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PRINCIPAL PARTS AND CANONICAL FACTORISATION OF HYPOELLIPTIC POLYNOMIALS IN TWO VARIABLES

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Introduction

The basis for this paper is Gorin's definition [5] of $\binom{k}{j}$-hypoellipticity. A polynomial $P(\xi)$, $\xi = (\xi_1, \ldots, \xi_n)$, is called $\binom{k}{j}$ - *hypoelliptic of tipe* $a_{jk} > 0$, if

$$(0.1) \qquad P(\xi) = 0, \quad \operatorname{Im} \xi_j = 0 \text{ for } j \neq k \Rightarrow |\xi_j| \leqslant C(1 + |\operatorname{Im} \xi_k|)^{a_{jk}} .$$

It is always assumed that $a_{ii} \geqslant 1$. Then the inequality

$$(0.2) \qquad |(\partial/\partial\xi_k)^i P(\xi)| \leqslant C(1 + |P(\xi)|)(1 + |\xi_j|)^{-i/a_{jk}} , \qquad (\xi \text{ real}, \ i = 1, 2, \ldots)$$

is a sufficient condition for (0.1) to hold, and also a necessary condition if $a_{jk} \geqslant 1$. If $P(\xi)$ is $\binom{k}{j}$-hypoelliptic for all k and j, then it is hypoelliptic in the ordinary sense (Hörmander [6]); if it is $\binom{k}{j}$-hypoelliptic for all k and for $j = 1, \ldots, n'$ with $n' > n$, then it is partially hypoelliptic in $x' = (x_1, \ldots, x_{n'})$(See Friberg [2]), so that all solutions of $P(D)u = 0$, $D = i^{-1}(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ are sums of derivatives of functions, infinitely differentiable in the x'-variables. Finally, if $P(\xi)$ is $\binom{k}{j}$-hypoelliptic for $k = n' + 1, \ldots, n$ and all j, then it can be shown that all solutions

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of $P(D)u = 0$, with support contained in a cylinder $|x'| \leqslant A$, are infinitely differentiable.

As in the related paper [3], we shall use the following notations. If $P(\xi) = \Sigma c_\alpha \xi^\alpha$, then we set the index set $(P) = \{\alpha; c_\alpha \neq 0\}$, while $(P)^*$ denotes the convex hull of $(P) \cup \{0\}$. The (upper) Newton surface (or polygon) $F(P)$ of P is then the union of the flat $(n-1)$ dimensional pieces of the boundary of $(P)^*$ bounding $(P)^*$ from above, i.e. which are not parts of the coordinate planes. If $F(P)$ consists of a single face segment, $F(P)$ will be called simple. Hypoelliptic polynomials with simple Newton polygon have been studied by Pini [7]. His main contribution was the introduction of a « principal part » containing all terms of $P(\xi)$ that are essential for the hypoellipticity. Cf. our Cor. 3.1. The purpose of the present paper is to extend and improve the results of Pini as far as possible for general $\binom{k}{j}$-hypoelliptic polynomial in two variables.

Our main tool in the two-dimensional case will be the construction of a polynomial P', the « $\binom{2}{1}$-hypoelliptic canonical factorization » of $P = P(\xi_1, \xi_2)$, whose zeros $\xi_2 = \Phi(\xi_1)$ have a finite Puiseux expansion, obtained by a suitable truncation of the Puiseux expansions of the zeros $P(\xi)$. It can be shown that P' is a product of $\binom{2}{1}$-hypoelliptic polynomials with simple Newton polygons, possibly multiplied by a polynomial in ξ_1 only. This factorization allows us to derive precise lower estimates for $|P(\xi)|$, and to define a « minimal » principal part for $P(\xi)$. Is turns out that the terms of $P(\xi)$ not belonging to this principal part are exactly the terms that are strictly weaker than $P(\xi)$, in the sense of Hörmander.

Since $\binom{2}{1}$-and $\binom{1}{2}$-hypoellipticity together imply hypoellipticity of $P(\xi)$, our results can be used to define a canonical factorization into hypoelliptic polynomials with simple Newton polygons, of any given hypoelliptic polynomial in two variables. We can also show that the $\binom{2}{1}$-and $\binom{1}{2}$-hypoelliptic principal parts coincide, hence define uniquely a hypoelliptic principal part. The special class of hypoelliptic polynomials in R^2, for which the principal part contains only terms with indices corresponding to points on the Newton polygon, was discussed in our previous paper [3].

In the more-dimensional case it is no longer possible to define a hypoelliptic canonical factorization. Consequently our results about principal parts for this case are still very incomplete.

1. The Newton algorithm.

Given a polynomial $P(\xi) = \Sigma c_\alpha \xi^\alpha$, $\xi \in R^2$, with index set $(P) = \{\alpha; c_\alpha \neq 0\}$, we can always find integers r, s, $h > 0$, with r and s relatively prime, such that

$$h = \max (r\alpha_1 + s\alpha_2), \alpha = (\alpha_1, \alpha_2) \in (P) . \tag{1.1}$$

Then every $\alpha \in (P)$ belongs to one of the lines $r\alpha_1 + s\alpha_2 = h - j$, $j = 0, 1, ..., h$. It follows that there are polynomials $\varphi_{0k}(u)$, such that

$$\tau^h P(\tau^{-r}, \tau^{-s}u) = f_0(\tau, u) = \sum_{k=0}^{h} \varphi_{0k}(u)\tau^k . \tag{1.2}$$

We want to determine the Puiseux expansions for the zeros $u = u(\tau)$ of $f_0(\tau, u) = 0$ for τ near 0, using the method of the Newton polygon. (See Bliss [1]). Suppose $\varphi_{00}(u)$ has a zero $c_0 \neq 0$ of multiplicity μ_0. Let

$$f_0(\tau, u) = f_0(\tau, c_0 + v) = g_0(\tau, v) ,$$

and construct the Newton polygon bounding the index set (g_0) of g_0 from below. The polygon determines a finite set of couples of positive integers (β_1, γ_1) and (r_1, s_1), r_1, s_1 relatively prime, such that

$$\begin{aligned} &f_0(\tau_1^{r_1}, c_0 + \tau_1^{s_1}u_1) = g_0(\tau_1^{r_1}, \tau_1^{s_1} u) = \tau_1^{\beta_1}u_1^{\gamma_1}f_1(\tau_1, u_1) , \\ &f_1(\tau_1, u_1) = \sum_{k=0}^{h_1} \varphi_{1k}(u_1)\tau_1^k , \qquad (h_1 > 0). \end{aligned} \tag{1.3}$$

We can now repeat the process, determining a zero c_1 of $\varphi_{10}(u_1)$ of multiplicity μ_1, and so on. After j iterations we have

$$u = c_0 + \tau_1^{s_1}(c_1 + \tau_2^{s_2}(c_2 + ... (c_{j-1} + \tau_j^{s_j}u_j) ...)) , \qquad (\tau = \tau_i^{r_1 \cdots r_i}) . \tag{1.4}$$

If $\gamma_j \neq 0$, we may choose $u_j = 0$, which gives a zero with a finite Puiseux expansion. Also, it may or may not happen that the process terminates after a finite number of iterations, with an $h_j = 0$, hence with $u_j = c_j$, $\varphi_{j0}(c_j) = 0$. If we introduce integers σ_0, σ_1 ..., such that

$$\sigma_0 = 0; \ \sigma_i = s_i + r_i\sigma_{i-1}, \ i = 1, 2, ..., \tag{1.5}$$

we can write (1.4) as

$$(1.6) \qquad u = \sum_{i=0}^{j} c_i \tau_i^{\sigma_i} + (u_j(\tau_j) - c_j)\tau_j^{\sigma_j}, \qquad (\tau = \tau_i^{r_1 \dots r_i}).$$

It can be shown (See Bliss [1]) that $r_i = \mu_i = 1$ for all i big enough, so that there is only a finite number of Puiseux expansions

$$(1.7) \qquad u = u(\tau) = \sum_0^\infty c_i \tau_i^{\sigma_i}, \qquad (\tau = \tau_i^{r_1 \dots r_i}),$$

generated by (1.6). Moreover, all these expansions converge for τ small enough, and together they represent all zeros $u = u(\tau)$ of $f_0(\tau, u)$ near $\tau = 0$. Now let ω_j be any one of unit the roots of order $\varrho_j = rr_1 \dots r_j$, and let $\omega_i = \omega_j^{r_{i+1} \dots r_j}$, so that ω_i is a unit root of order $\varrho_i = rr_1 \dots r_i$. Then, in view of (1.2), every expansion (1.6), (1.7) defines exactly ϱ_j zeros of $P(\xi) = P(\xi_1, \xi_2)$, « *conjugate at level* j ». These are of the form

$$(1.8) \qquad \xi_2 = \varphi_j(\tau_j) = \sum_0^\infty c_i \tau_j^{\varrho_j(\sigma_i/\varrho_i - s/r)} + \dots, \; (\omega_j \tau_j)^{-\varrho_j} = \xi_1,$$

or, if we define ξ_1^{1/ϱ_i} to be real for ξ_1 positive,

$$(1.9) \qquad \xi_2 = \sum_0^j c_i \omega_i^{\varrho_i \delta_i} \xi_1^{\varrho_i} + \dots, \; \delta_i = s/r - \sigma_i/\varrho_i, \qquad (\xi_1 > 0).$$

It should be noticed here that, in view of (1.5),

$$(1.10) \qquad \delta_i = \delta_{i-1} - s_i/\varrho_i = \dots = \delta_0 - \sum_1^i s_k/\varrho_k, \qquad (\delta_0 = s/r),$$

so that $\delta_0 > \delta_1 > \dots$, all the δ_1 being rational numbers, negative for i big enough. In contrast,

$$(1.11) \qquad \varrho_i \delta_i = sr_1 \dots r_i - \sigma_i$$

is an integer for every value of i.

Let now the couple (r, s) take on all the possible positive values determined by the Newton polygon $F(P)$ bounding the index set (P) « from above ». Then we obtain, through the algorithm just described, the Puisseux expansions of all those zeros $\xi_2 = \Phi(\xi_1)$ of $P(\xi)$, for which $|\xi_2|$ tends to infinity with $|\xi_1|$.

2. The canonical factorization.

Suppose that we want $P(\xi)$, $\xi = (\xi_1, \xi_2)$, to be $\binom{2}{1}$-hypoelliptic of type a_{12} in the sense of Gorin [5], i.e. such that

$$P(\xi_1, \xi_2) = 0, \ \mathrm{Im}\ \xi_1 = 0 \Rightarrow |\xi_1| \leqslant C(1 + |\mathrm{Im}\ \xi_2|)^{a_{12}}, \quad (a_{12} > 0)\ . \tag{2.1}$$

Then $|\mathrm{Im}\ \xi_2| \to \infty$ as $\xi_1 \to \pm\infty$, $P(\xi_1, \xi_2) = 0$, and we see that (2.1) is satisfied if and only if every zero $\xi_2 = \Phi(\xi_1)$ of $P(\xi)$ is of the form

$$\xi_2 = \Phi_c(\xi_1) = \sum_0^k c_i \xi_1^{\delta_i} + \mathrm{o}(1)\,|\xi_1|^{\delta_k} = \sum_0^l c'_i(-\xi)^{\delta_i} + \mathrm{o}(1)\,|\xi_1|^{\delta_l}\,, \tag{2.2}$$

with

$$\mathrm{Im}\ c_k \neq 0, \ \mathrm{Im}\ c'_l \neq 0, \ \text{for some } k,\ l \text{ with } \delta_k, \delta_l > 0\ . \tag{2.3}$$

DEFINITION 2.1: Let (2.2) be defined by the Newton algorithm of section 1. Then $c = c_0, c_1 \ldots$ is called a *minimal Newton sequence* (*of lenght* J) *if there are integers* k, l with*max* $(k, l) = J$, suc that

$$|\mathrm{Im}\ \Phi_c(\xi)| = |\mathrm{Im}\ c_k|\,|\xi_1|^{\delta_k}(1 + \mathrm{o}(1)) \qquad \text{as } \xi_1 \to +\infty\,, \quad (\mathrm{Im}\ c_k \neq 0)\,, \tag{2.4}$$

$$|\mathrm{Im}\ \Phi_c(\xi)| = |\mathrm{Im}\ c'_l|\,|\xi_1|^{\delta_l}(1 + \mathrm{o}(1)) \qquad \text{as } \xi_1 \to -\infty\,, \quad (\mathrm{Im}\ c'_l \neq 0)\,,$$

and if, for every zero $\Phi_{c,\omega}(\xi)$ conjugate to $\Phi_c(\xi)$ at level J,

$$|\mathrm{Im}\ \Phi_c(\xi)| = 0(1)\,|\mathrm{Im}\ \Phi_{c,\omega}(\xi)| \qquad \text{as } \xi_1 \to \pm\infty\ .$$

(In this definition we have made use of the fact that there is a certain arbitrariness in the relation (2.2) between the sequences $c_0, c_1, \ldots,$ and $c'_0, c'_1, \ldots$).

THEOREM 2.1: *The polynomial* $P(\xi)$ *is* $\binom{2}{1}$*-hipoelliptic if and only if for every minimal Newton sequence c the critical exponent* $\delta_J = \min(\delta_k, \delta_l)$, *given by* (2.4), *is strictly positive. If P is* $\binom{2}{1}$*-hypoelliptic, then it is also* $\binom{2}{2}$*-hypoelliptic, and the types are*

$$a_{12}(P) = \max_c\,(1/\delta_J), \quad a_{22}(P) = \max_c\,(\delta_0/\delta_J)\,, \tag{2.5}$$

with each maximum taken over all minimal sequences.

PROOF. *If* $c = c_0, c_1, \dots$ is a minimal sequence, then it follows from (2.2), (2.4) with, to be specific, $J = \max(k, l) = k$ that

$$
\begin{aligned}
&|\operatorname{Re}\xi_2| = |\operatorname{Re} c_0|\,|\xi_1|^{\delta_0}(1 + o(1))\,, && \text{as } |\xi_1| \to \infty \\
&|\operatorname{Im}\xi_2| = |\operatorname{Im} c_J|\,|\xi_1|^{\delta_J}(1 + o(1)) && \text{as } \xi_1 \to +\infty\,, \\
&|\operatorname{Im}\xi_2| \geqslant O(1)\,|\xi_1|^{\delta_J} && \text{as } \xi_1 \to -\infty\,.
\end{aligned}
\tag{2.6}
$$

Since by construction $\operatorname{Im} c_J \neq 0$, Theorem 2.1 is a direct consequence of (2.6) and the definition (0.1) of $\binom{k}{j}$-hypoellipticity. Note that $\operatorname{Re} c_0 \neq 0$ except possibly when $J = 0$. But then (2.5) still holds because of the assumption that $a_{jj} \geqslant 1$.

Now let c be a minimal sequence for $P(\xi)$ of length J, with $\delta_J > 0$. Denote by

$$
\Phi_{c,\omega,J}(\xi_1) = \sum_0^J c_i \omega^{\varrho_J \delta_i} \xi_1^{\delta_i}, \qquad \omega \in U_J = \{\omega;\ \omega^{\varrho_J} = 1\}\,,
\tag{2.7}
$$

a Puiseux expansion, truncated at level J, of any one of the zeros $\xi_2 = \Phi(\xi_1)$ of $P(\xi)$, conjugate to $\Phi_c(\xi_1)$ at level J. The product

$$
M_{c,J}(\xi) = \prod_{U_J} (\xi_2 - \Phi_{c,J,\omega}(\xi)) = \prod_{U_J} (\xi_2 - \sum_0^J c_{i,\omega} \xi_1^{\delta_i})
\tag{2.8}
$$

is then a symmetric polynomial in the zeros $\eta = \omega \xi_1^{1/\varrho_J}$ of the polynomial $\eta^{\varrho_J} - \xi_1$, hence a polynomial in ξ_1 (and ξ_2).

DEFINITION 2.2. Let every minimal Newton sequence c of lenght J for a $\binom{2}{1}$-hypoelliptic polynomial $P(\xi)$ represent an equivalence class (c) of minimal sequences, namely the ones that define zeros conjugate to $\Phi_c(\xi_1)$ at level J. Construct, for every equivalence class (c), a polynomial $M_{(c),J}(\xi)$ as in (2.8). Suppose that $P(\xi) = p(\xi_1)\xi_2^{m_2} +$ terms of lower degree in ξ_2. Then the product

$$
P'(\xi) = p(\xi_1) \prod_{(c)} M_{(c),J}(\xi)
\tag{2.9}
$$

is called the *canonical* $\binom{2}{1}$*-hypoelliptic factorization* of P(ξ), and the $M_{(c),J}$ are called *primitive* $\binom{2}{1}$*-hypoelliptic* polynomials (of length J).

In general $P' \neq P$, but the notation « canonical factorization » is motivated by the following.

LEMMA 2.1: *Let P' be the canonical $\binom{2}{1}$-hypoelliptic factorization of P. Then $P(\xi)$ and $P'(\xi)$ are strictly of the same strength, in the sense that $P(\xi)/P'(\xi) = 1 + 0(1) \mid \xi \mid^{-\theta}$ as $\mid \xi \mid \to \infty$, ξ real, for some $\theta > 0$. More exactly, let $\zeta = \min (\delta_J - \delta_{J+1})$, the minimum taken over all minimal sequences for P. Then*

$$(2.10) \qquad | P(\xi)/P'(\xi) - 1 | \leqslant C(| \xi_1 | + | \xi_2 |^{r/s})^{-\zeta} \text{ for } \xi \text{ real}, \ | \xi | \leqslant 1.$$

Moreover, P' is $\binom{2}{1}$ —, $\binom{2}{2}$-hypoellipic of the same types as P.

PROOF. That $a_{i2}(P') = a_{i2}(P)$, $i = 1, 2$, follows from Theorem 2.1, because P' and P have the same minimal sequences.

To prove (2.10), we write every zero of $P(\xi)$ in the form $\Phi_c = \Phi_{c,J} + \Phi^*_{c,J}$, supposing that the sequence c is conjugate at level J to a minimal sequence of length J. Then

$$P(\xi)/P'(\xi) = \prod_c \{1 - \Phi^*_{c,J}/(\xi_2 - \Phi_{c,J})\} = \\ = 1 + 0(1) \max_c \ | \Phi^*_{c,J} | \ / \ |\xi_2 - \Phi_{c,J}| \, .$$

But $| \xi_2 - \Phi_{c,J} | \geqslant A(| \xi_1 | + | \xi_2 |^{r/s})^{\delta_J}$ (this is a consequence of more accurate estimates given in the proof of Theorem 2.2), while

$$| \Phi^*_{c,J} | = | c_{J+1}\xi_1^{\delta_{J+1}} + \ldots | = 0(1) \ |\xi_1 |^{\delta_{J+1}}.$$

The lemma follows immediately.

COROLLARY 2.1: *Let $P(\xi)$ be a $\binom{2}{1}$-ÿypoelliptic polynomial in two variables. Let $s^i\alpha_1 + r^i\alpha_2 = h^i$, $i = 1, \ldots, N$, be the equations for the sides $F^i(P)$ of the Newton polygon $F(P)$. Write the canonical factorization of P as$P' = p(\xi_1) \Pi P^i(\xi)$, with $F(P^i)$ parallel to $F^i(P)$. Then*

$$| (\partial/\partial\xi_2)^k P(\xi) | \leqslant C(1 + | P(\xi) |)^{1-kb_2} \textit{ for } \xi \textit{ real},$$

$k = 1, \ldots,$ if we put

$$b_2 = \min_i \ (s^i/h^i) a_{22}(P^i) \, .$$

PROOF. Les us study the case $k = 1$, the general case offering no additional difficulties. We have $P(\xi) = p(\xi_1)\,\Pi\,(\xi_2 - \Phi(\xi_1))$, hence

$$(\partial/\partial\xi_2 P(\xi))/P(\xi) = \Sigma(\xi_2 - \Phi(\xi_1))^{-1}.$$

But $|\xi_2 - \Phi(\xi_1)| \geqslant A(\xi\,|\xi_1|^s + |\xi_2|^r)^{\delta_J/s}$, for $|\xi| \leqslant 1$, where s, r and δ_J depend on Φ. Also, $|P(\xi)| \leqslant C(|\xi_1|^s + |\xi_2|^r)^{h/sr}$ for the corresponding value of h. Consequently, $|\partial/\partial\xi_2 P(\xi)| \leqslant (1 + |P(\xi)|)^{1-b_2}$, where $b_2 = \min_{\Phi} r\delta_J/h = \min_{\Phi} (s/h)(\delta_J/\delta_0)$. Since $a_{22}(P^i) = \max_{\Phi_i} (\delta_0/\delta_J)$, for all Φ_i with $\delta_0 = s^i/r^i$, the proof of the lemma is complete.

Take as a simple example the following polynomial, used in a similar connection by Pini [7],

$$P(\xi) = \xi_1^4 + \xi_2^3 - i\xi_1^2\xi_2\,. \tag{2.11}$$

Here $r = 3$, $s = 4$, $c_0 = 1$; $r_1 = 1$, $s_1 = 2$, $c_1 = i/3$, and the only minimal sequence is of length $J = k = 1$, with

$$\Phi_{c,1,\omega} = -\,\omega\xi_1^{4/3} + (i/3)\omega^2\xi_1^{2/3}\,, \qquad (\omega^3 = 1)\,.$$

Then a simple computation gives

$$P'(\xi) = M_{c,1}(\xi) = P(\xi) - (i/3)^3\xi_1^2\,.$$

We easily prove (Cf. Theorem 2.2) that, for some $A > 0$,

$$(1 + |P(\xi)|) \geqslant A(|\xi_1|^4 + |\xi_2|^3)^{5/6} \qquad (\text{for } \xi \text{ real})\,. \tag{2.12}$$

Hence

$$P(\xi)/P'(\xi) = 1 + 0(1)(|\xi_1|^4 + |\xi_2|^3)^{-1/3} \qquad \text{as } |\xi| \to \infty.$$

Reversing the roles of ξ_1 and ξ_2, we find that (2.11) is also $\binom{1}{2}$-hypoelliptic. In fact, we have then $r = 4$, $s = 3$, $c_0' = 1$; $r_1 = 1$, $s_1 = 2$, $c_1 = -\,i/4$; $J = l = 1$, and

$$\Phi_{c,1,\omega} = \omega^3(-\,\xi_1)^{3/4} - (i/4)\omega(-\,\xi_1)^{1/4}\,, \qquad (\omega^4 = 1)$$

so that the corresponding $\binom{1}{2}$-hypoelliptic primitive polynomial is

$$P''(\xi) = P(\xi) + 2^{-3}\xi_2^2 + 2^{-8}\xi_2\,,$$

obviously, like P', strictly of the same strength as P. It now follows from Theorem 2.1 that (2.11) is hypoelliptic, with $a_{11} = 3$, $a_{21} = 4$; $a_{12} = 3/2$, $a_{22} = 2$, hence with $a_1 = 3$, $a_2 = 4$.

As a second example, let us consider polynomials parabolic in ξ_2 in the sense of Šilov, i.e. such that

$$P(\xi) = 0 \Rightarrow \operatorname{Im} \xi_2 \geqslant C \mid \operatorname{Re} \xi \mid^{\theta} - C_1 \qquad (\text{some } \theta > 0) .$$

Obviously such polynomials must be $\binom{2}{1}$-hypoelliptic. Let $\xi_2 = \Phi(\xi_1) = c_0\xi_1^{\delta_0} + \ldots + c_J\xi_1^{\delta_J} + \ldots$ be one of the zeros of $P(\xi)$, with $\operatorname{Im} c_i = 0$ for $i < J$. Then it is clear that $\delta_0, \ldots, \delta_{J-1}$ must be integers, because otherwise some of the zeros coniugate to $\Phi(\xi_1)$ could not satisfy the parabolicity condition. Moreover δ_J must be an even integer, because otherwise $\operatorname{Im} \Phi(\xi_1)$ and $\operatorname{Im} \Phi(-\xi_1)$ could not both tend to $+\infty$ with ξ_1. (These observations are originally due to V. M. Borok. See Gelfand-Šilov [4], p. 136). An immediate consequence is then, in view of Lemma 2.1, the following result

THEOREM 2.2: *Let $\xi = (\xi_1, \xi_2)$. Then $P(\xi)$ is parabolic in ξ_2 in the sense of Šilov if and only if is strictly of the same strength as a product of polynomials of the type*

$$S(\xi) = (\xi_2 - i\xi_1^{2p}) + S_1(\xi_1) ,$$

where S_1 is an arbitrary real polinomial.

Let us now return to the case of a primitive polynomial. Then we have the following basic estimate.

LEMMA 2.2: *Let $M(\xi) = M_{c,J}(\xi)$ be a primitive $\binom{2}{1}$-hypoelliptic polynomial as in* (2.8), *with*

$$\Phi_{c,J,\omega}(\xi) = \Phi_{\omega}(\xi) = \sum_0^J c_{i,\omega}\xi_1^{\delta_i} \, (\xi_1 > 0), = \sum_0^J c'_{i,\omega}(-\xi_1)^{\delta_i} \, (\xi > 0) .$$

(Here $_i = \delta s/r - \sum_0^i s_k/\varrho_k$, and $\omega^{\varrho} = 1$, with $\varrho = \varrho_J = rr_1 \ldots r_J$). Then $M(\xi)$ is of degree $m_1 = sr_1 \ldots r_J = \delta_0\varrho$ in ξ_1, and of degree $m_2 = rr_1 \ldots r_J = \varrho$ in ξ_2. Moreover,

$$(2.13) \qquad A(\mid \xi_1 \mid^{m_1} + \mid \xi_2 \mid^{m_2}) \geqslant \mid M(\xi) \mid \geqslant A_1 (\mid \xi_1 \mid^{m_1} + \mid \xi_2 \mid^{m_2})^{1-d}$$
$$\textit{for } \xi \textit{ real}, \ \mid \xi \mid \geqslant K ,$$

with $0 \leqslant d < 1$ *or, more exactly,*

$$(2.14) \qquad d = \sum_{1}^{J} (\delta_{i-1} - \delta_i)/\delta_0 \varrho_{i-1} \, .$$

Hence $d = 0$ *for* $J = 0$, *but* $0 < (1 - \delta_J/\delta_0)/m_2 \leqslant d \leqslant (1 - \delta_J\delta_0)/r < 1/r$ *for* $J > 0$.

If we add the restriction that, for some $k < J$,

$$(2.15) \qquad | \xi_2 - \Phi_\omega(\xi) | > \varepsilon \, | \xi_1 |^{\delta_k} \quad \textit{for all } \omega, \ \xi \textit{ real} \, ,$$

we get and improved estimate (2.13), *with d replaced by*

$$(2.16) \qquad d_k = \sum_{1}^{J} (\delta_{i-1,k} - \delta_{i,k})/\delta_0 \varrho_{i-1}, \ \delta_{i,k} = \max\,(\delta_i, \delta_k) \, .$$

PROOF. Suppose, for instance, that $J = \max\,(k, l) = k$. Then we notice that, for some $j = j(\omega)$, $0 \leqslant j + 1 \leqslant J$,

$$(2.17) \qquad | \xi_2 - \Phi_\omega(\xi) | \geqslant \max \left\{ \left| \xi_2 - \sum_{0}^{J} \operatorname{Re} c_{i,\omega} \xi_1^{\delta_i} \right| , \left| \sum_{j+1}^{J} \operatorname{Im} c_{i,\omega} \xi_1^{\delta_i} \right| \right\} ,$$

with $\operatorname{Im} c_{j+1,\omega} \neq 0$. Set now, with $\varepsilon \leqslant | c_j |$,

$$(2.18) \qquad V_{\omega,j,\varepsilon} = \{ \xi \in R^2 \, ; \ | \xi_2 - \Phi_\omega(\xi) | < \varepsilon \, | \xi_1 |^{\delta_j} \} , \qquad (j = j(\omega)).$$

Then $| \xi_2 - \Phi_{\omega_1}(\xi) | = 0(1) \, | \xi_1 |^{\delta_j}$ for $\xi_1 \to + \infty$, $\xi \in V_{\omega,j,\varepsilon}$, if and only if $c_{\omega_1, i} = c_{\omega, i}$ $(= \operatorname{Re} c_{\omega, i})$ for $i = 0, 1, \ldots, j - 1$. There are exactly $r_j \ldots$ $\ldots r_J = \varrho/\varrho_{j-1}$ zeros Φ_{ω_1} of this kind. But since $j < J$, exactly ϱ/ϱ_j of these zeros also satisfy $| \xi_2 - \Phi_{\omega_1}(\xi) | = o(1) \, | \xi_1 |^{\delta_j}$, namely those for which $c_{\omega_1, j} = c_{\omega, j}$ $(= \operatorname{Re} c_{\omega, j})$. Hence the estimate

$$(2.19) \qquad C_1 \, | \xi |^{\delta_j} \leqslant | \xi_2 - \Phi_{\omega_1}(\xi) | \leqslant C_2 \, | \xi_1 |^{\delta_j} ,$$

for some C_1, $C_2 > 0$, is valid in the domain $V_{\omega,j,\varepsilon}$ for exactly $\varrho(1/\varrho_{j-1} - 1/\varrho_j)$ zeros Φ_{ω_1}. On the other hand, outside the union $\bigcup_{\omega} V_{\omega,j,\varepsilon}$, $j(\omega) = j$, the estimate (2.19) is valid for all the ϱ/ϱ_{j-1} zeros Φ_{ω_1}. Consequently the best overall lower estimate for $| M |$ under the restriction (2.15) is $| M(\xi) | \leqslant B \, | \xi_1 |^{\varkappa}$, where

$$\varkappa = \sum_{0}^{J} \delta_{j,k}(\nu_j - \nu_{j+1}), \ \nu_j = \varrho/\varrho_{j-1}, \ \nu_{J+1} = 0 \, .$$

(Notice that the domain (2.15) is the complement of $\bigcup_\omega V_{\omega,k,\varepsilon}$). It follows that

$$|M(\xi)| \geqslant B(|\xi_1| + |\xi_2|^{r/s})^\varkappa \tag{2.20}$$

if $|\xi_1| \geqslant C|\xi_2|^{r/s}$. But if $|\xi_1| < C|\xi_2|^{r/s}$, with C small enough, then trivially

$$|M(\xi)| \geqslant B|\xi_2|^{m_2} \geqslant B_1(|\xi_1| + |\xi_2|^{r/s})^{\delta_0 m_2}.$$

Since $m_2 = \varrho = \nu_0$ is the number of zeros Φ_ω, we see that $\varkappa \leqslant \delta_0 \Sigma(\nu_j - \nu_{j-1}) \leqslant \delta_0 m_2$, so that

$$|M(\xi)| \geqslant B(|\xi_1| + |\xi_2|^{r/s})^\varkappa \equiv B_1(|\xi_1|^{m_1} + |\xi_2|^{m_2})^{\varkappa/m_1}$$

for ξ real, $|\xi| > 1$. Since the upper estimate in (2.13) is trivial, it remains only to prove (2.14). But by partial summation

$$\varkappa = \varkappa_k = \delta_0\nu_0 - \sum_1^J (\delta_{j-1,k} - \delta_{j,k})\nu_j =$$
$$= m_1\{1 - \sum_1^J (\delta_{j-1,k} - \delta_{j,k})/\delta_0\varrho_{j-1}\},$$

and (2.14) follows if we observe that $r = \varrho_0 \leqslant \varrho_1 \leqslant \ldots, \delta_J > 0$.

For instance, in the example (2.11) we have $\delta_0 = 4/3$, $\delta_1 = 2/3$, $J = 1$, and $r = 3$, so that $d = (1 - \delta_1/\delta_0)/r = 1/6$. Reversing the order of ξ_1 and ξ_2 we get $\delta_0 = 3/4$, $\delta_1 = 1/4$, $J = 1$, and $r = 4$, so that again $d = (1 - \delta_1/\delta_0)/r = 1/6$. (Cf. (2.12)).

Let now $P(\xi)$, $\xi \in R^2$, be a general $\binom{2}{1}$-hypoelliptic polynomial, and let $c = c_0, c_1, \ldots$ be a minimal Newton sequence for P. Constructing c by the Newton algorithm of section 1, we define $g_0, g_1, \ldots$ as in (1.3). The lower Newton polygon for g_{i-1} then determines a number of couples of relatively prime integers $(r_{ij}, s_{ij}) > 0$, (not necessarily all different), one of which is (r_i, s_i). To each couple (r_{ij}, s_{ij}) corresponds an exponent δ_{ij}, and μ_{ij} (complex) zeros for $P(\xi)$ of the form

$$\xi_2 = c_0\xi_1^{\delta_0} + \ldots + c_{i-1}\xi_1^{\delta_{i-1}} + c_{ij}\xi_1^{\delta_{ij}} + \ldots \tag{2.21}$$

In case P is primitive as in Lemma 2.1, the Newton polygon for each g_{i-1} is simple so that there are only r_i coefficients $c_{ij} = c_{i,\omega_j}$, all of multiplicity $\mu_i = \varrho/\varrho_i$ and with exponent $\delta_{ij} = \delta_i$.

DEFINITION 2.3: Let $P(\xi)$ be a $\binom{2}{1}$-hypoelliptic polynomial, $\xi \in R^2$, with a simple Newton polygon $F(P)$, and such that, for some $\mu_0 \geqslant 1$, r and s relatively prime,

$$(2.22) \qquad P(\xi) = (\xi_2^r - c_0^r \xi_1^s)^{\mu_0} + \Sigma' c_\alpha \xi^\alpha, \qquad \alpha \notin F(P),\ c_0 \neq 0\ .$$

Then P is called a *simple* $\binom{2}{1}$-hypoelliptic polynomial with *leading part* $P_0(\xi) = (\xi_2^r - c_0^r \xi_1^s)^{\mu_0}$.

Notice that every primitive $\binom{2}{1}$-hypoelliptic polynomial is simple, but the converse is not true. The importance of the simple polynomials is that if we use the canonical factorization (and Lemma 2.1) to write a given $\binom{2}{1}$-hypoelliptic $P(\xi)$ as equivalent to a product ΠS_λ of simple $\binom{2}{1}$-hypoelliptic polynomials, then in order to find a lower estimate for $|P(\xi)|$, it is sufficient to estimate each $|S_\lambda(\xi)|$ downwards. In contrast, the best lower estimate for a product of primitive $\binom{2}{1}$-hypoelliptic polynomials is in general better than the product of the lower estimates for each factor separately.

THEOREM 2.3: *Let a* $\binom{2}{1}$*-hypoelliptic* $P(\xi) = p(\xi_1)\xi_2^{m_2} + \ldots$ *have the canonical* $\binom{2}{1}$*-hypoelliptic factorization* $P' = p(\xi_1)\Pi M_{(c),J}$, *and group together the primitive factors to write* P'/p *as a product* $\prod\limits_\lambda S_\lambda$ *of simple* $\binom{2}{1}$*-hypoelliptic factors* $S_\lambda = \Pi M_{(c)_\lambda,J}$ *with relatively prime leading parts* $(\xi_2^r - c_0\xi_1)^{\mu_0}$. *Set*

$$(2.23) \quad d_\lambda = \max_{(c)\lambda} \Big\{ \sum_{i=1}^{J} \sum_j (\delta_{i-1} - \delta_{i,j})\mu_{ij} r_{ij} \Big\} / \mu_0 s,\ \delta_{i,j} = \max(\delta_i, \delta_{ij})\ .$$

Then

$$(2.24) \qquad (1 - \delta_J/\delta_0)/\mu_0 r \leqslant d_\lambda \leqslant (1 - \delta_J/\delta_0)/r\ ,$$

and

$$(2.25) \qquad A \geqslant |P(\xi)| / \{p(\xi)\prod_\lambda (|\xi_1|^{s'} + |\xi_2|^{r'})^{\mu_0'}\} \geqslant A_1 H(\xi)$$

for ξ *real,* $|\xi| \leqslant K$, *where* $H(\xi) = (\,|\,\xi_1\,|^s + |\,\xi_2\,|^r)^{-\mu_0 d_\lambda}$ *when* $|\,\xi_1^r - c_0^r \xi_2^s\,| < \varepsilon\,|\,\xi_1\,|^r$ *for some real* c_0 *and some* ε *small enough, while* $H(\xi) = 1$ *outside the union of all such sets.*

PROOF. In view of the proof of Lemma 2.1, it is enough to prove (2.25) for the case when $P = P'$. Further, it is evident that there are constants A_λ such that $|\,S_\lambda(\xi)\,| \geqslant A_\lambda(\,|\,\xi_1\,|^s + |\,\xi_2\,|^r)^{\mu_0}$ (s, r, and μ_0 depending on λ) for all λ except one, at most, at every real point ξ, $|\,\xi\,| \geqslant K$. Hence it is enought to prove that

$$(2.26) \qquad |\,S_\lambda(\xi)\,| \geqslant A_\lambda(\,|\,\xi_1\,|^s + |\,\xi_2\,|^r)^{\mu(1-d_\lambda)}\,, \qquad |\,\xi\,| \geqslant K\,.$$

This can be done easily by the same reasoning as in the proof of Lemma 2.2 (Cf. also (2.16)). We find that (2.26) is valid with

$$(2.27) \qquad d_\lambda = \max_{(c)\lambda} \sum_{i=1}^{J} \sum_j{}' (\delta_0 - \delta_{i,j})\mu_{ij} r_{ij}/\mu_0 s, \qquad \delta_{i,j} = \max\,(\delta_i, \delta_{ij})\,,$$

where Σ' for $i > J$ means that the summation does not include the index j for which $c_{ij} = c_i$. Since $\sum_j \mu_{ij} r_{ij} = \sum_j{}' \mu_{ij} r_{ij} + \sum_j \mu_{i+1,j} r_{i+1,j}$ (2.26) clearly implies (2.23). Finally, we can derive the estimates for d_λ from (2.27), if we observe that $\delta_{ij} \geqslant \delta_J$, and that $\Sigma\Sigma' \mu_{ij} r_{ij} = \mu_0$.

Let us now recall (See Friberg [2]), that $P(\xi)$, $\xi = (\xi_1, \xi_2)$, is called *partially hypoelliptic in* ξ_1 *if*

$$(2.28) \qquad P(\xi + i\eta) = 0, \qquad |\,\xi_1\,| \to \infty \Rightarrow |\,\eta\,| \to \infty\,.$$

An equivalent condition is that P is both $\binom{2}{1}$- and $\binom{1}{1}$-hypoelliptic. (Gorin [5]). But if P is $\binom{2}{1}$-hypoelliptic, then we know that P is equivalent to a $\binom{2}{1}$-hypoelliptic polynomial $P' = p(\xi_1)P_1(\xi)$, where the Newton polygon $F(P_1)$ has only sides with positive normals. Hence all the zeros $\xi_1 = \Phi(\xi_2)$ of $P_1(\xi)$ are of the form $\xi_1 = \sum_0^\infty c_i \xi_2^{\delta_i}$, with $\delta_0 > 0$.

With $\eta_2 = 0$ in (2.28), we see that P_1, and then P, can be $\binom{1}{1}$-hypoelliptic only if Im $c_i \neq 0$ for some i with $\delta_i > 0$, i.e. only if P_1 is $\binom{1}{2}$-hypoelliptic. It is now easy to complete the proof of the following

THEOREM 2.4: *Let* $\xi = (\xi_1, \xi_2)$, $P(\xi) = p_1(\xi_1)\xi_2^{m_2} + \ldots$. *Then* P *is* *partially hypoelliptic in* ξ_1 *if and only if* P *is* (*strictly*) *of the same strength as a polynomial*

$$P'(\xi) = p(\xi_1)P_1(\xi),\ P_1(\xi) \text{ hypoelliptic.} \tag{2.29}$$

Polynomials of the type (2.29) have in fact been used earlier as examples of partially hypoelliptic polynomials (Friberg [2], Gorin [5]).

3. The principal part.

DEFINITION 3.1: Let a $\binom{2}{1}$-hypoelliptic polynomial $P(\xi) = \sum_{(P)} c_\alpha \xi^\alpha$ have the canonical factorization $P' = p(\xi_1)\Pi S_\lambda$. Suppose that $r^n\alpha_1 + s^n\alpha_2 = h^n$, $n = 1, \dots, N$, are the equations for the sides of the Newton polygon $F(P) = F(P')$, with r^n, s^n relatively prime. For given n, consider all $S_\lambda = S_\lambda^n$ with Newton polygon given by an equation $r^n\alpha_1 + s^n\alpha_2 = h_\lambda^n$, so that $h^n = \sum_\lambda h_\lambda^n$, and define $d_\lambda = d_\lambda^n$ as in (2.28). Set

$$H(P) = \bigcup_{n=1}^{N} \{\alpha \in (P);\ h^n \geqslant r^n\alpha_1 + s^n\alpha_2 \geqslant h^n - \max_\lambda h_\lambda^n d_\lambda^n\}\,. \tag{3.1}$$

Then the polynomial

$$P_H(\xi) = \sum_{H(P)} c_\alpha \xi^\alpha \tag{3.2}$$

is called the $\binom{2}{1}$-*hypoelliptic principal part* of $P(\xi)$.

The definition is partly motivated by Theorem 2.3, which shows that $P(\xi) - P_H(\xi)$ is strictly weaker than $P_H(\xi)$. But we can prove more:

THEOREM 3.1: *Let* P, Q *be* $\binom{2}{1}$-*hypoelliptic, with coinciding principal parts,* $P_H = Q_H$. *Then* P *and* Q *have identical minimal Newton sequences* (*and are consequently* $\binom{2}{1}$-*hypoelliptic of the same type*).

PROOF. It is enough to show that the minimal sequences of P depend only on the coefficients c_α of P with $\alpha \in H(P)$. Omitting the indices n, let $r\alpha_1 + s\alpha_2 = h$ be the equation of one side in $F(P)$, and define $f_i(\tau_i, u_i)$, $i = 0, 1, \dots$, and $\varphi_{ik}(u)$, $i = 0, 1, \dots$, $k = 0, \dots, h_i$, as in (1.2), (1.3). We notice that $\varphi_{0k}(u)$ is determined entirely by the coefficients c_α of P with $r\alpha_1 + s\alpha_2 = h - k$. On the other hand, to compute the coefficients $c_0, \dots, c_J$ of a minimal Newton sequence, we need only know $\varphi_{00}, \dots, \varphi_{J0}$. But we have

$$(3.3)\qquad f_0(\tau_J^{\varrho_J/r}, c_0 + \dots + \tau_J^{\sigma_J} u_J) = \sum_0^h \varphi_{0k}(c_0 + \dots + \tau_J^{\sigma_J} u_J)\tau_0^k = \\ = (\tau_1^{\beta_1} u_1^{\gamma_1}) \dots (\tau_J^{\beta_J} u_J^{\gamma_J}) f_J(\tau_J, u_J),\ \tau_i^{\varrho_i} = \tau_j^{\varrho_j},\ u_i = c_i + \dots$$

Since $f_J(\tau_J, u_J) = \varphi_{J0}(u_J) + 0(1)\tau_J$, it follows that to determine φ_{J0}, for instance, we need only know $\varphi_{00}, \dots, \varphi_{0I}$, where

$$(3.4)\qquad I = [r \sum_1^J \beta_i/\varrho_i]\ ,$$

the entire part of $r\ \Sigma\beta_i/\varrho_i$. Now it is easy to check from the Newton polygon for $f_{i-1}(\tau_{i-1}, c_{i-1} + v_i)$ that

$$(3.5)\qquad \beta_i/r_i = \sum_j \mu_{ij} r_{ij} \min (s_{ij}, s_i r_{ij}/r_i)\ .$$

Since also, in view of (1.9), when $\delta_{i,j} = \max(\delta_i, \delta_{ij})$,

$$(3.6)\qquad \delta_{i-1} - \delta_{i,j} = (1/\delta_{i-1}) \min (s_{ij}/r_{ij}, s_i/r_i)\ ,$$

we see on comparison of formulas (2.23) and (3.5) that

$$(3.7)\qquad d_\lambda = \max_{(c)\lambda} \{(\sum_1^J \beta_i/\varrho_i)/\mu_0 s\}\ .$$

Consequently, in view of (3.3) to determine all the minimal sequences belonging to S_λ, we must know φ_{0i} for $i \leqslant I_\lambda$, with

$$I_\lambda = [\mu_0 s r d_\lambda] = [h_\lambda d_\lambda]\ .$$

It follows that all minimal sequences for P are determined by the coefficients c_α with $\alpha \in H(P)$.

COROLLARY 3.1: *Let $r^n\alpha_1 + s^n\alpha_2 = h^n$, $n = 1, \dots, N$, be the sides of the Newton polygon $F(P)$ for a given polynomial $P(\xi) = \sum_{(P)} c_\alpha \xi^\alpha$, $\xi \in R^2$. Suppose the maximal multiplicity of a zero of $\varphi_{00}(u) = \varphi_{00}^n(u)$ is μ_0^n, φ_{00} being defined by* (1.2). *Let $Q(\xi) = \sum_{(Q)} c_\alpha \xi^\alpha$, with*

$$(3.8)\qquad (Q) = \bigcup_{n=1}^N \{\alpha \in (P);\ h^n \geqslant r^n\alpha_1 + s^n\alpha_2 > h^n - \mu_0^n s^n\}\ .$$

Then P is $\binom{2}{1}$-hypoelliptic if and only if Q is $\binom{2}{1}$-hypoelliptic.

PROOF. All we have to do is observe that, according to the estimates (2.24),

$$(3.9) \qquad \max_{\lambda} h_{\lambda}^{n} d_{\lambda}^{n} = \max_{\lambda} \mu_0 r^n s^n d_{\lambda}^{n} < \mu_0 s^n .$$

THEOREM 3.2: *Let* $P(\xi) = \sum_{(P)} c_\alpha \xi^\alpha$, $\xi \in R^2$, *and set* $Q(\xi) = \sum_{(Q)} c_\alpha \xi^\alpha$, *with*

$$(3.10) \qquad (Q) = \bigcup_{n=1}^{N} \{\alpha \in (P);\ h^n \geqslant r^n \alpha^1 + s^n \alpha^2 \geqslant h^n - \mu_0^n \min(s^n, r^n)\} .$$

Then P is both $\binom{2}{1}$- and $\binom{1}{2}$-hypoelliptic if and only if the same is the case for Q. If this condition is satisfied, then P is also hypoelliptic in the ordinary sense and strictly of the strength as a product $P' = \Pi S_\lambda$, where every S_λ is a simple $\binom{2}{1}$-hypoelliptic and $\binom{1}{2}$-hipoelliptic polynomial at the same time. Moreover, the $\binom{2}{1}$- and $\binom{1}{2}$-hypoelliptic principal parts of P coincide, as well as those of each S_λ.

PROOF. Suppose $P(\xi)$, $\xi \in R^2$, is both $\binom{2}{1}$- and $\binom{1}{2}$-hypoelliptic. Then, in view of Theorem (2.1), P is also $\binom{2}{2}$- and $\binom{1}{1}$-hypoelliptic, hence hypoelliptic in the ordinary sense. Now let $P' = p(\xi_1)\Pi S_\lambda(\xi)$ be the $\binom{2}{1}$-hypoelliptic factorization of P. Then since P and P' are strictly equally strong (Lemma 2.1), it follows that P' is hypoelliptic. Every factor of a hypoelliptic polynomial being hypoelliptic, this means that each S_λ is hypoelliptic, and that $p(\xi_1)$ is a constant C. Let now d_λ be the maximum of the numbers d for which $|S_\lambda(\xi)| \geqslant A(|\xi_1|^s + |\xi_2|^r)^{\mu_0(1-d)}$ all real ξ, $|\xi| \geqslant K$, where s, r and μ_0 are determined by the leading part $(\xi_2^r - c_0^r \xi_1^s)^{\mu_0}$ of S_λ. Then, due to Theorem 2.3, we know that $d_\lambda < 1/r$. But S_λ is also simple $\binom{1}{2}$-hypoelliptic, hence we must also have $d_\lambda < 1/s$. Further, since every d_λ is the same, wherther it is determined with start from the $\binom{2}{1}$-hypoellipticity or from the $\binom{1}{2}$-hypoellipticity, it follows that $P_H(\xi)$ is not only the $\binom{2}{1}$-hypoelliptic but also the $\binom{1}{2}$-hypoelliptic principal part of $P(\xi)$. Finally, the equivalence of P and Q is proved as in Corollary 3.1.

In view of Theorem 3.2, if $P(\xi)$, $\xi \in R^2$, is hypoelliptic, we can justly call $P_H(\xi)$ the (*hypoelliptic*) *principal part* of P, and, for instance, $P' = \Pi(S_\lambda)_H$ a *canonical hypoelliptic factorization* of P. It may be worth noticing, that in spite of the truncation of S_λ to $(S_\lambda)_H$ (which is made for the sake of symmetry), in general $P_H \neq c\Pi(S_\lambda)_H$. This is obvious from the following example.

Suppose that P has the hypoelliptic factorization $P' = c\Pi S_\lambda$, where each S_λ is equal to its leading part $(\xi_2^r - c_0^r \xi_1^s)^{\mu_0}$, i.e. suppose that every minimal sequence for P (with respect to $\binom{2}{1}$- or $\binom{1}{2}$-hypoellipticity) is of length $J = 0$. Then obviously $d_\lambda = 0$, for all λ, and (2.25) becomes

$$A \geqslant |\, P(\xi) \,| \,/ \prod_\lambda (\, |\, \xi_1 \,|^s + |\, \xi_2 \,|^r)^{\mu_0} \geqslant A_1, \quad \xi \text{ real}, \ |\, \xi \,| \geqslant K \,. \tag{3.10}$$

Also $P_H(\xi) = P_F(\xi) = \sum_F c_\alpha \xi^\alpha$, where $F = F(P)$ is the Newton polygon of P. In general $P_H(\xi) \neq c\Pi S_\lambda = c\Pi(\xi_2^r - c_0^r \xi_1^s)^{\mu_0}$, as is easily checked. Now, (3.10) is exactly the definition of a multi-quasielliptic polyomial, in the sense of Friberg [3]. Hence we get from the preceding discussion and from Theorem 2.1 the following result (Cf. [3]).

THEOREM 3.3: *If* $\xi \in R^2$, *then* $P(\xi)$ *is multi-quasielliptic, i.e. satisfies an estimate* (3.10), *if and only if one of the following two* (*equivalent*) *conditions is satisfied*:

$$\text{i) } P_H = P_F \,, \qquad \text{ii) } a_{11}(P) = a_{22}(P) = 1 \,.$$

4. Sufficient conditions for hypoellipticity.

Let us now drop the condition $\xi = (\xi_1, \xi_2)$. It is then no longer possible to extend the results of section 2 concerning the canonical $\left(\binom{k}{j}\text{-}\right)$ hypoelliptic factorization of a $\left(\binom{k}{j}\text{-}\right)$ hypoelliptic polynomial. Counter-examples were given in Friberg [3], all of them *multiquasielliptic* in the generalized sense that they satisfy an estimate of the type

$$A \geqslant |\, P(\xi) \,| \,/ \Sigma \,|\, \xi^{\alpha^i} \,| \geqslant A_1 > 0 \text{ for } \xi \text{ real}, \ |\, \xi \,| \geqslant K \,. \tag{4.1}$$

(Here the α^i are a finite number of multi-indices $\geqslant 0$.) Nevertheless, it is sometimes possible also in the more-dimensional case to find a principal part of a $\left(\binom{k}{j}\text{-}\right)$ hypoelliptic polynomial. For instance for a multi-quasielliptic polynomial (4.1), the principal part is always $P_F = \sum_{F(P)} c_\alpha \xi^\alpha$, as in the two-dimensional case. (Friberg [3]).

To simplify the exposition, we shall in what follows mostly restrict our attention to the case when $P(\xi)$ has a simple Newton surface $F(P)$, given by an equation

$$\Sigma \alpha_i / m_i = 1 \quad \text{for all } \alpha \in F(P)\,. \tag{4.2}$$

THEOREM 4.1: *Suppose $F(P)$ is given by* (4.2), *and that P satisfies, for ξ real, $|\xi| \geqslant K$, an estimate*

$$A \geqslant |P(\xi)| \,/\, \Sigma\, |\xi_i|^{m_i} \geqslant A_1(\Sigma\, |\xi_1|^{m_i})^{-d}\,. \tag{4.3}$$

Then $P(\xi)$ is $\binom{k}{j}$-hypoelliptic for all j if $d < 1/m_k$, hypoelliptic if $d <$ $< \min\,(1/m_k)$.

The proof is trivial, because we have

$$|(\partial/\partial\xi_k)^l P(\xi)| \leqslant C(\Sigma\, |\xi_i|^{m_i})^{1-l/m_k} \leqslant C_1(\Sigma\, |\xi_1|^{m_i})^{d-l/m_k}\, |P(\xi)| \tag{4.4}$$

for $|\xi| \geqslant K$, when P satisfies (4.3). Clearly, in case of a non-simple $F(P)$, (4.3) must be replaced by an estimate of the type

$$A \leqslant |P(\xi)| \,/\, \Sigma\, |\xi^{\alpha^i}| \leqslant A_1(\Sigma\, |\xi^{\alpha^i}|\,)^{-d}, \qquad A_1 > 0\,, \tag{4.5}$$

with $d > 0$, but small enough. We notice that (4.5) defines a class of hypoelliptic polynomials slightly larger than the class of all multi-quasielliptic polynomials.

COROLLARY 4.1: *Let $\xi = (\xi_1, \xi_2)$, and suppose that $S(\xi) = (\xi_2^r - c_0^r \xi_1^s) +$ $+$ terms $c_\alpha \xi^\alpha$ with $r\alpha_1 + s\alpha_2 < rs$. If $S(\xi)$ is $\binom{2}{1}$-hypoelliptic, then it is simple $\binom{2}{1}$-hypoelliptic, with leading part of multiplicity $\mu_0 = 1$, and*

$$|S(\xi)| \geqslant A(\,|\xi_1|^s + |\xi_2|^r)^{1-d}, \quad d = (1 - \delta_J/\delta_0)/r > 0\,, \quad \textit{for } \xi \textit{ real}, \ |\xi| \geqslant K\,. \tag{4.5}$$

Conversely, if $S(\xi)$ satisfies (4.5), *then S is* $\binom{2}{1}$- and $\binom{2}{2}$-*hypoelliptic, with*

$$a_{12}(S) = (s/r - sd)^{-1} = 1/\delta_J, \quad a_{22}(S) = (1 - rd)^{-1} = \delta_0/\delta_J\,. \tag{4.6}$$

Moreover, if $\delta_0 - \delta_J < 1$, *for instance if* $s < r$, *then* (4.5) *implies that* S *is hypoelliptic, with*

$$a_{21}(S) = (1 - sd)^{-1}, \quad a_{11}(S) = (r/s - rd)^{-1}. \tag{4.7}$$

PROOF. The value of d follows from (2.24). Conversely, if S satisfies (4.5), then we can prove as in (4.4) that for $|\xi| \geqslant 1$

$$| (\partial/\partial\xi_2)^l P(\xi) | \leqslant C(| \xi_1 |^s + | \xi_2 |^r)^{d-l/r},$$

$$| \partial/\partial\xi_1 P)^l(\xi) | \leqslant C(| \xi_1 |^s + | \xi_2 |^r)^{d-l/s}.$$

The first estimate gives the values of a_{12} and a_{22}, the same as were computed in Theorem 2.1. The second inequality implies the $\binom{1}{2}$- and $\binom{1}{1}$-hypoellipticity, provided that a_{21} and a_{11}, as given by (4.7), are positive.

A first example is the polynomial (2.11), for which $\delta_0 - \delta_J = 2/3$. As a second example, consider a primitive Šilov-parabolic polynomial $S(\xi) = (\xi_2 - i\xi_1^{2p}) + S_1(\xi_1)$, S_1 real, degree $S_1 = m_1$. Here $\delta_0 = \max(2p, m_1)$, and $\delta_J = 2p$. Hence $\delta_0 - \delta_J < 1$ if and only if $m_1 \leqslant 2p$. Consequently a Šilov-parabolic polynomial is in general not hypoelliptic. This means that the definition of parabolicity given by Hörmander [6, p. 152] is more restrictive than Šilov's definition.

Let us return to the general more- dimensional case. Generalizing an observation due to Hörmander [6, p. 103] we have the following result, showing the existence of hypoelliptic polynomials with simple Newton surface and an almost arbitrary (real) leading part. Let

$$P = Q + iR_\gamma, \quad Q = Q_1^{\mu_1} \dots Q_N^{\mu_N}, \tag{4.6}$$

where the Q_i are real polynomials with every $F(Q_i)$ parallel to $F(P)$, $\Sigma a_j/m_j \leqslant 1$ for $\alpha \in (P)$, and where R_γ is real quasielliptic, $\Sigma\alpha_j/m_j = \gamma < 1$ for $\alpha \in (R_\gamma)$. Then P is $\binom{k}{j}$-hypoelliptic for all j provided that

$$\gamma > 1 - \mu/m_k, \quad \mu = \min \mu_i. \tag{4.7}$$

For the proof we first notice that, for instance,

$$| \partial/\partial\xi_k R_\gamma(\xi) | = o(1) | R_\gamma(\xi) | = o(1) | P(\xi) | \text{ as } | \xi | \to \infty,\ \xi \text{ real}, \tag{4.8}$$

because $|P| \geqslant \max(|Q|, |R|)$. Next we write $\mu_i = \varepsilon_i \mu$, $i = 1, ..., N$, so that $\min \varepsilon_i = 1$, and $|Q| = (\Pi |Q_i|^{\varepsilon_i})^\mu$. It follows that $|\partial/\partial\xi_k Q(\xi)|$ can be estimated by a sum of terms like

$$|Q \cdot \partial/\partial\xi_k Q_l/Q_l| = |Q|^{1-1/\mu} \cdot \prod_{i \neq l} |Q_i|^{\varepsilon_i} \cdot |Q_l|^{\varepsilon_l - 1} |\partial/\partial\xi_k Q_l| \leqslant$$

$$\leqslant A |Q|^{1-1/\mu}(\Sigma |\xi_j|^{m_j})^{1/\mu - 1/m_k} \leqslant A_1 |Q|^{1-1/\mu} |R_\gamma|^{(1/m - 1/m_k)/\gamma}$$

for ξ real, $|\xi|$ large enough. Consequently

$$|\partial/\partial\xi_k Q| = o(1)|P| \text{ as } |\xi| \to \infty,$$

if we assume that $(1/\mu - 1/m_k)/\gamma < 1/\mu$, or $\gamma > 1 - \mu/m_k$, which is precisely (4.7). We notice that if Q is not itself $\binom{k}{j}$-hypoelliptic, then R_γ must be considered to belong to the $\binom{k}{j}$-hypoelliptic principal part of P, for any sensible definition of the principal part in the more-dimensional case. (Cf. Pini [7], p. 11). It is also easy to find examples where $\gamma \leqslant 1 - \mu/m_k$ and P is not $\binom{k}{j}$-hypoelliptic.

Consider now instead a polynomial $P = \sum_0^N P_i$, $P_i = \prod_l (Q_{il})^{\mu_{il}}$, Q_{il} positive semi-definite, and suppose that $\sum_1^n \alpha_j/m_j \leqslant \gamma_i \leqslant \gamma_0$ for $\alpha \in (P_i)$. Set $\mu_i = \min_l \mu_{il}$. Then the estimate

$$(\sum_j |\xi_j|^{m_j})^\gamma \leqslant C(1 + |P(\xi)|), \quad \gamma > \max_i (\gamma_i - \mu_i/m_k),$$

is a sufficient condition for $\binom{k}{j}$-hypoellipticity. (The proof is the same as in the preceding example). Notice that this result indicates that, as in the two-dimensional case, the form of the principal part of P does not depend exlusively on the leading part P_0.

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