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Some integrals involving products of Bessel and Legendre functions - II

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SOME INTEGRALS INVOLVING PRODUCTS OF BESSEL AND LEGENDRE FUNCTIONS - II

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Sunto: In the present paper the integral

$$\int_0^c x^{\varrho-1}(c^2 - x^2)^{-(1/2)\sigma} R\left(\alpha, \beta, \gamma, \frac{x^2}{a^2}\right) R\left(\lambda, \mu, \eta, \frac{x^2}{b^2}\right) P_\nu^\sigma\left(\frac{x}{c}\right) dx,$$

where c is real, *non*-zero and finite, $\mathrm{Re}\,(\varrho) > 0$, $\mathrm{Re}\,(\sigma) < 1$, is evaluated in terms of a double hypergeometric series and its several interesting special cases are discussed. A number of known results are also exhibited as necessary consequences of this integral.

1. In an earlier paper [5] we have proved that, for real, *non*-zero and finite values of c, the integral [loc. cit. p. 419]

$$(1.1) \qquad \int_0^c x^{\varrho-1}(c^2 - x^2)^{-(1/2)\sigma} J_\lambda\left(\frac{2x}{a}\right) J_\mu\left(\frac{2x}{b}\right) P_\nu^\sigma\left(\frac{x}{c}\right) dx$$

$$= \frac{2^{\sigma-1}c^{\delta-\sigma}\Gamma\left(\frac{1}{2}\,\delta\right)\Gamma\left(\frac{1}{2}\,\delta+\frac{1}{2}\right)}{a^\lambda b^\mu \Gamma(\lambda+1)\Gamma(\mu+1)\Gamma\left\{\frac{1}{2}\,(\delta-\nu-\sigma+1)\right\}\Gamma\left\{\frac{1}{2}\,(\delta+\nu-\sigma+2)\right\}} \cdot$$

$$\cdot F\left[\begin{matrix} \frac{1}{2}\delta,\ \frac{1}{2}\,\delta+\frac{1}{2} \ : \ - \ ; \ - \ ; \\ \frac{1}{2}\,(\delta-\nu-\sigma+1),\ \frac{1}{2}(\delta+\nu-\sigma+2) : \lambda+1;\ \mu+1; \end{matrix}\ -\frac{c^2}{a^2},\ -\frac{c^2}{b^2}\right],$$

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where $\delta = \lambda + \mu + \varrho$, Re $(\delta) > 0$, Re $(\sigma) < 1$. Here as well as in what follows the notation for double hypergeometric functions is due to Burchnall and Chaundy [3, p, 112] in preference, for the sake of brevity, to that of Kampé de Fériet [2, p. 150] introduced earlier.

Put

$$R(\lambda, \mu, \nu, z) = \sum_{m=0}^{\infty} \frac{(-)^m(\lambda + m + 1)_m}{m\,!\,\Gamma(\mu + m + 1)\Gamma(\nu + m + 1)} z^m, \tag{1.2}$$

then the known formula [7, p. 151]

$$\left(\frac{1}{2} z\right)^{\mu+\nu} = \frac{\Gamma(\mu + 1)\Gamma(\nu + 1)}{\Gamma(\mu + \nu + 1)} \sum_{n=0}^{\infty} \frac{(\mu + \nu + 2n)\Gamma(\mu + \nu + n)}{n\,!} J_{\mu+n}(z)\, J_{\nu+n}(z)$$

admits of the generalization

$$\sum_{n=0}^{\infty} \frac{(\lambda + 2n)\Gamma(\lambda + n)}{n\,!} z^n R(\lambda + 2n, \mu + n, \nu + n, z) = \frac{\Gamma(\lambda + 1)}{\Gamma(\mu + 1)\Gamma(\nu + 1)}, \tag{1.3}$$

since we observe that [7, p. 147]

$$J_\mu(2z)J_\nu(2z) = z^{\mu+\nu}R(\mu + \nu, \mu, \nu, z^2)\,. \tag{1.4}$$

For a detailed discussion of the various properties of the entire function $R(\lambda, \mu, \nu, z)$ see [1] and [6].

Now restrict c in the manner stated earlier and make use of the formula [4, p. 314]

$$\int_0^1 x^{\lambda-1}(1 - x^2)^{-(1/2)\mu}P_\nu^\mu(x)\,\mathrm{d}x = \frac{\pi^{1/2}2^{\mu-\lambda}\Gamma(\lambda)}{\Gamma\left\{\frac{1}{2}(\lambda - \mu - \nu + 1)\right\}\Gamma\left\{\frac{1}{2}(\lambda - \mu + \nu + 2)\right\}},$$

$$\mathrm{Re}\,(\lambda) > 0, \quad \mathrm{Re}\,(\mu) < 1;$$

so that

$$\int_0^c x^{\varrho-1}(c^2 - x^2)^{-(1/2)\sigma}R\left(\alpha, \beta, \gamma, \frac{x^2}{a^2}\right) R\left(\lambda, \mu, \eta, \frac{x^2}{b^2}\right) P_\nu^\sigma\left(\frac{x}{c}\right) \mathrm{d}x$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-)^{r+s}\left(\frac{1}{a}\right)^{2r}\left(\frac{1}{b}\right)^{2s}(\alpha+r+1)_r(\lambda+s+1)_s}{r\,!\,s\,!\,\Gamma(\beta+r+1)\Gamma(\gamma+r+1)\Gamma(\mu+s+1)\Gamma(\eta+s+1)} \cdot$$

$$\cdot \int_0^c x^{\varrho+2r+2s-1}(c^2-x^2)^{-(1/2)\sigma} P_\nu^\sigma\left(\frac{x}{c}\right) \mathrm{d}x,$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-)^{r+s}2^{\sigma-1}c^{\varrho-\sigma}(\alpha+r+1)_r(\lambda+s+1)_s}{r\,!\,s\,!\,\Gamma(\beta+r+1)\Gamma(\gamma+r+1)\Gamma(\mu+s+1)\Gamma(\eta+s+1)} \cdot$$

$$\cdot \frac{\Gamma\left\{\frac{1}{2}(\varrho+2r+2s)\right\}\Gamma\left\{\frac{1}{2}(\varrho+2r+2s+1)\right\}\left(\frac{c}{a}\right)^{2r}\left(\frac{c}{b}\right)^{2s}}{\Gamma\left\{\frac{1}{2}(\varrho-\sigma-\nu+2r+2s+1)\right\}\Gamma\left\{\frac{1}{2}(\varrho-\sigma+\nu+2r+2s+2)\right\}},$$

and therefore

$$(1.5)\qquad \int_0^c x^{\varrho-1}(c^2-x^2)^{-(1/2)\sigma} R\left(\alpha,\beta,\gamma,\frac{x^2}{4a^2}\right) R\left(\lambda,\mu,\eta,\frac{x^2}{4b^2}\right) P_\nu^\sigma\left(\frac{x}{c}\right)\mathrm{d}x$$

$$= \frac{2^{\sigma-1}c^{\varrho-\sigma}\Gamma\left(\frac{1}{2}\varrho\right)\Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}\right)}{\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\mu+1)\Gamma(\eta+1)\Gamma\left\{\frac{1}{2}(\varrho-\sigma-\nu+1)\right\}\Gamma\left\{\frac{1}{2}(\varrho-\sigma+\nu+2)\right\}}\cdot$$

$$\cdot F\left[\begin{matrix}\frac{1}{2}\varrho,\ \frac{1}{2}\varrho+\frac{1}{2} : \frac{1}{2}\alpha+\frac{1}{2},\ \frac{1}{2}\alpha+1;\ \frac{1}{2}\lambda+\frac{1}{2},\ \frac{1}{2}\lambda+1; \\ \frac{1}{2}(\varrho-\sigma-\nu+1), \frac{1}{2}(\varrho-\sigma+\nu+2) : \alpha+1, \beta+1, \gamma+1; \lambda+1, \mu+1, \eta+1;\end{matrix}\ -\frac{c^2}{a^2}, -\frac{c^2}{b^2}\right],$$

provided that Re $(\varrho) > 0$ and Re $(\sigma) < 1$.

For $\sigma = 0$, (1.5) gives us the formula

$$(1.6)\qquad \int_0^c x^{\varrho-1} R\left(\alpha,\beta,\gamma,\frac{x^2}{4a^2}\right) R\left(\lambda,\mu,\eta,\frac{x^2}{4b^2}\right) P_\nu\left(\frac{x}{c}\right)\mathrm{d}x =$$

$$= \frac{c^\varrho\,\Gamma\left(\frac{1}{2}\varrho\right)\Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}\right)}{2\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\mu+1)\Gamma(\eta+1)\Gamma\left\{\frac{1}{2}(\varrho-\nu+1)\right\}\Gamma\left\{\frac{1}{2}(\varrho+\nu+2)\right\}}\cdot$$

$$\cdot F\left[\begin{matrix}\frac{1}{2}\varrho,\ \frac{1}{2}\varrho+\frac{1}{2}:\frac{1}{2}\alpha+\frac{1}{2},\ \frac{1}{2}\alpha+1;\ \frac{1}{2}\lambda+\frac{1}{2},\ \frac{1}{2}\lambda+1;\\ \frac{1}{2}(\varrho-\nu+1),\ \frac{1}{2}(\varrho+\nu+2):\alpha+1,\beta+1,\gamma+1;\lambda+1,\mu+1,\eta+1;\end{matrix}\ -\frac{c^2}{a^2},\ -\frac{c^2}{b^2}\right],$$

where, as in the earlier case, Re $(\varrho) > 0$.

2. Set $\alpha = \beta + \gamma$, $\lambda = \mu + \eta$ and change the notation slightly. In view of (1.4), the formula (1.5) will then express an integral involving the product

$$J_\lambda\left(\frac{x}{a}\right) J_\mu\left(\frac{x}{a}\right) J_\xi\left(\frac{x}{b}\right) J_\eta\left(\frac{x}{b}\right) P_\nu^\sigma\left(\frac{x}{c}\right)$$

in terms of a hypergeometric series, and we have

$$\text{(2.1)}\quad \int_0^c x^{\varrho-1}(c^2-x^2)^{-(1/2)\sigma} J_\lambda\left(\frac{x}{a}\right) J_\mu\left(\frac{x}{a}\right) J_\xi\left(\frac{x}{b}\right) J_\eta\left(\frac{x}{b}\right) P_\nu^\sigma\left(\frac{x}{c}\right) \mathrm{d}x$$

$$= \frac{2^{\sigma-\delta+\varrho-1} c^{\delta-\sigma}\,\Gamma\left(\frac{1}{2}\delta\right)\Gamma\left(\frac{1}{2}\delta+\frac{1}{2}\right)}{a^{\lambda+\mu}b^{\xi+\eta}\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma(\xi+1)\Gamma(\eta+1)\Gamma\left\{\frac{1}{2}(\delta-\sigma-\nu+1)\right\}\Gamma\left\{\frac{1}{2}(\delta-\sigma+\nu+2)\right\}}\,\cdot$$

$$\cdot F\left[\begin{matrix}\frac{1}{2}\delta,\ \frac{1}{2}\delta+\frac{1}{2} & :\frac{1}{2}(\lambda+\mu+1),\ \frac{1}{2}(\lambda+\mu+2); & \frac{1}{2}(\xi+\eta+1),\ \frac{1}{2}(\xi+\eta+2);\\ \frac{1}{2}(\delta-\sigma-\nu+1),\ \frac{1}{2}(\delta-\sigma+\nu+2) & :\lambda+1,\ \mu+1,\ \lambda+\mu+1; & \xi+1,\ \eta+1,\ \xi+\eta+1\ ;\end{matrix}\ -\frac{c^2}{a^2},\ -\frac{c^2}{b^2}\right],$$

where $\delta = \lambda + \mu + \xi + \eta + \varrho$, Re $(\delta) > 0$ and Re $(\sigma) < 1$.

Similarly, (1.6) is reduced to the form

$$(2.2)\quad \int_0^c x^{\varrho-1} J_\lambda\left(\frac{x}{a}\right) J_\mu\left(\frac{x}{a}\right) J_\xi\left(\frac{x}{b}\right) J_\eta\left(\frac{x}{b}\right) P_\nu^\sigma\left(\frac{x}{c}\right) \mathrm{d}x =$$

$$= \frac{2^{\varrho-\delta-1} c^\delta \Gamma\left(\frac{1}{2}\delta\right) \Gamma\left(\frac{1}{2}\delta + \frac{1}{2}\right)}{a^{\lambda+\mu} b^{\xi+\eta} \Gamma(\lambda+1)\Gamma(\mu+1)\Gamma(\xi+1)\Gamma(\eta+1)\Gamma\left\{\frac{1}{2}(\delta-\nu+1)\right\}\Gamma\left\{\frac{1}{2}(\delta+\nu+2)\right\}} \cdot$$

$$\cdot F\left[\begin{matrix} \frac{1}{2}\delta, \frac{1}{2}\delta + \frac{1}{2} & : \frac{1}{2}(\lambda+\mu+1), \frac{1}{2}(\lambda+\mu+2); \\ \frac{1}{2}(\delta-\nu+1), \frac{1}{2}(\delta+\nu+2) : & \lambda+1, \mu+1, \lambda+\mu+1 \quad ; \end{matrix}\right.$$

$$\left.\begin{matrix} \frac{1}{2}(\xi+\eta+1), \frac{1}{2}(\xi+\eta+2); \\ \xi+1, \eta+1, \xi+\eta+1 \quad ; \end{matrix} \quad -\frac{c^2}{a^2}, -\frac{c^2}{b^2}\right],$$

valid if Re $(\delta) > 0$.

3. When $a = b$, the double series in (1.5) equals

$$\sum_{k=0}^{\infty}\left\{\frac{\left(\frac{1}{2}\varrho\right)_k \left(\frac{1}{2}\varrho+\frac{1}{2}\right)_k \left(\frac{1}{2}\lambda+\frac{1}{2}\right)_k \left(\frac{1}{2}\lambda+1\right)_k \left(-\frac{c^2}{a^2}\right)^k}{k!\left(\frac{1}{2}\varrho-\frac{1}{2}\sigma-\frac{1}{2}\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}\varrho-\frac{1}{2}\sigma+\frac{1}{2}\nu+1\right)_k (\lambda+1)_k(\mu+1)_k(\eta+1)_k}\cdot\right.$$

$$\left.\cdot \sum_{r=0}^{k} \frac{(-k)_r \left(\frac{1}{2}\alpha+\frac{1}{2}\right)_r \left(\frac{1}{2}\alpha+1\right)_r (-\lambda-k)_r(-\mu-k)_r(-\eta-k)_r}{r!\,(\alpha+1)_r(\beta+1)_r(\gamma+1)_r \left(\frac{1}{2}-\frac{1}{2}\lambda-k\right)_r \left(-\frac{1}{2}\lambda-k\right)_r}\right\},$$

and this simplifies as a ${}_4F_5$ if we further set

$$\beta = \gamma + \frac{1}{2} = \frac{1}{2}\alpha \quad \text{and} \quad \mu = \eta + \frac{1}{2} = \frac{1}{2}\lambda\,.$$

Therefore, a special case of (1.5) is

$$(3.1)\quad \int_0^c x^{\varrho-1}(c^2-x^2)^{-(1/2)\sigma} R\left(\lambda, \frac{1}{2}\lambda, \frac{1}{2}\lambda-\frac{1}{2}, \frac{x^2}{16a^2}\right)\cdot$$

$$\cdot R\left(\mu, \frac{1}{2}\mu, \frac{1}{2}\mu-\frac{1}{2}, \frac{x^2}{16a^2}\right) P_\nu^\sigma\left(\frac{x}{c}\right) \mathrm{d}x =$$

$$= \frac{2^{\lambda+\mu+\sigma-1} c^{\varrho-\sigma} \Gamma\left(\frac{1}{2}\varrho\right)\Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}\right)}{\pi\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma\left\{\frac{1}{2}(\varrho-\sigma-\nu+1)\right\}\Gamma\left\{\frac{1}{2}(\varrho-\sigma+\nu+2)\right\}}\cdot$$

$$\cdot {}_4F_5\left[\begin{matrix}\frac{1}{2}\varrho, \frac{1}{2}\varrho+\frac{1}{2}, \frac{1}{2}(\lambda+\mu+1), \frac{1}{2}(\lambda+\mu+2) & ; \\ \frac{1}{2}(\varrho-\sigma-\nu+1), \frac{1}{2}(\varrho-\sigma+\nu+2), \lambda+1, \mu+1, \lambda+\mu+1; & \end{matrix} -\frac{c^2}{a^2}\right],$$

where $\mathrm{Re}\,(\varrho) > 0$ and $\mathrm{Re}\,(\sigma) < 1$.

The last formula when re-written by virtue of the relation [1, p. 911]

$$(3.2)\qquad R\left(2\nu, \nu, \nu-\frac{1}{2}, z^2\right) = \frac{z^{-2\nu}}{\pi^{1/2}} J_{2\nu}(4z),$$

gives us the known result [5, p. 420 (2.1)]

$$(3.3)\quad \int_0^c x^{\varrho-1}(c^2-x^2)^{-(1/2)\sigma} J_\lambda\left(\frac{x}{a}\right) J_\mu\left(\frac{x}{a}\right) P_\nu^\sigma\left(\frac{x}{c}\right) \mathrm{d}x =$$

$$= \frac{2^{\sigma-1} c^{\delta-\sigma} \Gamma\left(\frac{1}{2}\delta\right)\Gamma\left(\frac{1}{2}\delta+\frac{1}{2}\right)}{a^{\lambda+\mu}\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma\left\{\frac{1}{2}(\delta-\sigma-\nu+1)\right\}\Gamma\left\{\frac{1}{2}(\delta-\sigma+\nu+2)\right\}}\cdot$$

$$\cdot {}_4F_5\left[\begin{matrix}\frac{1}{2}\delta, \frac{1}{2}\delta+\frac{1}{2}, \frac{1}{2}(\lambda+\mu+1), \frac{1}{2}(\lambda+\mu+2) & ; \\ \frac{1}{2}(\delta-\sigma-\nu+1), \frac{1}{2}(\delta-\sigma+\nu+2), \lambda+1, \mu+1, \lambda+\mu+1 ; & \end{matrix} -\frac{c^2}{a^2}\right],$$

which generalizes Bailey's integral [4, p. 338] and is valid when $\mathrm{Re}\,(\delta) > 0$, $\delta = \lambda + \mu + \varrho$, $\mathrm{Re}\,(\sigma) < 1$.

In a similar way, (1.6) reduces to the formula [5, p. 421 (2.2)]

$$(3.4)\quad \int_0^c x^{\varrho-1} J_\lambda\left(\frac{x}{a}\right) J_\mu\left(\frac{x}{a}\right) P_\nu\left(\frac{x}{c}\right) dx$$

$$= \frac{c^\delta \Gamma\left(\frac{1}{2}\delta\right) \Gamma\left(\frac{1}{2}\delta + \frac{1}{2}\right)}{2a^{\lambda+\mu}\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma\left\{\frac{1}{2}(\delta-\nu+1)\right\}\Gamma\left\{\frac{1}{2}(\delta+\nu+2)\right\}} \cdot$$

$$\cdot\, {}_4F_5\left[\begin{matrix} \frac{1}{2}\delta, \frac{1}{2}\delta+\frac{1}{2}, \frac{1}{2}(\lambda+\mu+1), \frac{1}{2}(\lambda+\mu+2) \;; \\ \frac{1}{2}(\delta-\nu+1), \frac{1}{2}(\delta+\nu+2), \lambda+1, \mu+1, \lambda+\mu+1; \end{matrix} -\frac{c^2}{a^2}\right],$$

$$\operatorname{Re}(\delta) > 0;$$

whose particular case in which $a = 1$, $\varrho = \lambda$ and ν is an integer corresponds to Bose's result [4, p. 337 (31)].

Since [7, p. 150]

$$(3.5)\qquad J_\mu(z) J_\nu(z) = \frac{2}{\pi}\int_0^{(1/2)\pi} J_{\mu+\nu}(2z\cos\theta)\cos(\mu-\nu)\theta\, d\theta,$$

from the formulae (3.3) and (3.4) we also have

$$(3.6)\quad {}_4F_5\left[\begin{matrix} \frac{1}{2}\delta, \frac{1}{2}\delta+\frac{1}{2}, \frac{1}{2}(\lambda+\mu+1), \frac{1}{2}(\lambda+\mu+2) \;; \\ \frac{1}{2}(\delta-\sigma-\nu+1), \frac{1}{2}(\delta-\sigma+\nu+2), \lambda+1, \mu+1, \lambda+\mu+1; \end{matrix} -\frac{c^2}{a^2}\right]$$

$$= \frac{\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma\left\{\frac{1}{2}(\delta-\sigma-\nu+1)\right\}\Gamma\left\{\frac{1}{2}(\delta-\sigma+\nu+2)\right\}}{2^{\sigma-1} c^{\delta-\sigma}\Gamma\left(\frac{1}{2}\delta\right)\Gamma\left(\frac{1}{2}\delta+\frac{1}{2}\right)} \cdot$$

$$\cdot\, \frac{a^{\lambda+\mu}}{\pi}\int_0^c\int_0^{(1/2)\pi} t^{\varrho-1}(c^2-t^2)^{-(1/2)\sigma} J_{\lambda+\mu}\left(\frac{2t}{a}\cos\theta\right) P_\nu^\sigma\left(\frac{t}{c}\right)\cos(\lambda-\mu)\theta\, dt\, d\theta,$$

and

$$(3.7)\quad {}_4F_5\left[\begin{matrix}\frac{1}{2}\delta,\ \frac{1}{2}\delta+\frac{1}{2},\ \frac{1}{2}(\lambda+\mu+1),\ \frac{1}{2}(\lambda+\mu+2)\ ; \\ \frac{1}{2}(\delta-\nu+1),\ \frac{1}{2}(\delta+\nu+2),\ \lambda+1,\ \mu+1,\ \lambda+\mu+1\ ;\end{matrix}\ -\frac{c^2}{a^2}\right]$$

$$=\frac{\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma\left\{\frac{1}{2}(\delta-\nu+1)\right\}\Gamma\left\{\frac{1}{2}(\delta+\nu+2)\right\}}{c^{\delta}\,\Gamma\left(\frac{1}{2}\delta\right)\Gamma\left(\frac{1}{2}\delta+\frac{1}{2}\right)}\cdot$$

$$\cdot\frac{4a^{\lambda+\mu}}{\pi}\int_0^c\int_0^{(1/2)\pi} t^{\varrho-1}J_{\lambda+\mu}\left(\frac{2t}{a}\cos\theta\right)P_{\nu}\left(\frac{t}{c}\right)\cos(\lambda-\mu)\theta\,dt\,d\theta,$$

respectively, provided $\mathrm{Re}\,(\lambda+\mu)>-1$ in addition to the conditions already stated.

4. Following the method illustrated in § 1 we also have

$$(4.1)\quad \int_0^a x^{\varrho-1}(a^2-x^2)^{-(1/2)\mu}P_{\nu}^{\mu}\left(\frac{x}{a}\right)R\left(\alpha,\beta,\gamma,\frac{x^2}{4b^2}\right)dx$$

$$=\frac{2^{\mu-1}a^{\varrho-\mu}\,\Gamma\left(\frac{1}{2}\varrho\right)\Gamma\left(\frac{1}{2}\varrho+\frac{1}{2}\right)}{\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma\left\{\frac{1}{2}(\varrho-\mu-\nu+1)\right\}\Gamma\left\{\frac{1}{2}(\varrho-\mu+\nu+2)\right\}}\cdot$$

$$\cdot\,{}_4F_5\left[\begin{matrix}\frac{1}{2}\varrho,\ \frac{1}{2}\varrho+\frac{1}{2},\ \frac{1}{2}\alpha+\frac{1}{2},\ \frac{1}{2}\alpha+1\ ; \\ \frac{1}{2}(\varrho-\mu-\nu+1),\ \frac{1}{2}(\varrho-\mu+\nu+2),\ \alpha+1,\ \beta+1,\ \gamma+1\ ;\end{matrix}\ -\frac{a^2}{b^2}\right],$$

provided that $\mathrm{Re}\,(\varrho)>0$ and $\mathrm{Re}\,(\mu)<1$, and the corresponding formula for $\mu=0$ is

$$(4.2)\qquad \int_0^a x^{\varrho-1} P_\nu\left(\frac{x}{a}\right) R\left(\alpha, \beta, \gamma, \frac{x^2}{4b^2}\right) \mathrm{d}x =$$

$$= \frac{a^\varrho \Gamma\left(\frac{1}{2}\varrho\right) \Gamma\left(\frac{1}{2}\varrho + \frac{1}{2}\right)}{2\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma\left\{\frac{1}{2}(\varrho-\nu+1)\right\}\Gamma\left\{\frac{1}{2}(\varrho+\nu+2)\right\}} \cdot$$

$$\cdot\ {}_4F_5\left[\begin{matrix}\frac{1}{2}\varrho, \frac{1}{2}\varrho+\frac{1}{2}, \frac{1}{2}\alpha+\frac{1}{2}, \frac{1}{2}\alpha+1 & ; \\ \frac{1}{2}(\varrho-\nu+1), \frac{1}{2}(\varrho+\nu+2), \alpha+1, \beta+1, \gamma+1; & \end{matrix} -\frac{a^2}{b^2}\right],$$

valid when Re $(\varrho) > 0$.

It is not difficult to exhibit (4.1) and (4.2) as the limiting cases of the formulae (1.5) and (1.6) respectively, when $b \to \infty$.

In the special case $\alpha = \beta + \gamma$ of these integrals if we make use of the relation (1.4) we shall again have the formulae (3.3) and (3.4).

On the other hand, if in (4.1) we set $\beta = \gamma + \frac{1}{2} = \frac{1}{2}\alpha$, and employ the relation (3.2), we get

$$(4.3)\qquad \int_0^a x^{\varrho-1}(a^2 - x^2)^{-(1/2)\mu} J_\lambda\left(\frac{2x}{\xi}\right) P_\nu^\mu\left(\frac{x}{a}\right) \mathrm{d}x$$

$$= \pi^{1/2} \frac{\left(\frac{1}{2}a\right)^{\varrho+\lambda-\mu}}{\xi^\lambda} R\left(\varrho+\lambda-1, \frac{1}{2}-\mu, \lambda, \frac{a^2}{4\xi^2}\right),$$

where Re $(\varrho + \lambda) > 0$, Re $(\mu) < 1$ and the parameters are constrained by means of

$$\varrho + \lambda + \mu - \nu = 2\,.$$

Choose $\varrho = \frac{3}{2} - \mu$ so that $\lambda = \nu + \frac{1}{2}$ and make use of the relation (1.4). The special case $\xi = 2$ of (4.3) then leads to the known formula [4, p. 337]

$$(4.4)\qquad \int_0^a x^{(1/2)-\mu}(a^2-x^2)^{-(1/2)\mu} P_\nu^\mu\left(\frac{x}{a}\right) J_{\nu+(1/2)}(x)\,\mathrm{d}x =$$

$$= \left(\frac{1}{2}\pi\right)^{1/2} a^{1-\mu} J_{(1/2)-\mu}\left(\frac{1}{2}a\right) J_{\nu+(1/2)}\left(\frac{1}{2}a\right),$$

which holds whenever $\mathrm{Re}(\mu-\nu)<2$ and $\mathrm{Re}(\mu)<1$.

By virtue of (3.5) the last formula gives us the interesting result

$$(4.5)\qquad \int_0^{(1/2)\pi} J_{\nu-\mu+1}(a\cos\theta)\cos(\mu+\nu)\theta\,\mathrm{d}\theta =$$

$$= a^{\mu-1}\left(\frac{1}{2}\pi\right)^{1/2}\int_0^a x^{(1/2)-\mu}(a^2-x^2)^{-(1/2)\mu} P_\nu^\mu\left(\frac{x}{a}\right) J_{\nu+(1/2)}(x)\,\mathrm{d}x,$$

which holds under the constraints stated earlier.

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