v \in N^* \end{cases}$$

and that:

$$(4.20)\quad \begin{cases} \mathcal{B}^*U^*=\{f \times \varphi_{l_0} \times \ldots \times \varphi_{l_{m-1}} \in L^2(\Omega) \times \prod_{j=0}^{m-1} H^{1/2+l_j\ -2m}(\Gamma) \\ \text{such that} \\ \int_\Gamma f(x)v(x)dx + \sum_{j=0}^{m-1} \int_\Omega \varphi_{l_j}(x)\gamma_{l_j}v(x)d\sigma(x)=0 \text{ for all } v \in N. \end{cases}$$

Under what conditions may we apply Corollary 3.3. We already know that the injection from $H^{2m}(\Omega)$ into $L^2(\Omega)$ is compact. We need

4.6. *Examples of Closed Unbounded Operators: Elliptic Operators.*

In order to apply Corollary 3.3, we have assume that the inequality (3.22) holds, i.e., that $(D, \wedge)$ is closed by Proposition 2.2. Such an a priori estimate holds when $\mathcal{B}$ is an « *elliptic* » boundary value operator. (See for instance J. L. Lions - E. Magenes [2], Chapter 2, § 5).

Let *us* associate Λ defined by (4.4) the polynomial

$$(4.21) \qquad \wedge_0(x, \xi) = \sum_{|p+q|=2m} (1-)^{|q|} \xi^q (a_{pq}(x)\xi^p); \xi = (\xi_1, \ldots, \xi_n)$$

and let us consider boundary operators δ_{k_j} defined by (4.15).

We will say that $= \wedge \times \delta_{k_0} \times \ldots \times \delta_{k_{m-1}}$ is « *elliptic* » iff the following conditions hold:

(4.22)

- *i*) $\wedge_0(x, \xi) \neq 0$ for all $\xi \in R^n$; $\xi \neq 0$
- *ii*) for all pairs (ξ', ξ), the polynomial $\wedge_0(\xi + \tau\xi')$ has m roots τ_j^+ such that $Im\tau_j^+ > 0$
- *iii*) for all $x \in \Gamma$ and for all tangent vector ξ to Γ at x, the polynomial $\sum b_k^l(x)(\xi + \tau\xi')^l \; (0 \leq j \leq m-1)$ are linearly independeent modulo the polynomial
$$\prod_{j=1}^{m} (\tau - \tau_j^+)$$
where τ_j^+ are the roots defined in Condition *ii*).

If the coefficients $a_{pq}(x)$, $b_{k_j}^l(x)$ and the manifold Γ are indefinitely differentiable, a famous Theorem (Agmon-Douglis-Nirenberg) imply that $\mathfrak{F}$ is closed:

$$(4.23) \qquad \| u \|_{2m} \leq c[\| \wedge u \|_0^2 + \sum \| \gamma_{k_j} u \|^2_{2m-k_j-1/2} + \| u \|^2_{2m-1}]^{1/2}$$

where we have set $\| u \|_k = \| u \|_{H^k(\Omega)}$ and $\| \varphi \|_k = \| \varphi \|_{H^k(\Gamma)}$.

NOTE 4.1. Let *us* notice that, by Theorem 2.2, we do not need the « dual a priori estimates » in order to obtain the conclusion of Corollary 3.3.

4.7. *Other Choice of the Spaces E and K.*

We started with the simplest choice $E = K = L^2(\Omega)$. But « smaller » will be the spaces E and K, « larger » will be the spaces K' and E', and thus, « larger » will be the space U^*. This will permit *us* to solve boundary value problem P^* for « irregular data ». (See J. L. Lions - E. Magenes, Chapter 2, § 6). Let us replace $E=L^2(\Omega)$ by

(4.24) $$E=H^r(\Omega);\ r\geq 0$$

and $U=H^{2m}(\Omega)$ by

(4.25) $$U=H^{2m+r}(\Omega).$$

We will choose

(4.26) $$U_0=H^{2m+r}(\Omega)\cap H_J^{2m}(\Omega)=\ker\beta;\ \beta=\gamma_0\times\ \dots\ \times\gamma_{2m-1}$$

where β maps u onto Φ defined by

(4.27) $$\Phi=\prod_{j=0}^{2m-1}H^{2m+r-j-1/2}(\Gamma).$$

The problem is now to find a normal space of distributions K (i.e., in which $D(\Omega)$ si dense) containing U. Such spaces exist: Let $p(x)$ be a positive function on Ω, equal to 0 on Γ, such that

(4.28) $$\lim_{x\leftarrow x_0}(p(x)/d(x,\ \Gamma)\neq 0 \text{ for all } x_0\in\Gamma$$

where $d(x,\ \Gamma)$ is the distance from x to the boundary Γ, assumed sufficiently smooth.

Let us set

(4.29) $$\equiv^k(\Omega)=\{u\in L^2(\Omega) \text{ such that } \rho^{|j|}D^ju\in L^2(\Omega) \text{ for } |j|\leq k\}$$

equipped with the norm

$$\Big(\sum_{|j|\leq k}||\,p^{|j|}D^ju\,||_0^2\Big)^{1/2},$$

Thus (See J. L. Lions - E. Magenes, Chapter 2, § 6) $\equiv^k\Omega$ is a Hilbert space in which $D(\Omega)$ is dense, satisfying:

(4.30) $$H^k(\Omega)\subset\equiv^k(\Omega)\subset H^{k-1}(\Omega);\ \ \equiv^{-k}(\Omega)=\equiv^k(\Omega)'.$$

If we choose $K=\Xi^{2m+r}(\Omega)$, our main assumption (1,2) will be satisfied.

The domain U^*:

(4.31) $$U^* \doteq \{u \in H^r(\Omega)' \text{ such that } \Lambda^* u \in \Xi^{-(2m+r)}(\Omega)\}$$

is a Hilbert space. By Theorem 1.1, we may construct adjoint boundary operators γ_j^* mapping U^* into $H^{-2m-r+j+1/2}(\Gamma)$ such that Green's formula holds again:

(4.32) $$\int_\Omega u \cdot \Lambda v dx - \int_\Omega \Lambda^* u \cdot v dx = \Sigma \int_\Gamma \gamma_j^* u \cdot \gamma_j v dx$$

where u belongs to U^* and v to $U=H^{2m+r}(\Omega)$. The adjoint operator maps U^* into

$$\Xi^{-(2m+r)}(\Omega) \times \prod_{j=0}^{m-1} H^{-2m-r+lj+1/2}(\Gamma)$$

NOTE 4.2. The adjoint boundary operators γ_j^* are extensions to U^* of the operators γ_j^* introduced in Section 4.2. (See Note 1.8).

NOTE 4.3. There exists other choices of spaces E; for instance, since $H^r(\Omega)$ is not a normal space of distributions, it may be convenient to choose $E=H_0^r(\Omega)$ and $U=\{u \in H^{2m+r}(\Omega)$ such that $\Lambda u \in H_0^r(\Omega)\}$. We then have to verify that $\beta=\gamma_0 \times \ldots \times \gamma_{2m-1}$ maps this new space U *onto* the space Φ defined by (4.27).

Theorem 3.2 implies: If the range $\mathfrak{F}(H^{2m+r}(\Omega))$ is close into

$$H^r(\Omega) \times \prod_{j=0}^{m-1} H^{2m+r-k_j-1/2}(\Gamma)$$

then the adjoint operator $\mathfrak{F}^*$ defines an isomorphism

(4.23) $$\text{form } U^*/N^* \text{ onto } \{\Xi^{-2m-r}(\Omega) \times \Pi \ {}^{-2m-r+l_j+1/2}(\Gamma), \beta_1, N\}.$$

We thus get a theorem of existence of a boundary value problem when the « data » belongs to « large » spaces of distributions on Ω and Γ.

NOTE 4.4. Let $\mathfrak{S}$ be « elliptic ». The estimate (4.23) holds when we replace $E=L^2(\Omega)$ by $E=H^r(\Omega)$ for any integer $r\geq 0$. This implies that the kernels N and N^* are contained in $D(\Omega)$ (and are finite dimensional spaces). Therefore, the ranges of $\mathfrak{S}$ are closed for any r (we apply Corollary 3.3 with $K=\Xi^{2m+r-1}(\Omega)$ for instance) and the above conclusion holds. (See Note 3.1).

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