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Satish Chandra Chakrabarti

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ON HIGHER DIFFERENCES

Nota IV (*) *di* SATISH CHANDRA CHAKRABARTI (*a Calcutta*)

1. *Introduction.*

In the last paper [1]) it was shown how to express the differential coefficient $d^k u_x / dx^k$ in terms of the operators A^n. In the present paper it is shown that the same differential coefficient, viz, $d^k u_x / dx^k$ can be expressed also in terms of the operators A_n. While working with this problem some algebraic identities come across which are also considered here. Finally we show that the Theories of Higher Differences may also be applied to Interpolations and Statistics.

2. *Notations.*

Besides the notations used in eariler Notes on the same subject, some new notations are used here which are given below:

(*a*). $\begin{pmatrix} 1, a^n \\ a^r \end{pmatrix}_p$ = sum of the products of the elements, taken p at a time, of the series

$$1, \quad a, \quad a^2, \quad \dots \quad a^n$$

in which the element a^r is absent; $r \leqslant n$.

(*) Pervenuta in redazione il 10 luglio 1961.
Indirizzo dell'A.: Jadabpur University, Calcutta (India).

[1]) CHAKRABARTI S. C., *On Higher Differences*, Note IV. Rend. Sem. Padova. XXX (1960) 309-15.

Thus

$$\begin{pmatrix} 1, & a^4 \\ & a^2 \end{pmatrix}_2 = a + a^3 + 2a^4 + a^5 + a^7$$

where the series is 1, a, a^3, a^4.

We take $\begin{pmatrix} 1, & a^n \\ & a^r \end{pmatrix}_0 = 1$ and $\begin{pmatrix} 1, & a^n \\ & a^r \end{pmatrix}_p = 0$, if $p > n$ or negative.

(*b*). $\begin{pmatrix} 1, & a^n \\ a^r, & a^s \end{pmatrix}_p$ = sum of the products of the elements, taken p at a time, of the series

$$1\,, \quad a\,, \quad a^2\,, \quad \ldots \quad a^n$$

in which the elements from a^r to a^s are absent. $r < s \leqslant n$

Thus

$$\begin{pmatrix} 1, & a^6 \\ a^2, & a^4 \end{pmatrix}_3 = a^6 + a^7 + a^{11} + a^{12}$$

the series here is 1, a, a^5, a^6.

We take

$$\begin{pmatrix} 1, & a^n \\ a^r, & a^s \end{pmatrix}_0 = 1 \quad \text{and} \quad \begin{pmatrix} 1, & a^n \\ a^r, & a^s \end{pmatrix}_p = 0$$

if $p > n + r - s$ or negative

(*c*). ${}^nN_p$ = sum of the products of the first n natural numbers taken p at a time.

[It is to be noted that in this paper, we generally deal with integers only, if not otherwise stated]

3. Lemma. *If*

$$\lambda_3 = \begin{vmatrix} a & a^2 & a^3 \\ a^2 & a^4 & a^6 \\ a^3 & a^6 & a^9 \end{vmatrix}_3$$

then similarly formed

(1) $$\lambda_n = (a^{n-1})_{n-1}(a^{n-2})_{n-2} \cdots (a^1)_1 a^{(1/6)n(n+1)(n+2)}$$

THEOREM. *If* $(a^n)_{m,r}$ *denotes the determinant of the n th order obtained from* λ_{n+1} *by deleting the m th column and the r th row, then*

$$(a^n)_{m,r} = {}^nS_{r-1}\lambda_n \binom{1,\ a^n}{a^{n-r-1}}_{m-1} a^l \tag{2}$$

where

$$l = (n+1)(n-r-m+3) - 1$$

4. Some important results: $fk \leqslant n$, then

$$\sum_{p=0}^{k} (-)^p \binom{1,\ a^n}{a^{n-p}}_k {}^kS_p = (-)^k \frac{a^{kn}}{{}^{k+1}S_{k+1}} \cdot (a^k)_k \tag{3} \qquad (i)$$

$$\sum_{p=0}^{n} (-)^p \binom{1,\ a^n}{a^{n-p}}_k {}^nS_p = 0 \tag{4} \qquad (ii)$$

[Obtain the first element of the r th order of differences (multipliers; 1, a, a^2 etc) from the series in which $u_p = \binom{1,\ a^n}{a^{n-p}}_k$, $p = 0, 1$ etc with the help of the selfevident formulæ like

$$\binom{1,\ a^n}{a^n}_k = \binom{1,\ a^n}{a^{n-1},\ a^n}_k + a^{n-1}\binom{1,\ a^n}{a^{n-1},\ a^n}_{k-1} \tag{5}$$

From this element follow readily (3) and (4)]

$$\sum_{p=0}^{n-k} (-)^p \frac{1}{a^{(n-k)p}} \binom{1,\ a^n}{a^{n-p}}_k {}^{n-k}S_p = \frac{a^{nk}(a^{n-k})_{n-k}}{{}^{n+1}S_{n+1}} \tag{6} \qquad (iii)$$

$$\sum_{p=0}^{n} (-)^p \frac{1}{a^{np}} \binom{1,\ a^n}{a^{n-p}}_k {}^nS_p = 0 \tag{7} \qquad (iv)$$

Find the first element of the r th order of differences from the series $u_p = \dfrac{1}{a^{p(n+1)}}\binom{1,\ a^n}{a^{n-p}}_k$ (multipliers: a^n, a^{n-1}, a^{n-2}, ...), with the help of formulæ like (5) and then we have

$$\text{(8)} \qquad \sum_{p=0}^{r} (-)^p \frac{1}{a^{p(n+1)}} \binom{1,\, a^n}{a^{n-p}}_k {}^rR_p = \frac{(a^r)_r}{{}^{r+1}S_{r+1}} \binom{1,\, a^n}{a^{n-r},\, a^n}_k$$

where ${}^rR_p$ = sum of the products of r factors a^n, a^{n-1}, ... a^{n-r+1} taken p at a time.

Since ${}^rR_p = a^{p(n-r+1)}\, {}^rS_p$, (8) reduces to

$$\text{(9)} \qquad \sum_{p=0}^{r} (-)^p \frac{1}{a^{pr}} \binom{1,\, a^n}{a^{n-p}}_k {}^rS_p = \frac{(a^r)_r}{{}^{r+1}S_{r+1}} \binom{1,\, a^n}{a^{n-r},\, a^n}_k$$

From (9) readily follow (6) and (7).

$$\text{(10)} \qquad \sum_{p=0}^{n} (-)^p \frac{1}{a^{(n+1)p}} \binom{1,\, a^n}{a^{n-p}}_k {}^nS_p = \frac{(a^{n+1})_{n-k} (a^n)_k\, {}^{k+1}S_{k+1}}{a^n\; {}^{n+1}S_{n+1}} \qquad (v)$$

5. - Theorem.

$$\text{(11)} \quad {}^nN_r = \begin{vmatrix} \frac{\Phi(n,\, n+1)}{n!} & 1 & & \\ \frac{\Phi(n-1,\, n+1)}{(n-1)!} & \frac{\Phi(n-1,\, n)}{(n-1)!} & 1 & \\ \cdots & \cdots & \cdots & \\ \frac{\Phi(n-r+1,\, n+1)}{(n-r+1)!} & \frac{\Phi(n-r+1,\, n)}{(n-r+1)!} & \cdots & \frac{\Phi(n-r+1,\, n-r+2)}{(n-r+1)!} \end{vmatrix}_r$$

Here the recurrence formula is

$$\text{(12)} \qquad \sum_{p=0}^{k-r} (-)^p \Phi(r,\, k-p)\, {}^nN_p = 0$$

$K7r$ and $r \leqslant n \leqslant k-1$ (See art. 6, Note iv)

[In (12), put $k = n + 1$ and $r = n,\, n-1,\, n-2$ etc. Then eliminate]

Note (*i*): $\Phi(r, n)$ is divisible by $\Phi(r, r)$

Note (*ii*): A table for the values of $\Phi(r, n)$ may be easily obtained by the formula

$$(13) \qquad \Phi(r, n) = r\Phi(r, n-1) + r\Phi(r-1, n-1)$$

To prove this formula, write out the two terms on the right side in full and then apply the identities

$${}^{r}C_k - {}^{r-1}C_{k-1} = {}^{r-1}C_k \quad \text{and} \quad {}^{r-1}C_k r(r-k)^{n-1} = {}^{r}C_k(r-k)^n]$$

Note (*iii*): A table for the values of ${}^nN_r$ may be obtained by (11), Art. 5 or by the formula

$${}^nN_r = {}^{n-1}N_r + n\,{}^{n-1}N_{r-1}$$

N.B.: The reader is advised to obtain the two tables mentioned above before embarking on the study of the subject.

6. THEOREM:

$$(14) \qquad \sum_{p=0}^{n=r} (-)^p \frac{\Phi(r+p, n)}{(r+p)^{(p)}}\, {}^{r+p-1}N_p = 0$$

[This theorem is also the recurrence formula for the result (11), Art. 5. To show this, put $n = r+1, r+2, \dots r+m$ in (14) and then eliminating NS', transform the resulting recurrent.]

7. - THEOREM.

If

$$u_x = \sum_{p=1}^{n} \alpha_p x^p$$

then

$$(15) \quad \frac{\alpha^k u_0}{\alpha x^k} = (-)^{k-1} \frac{\alpha_k k!}{{}^{k+1}S_{k+1} x^k} \cdot$$

$$\cdot \left[x^k + \frac{a^{k-1}}{(a^{k-1})_{k-1}} \sum_{r=1}^{k} \left\{ \sum_{p=0}^{k-r} (-)^{p-r+1}\, {}^{p+r}O_r^{-} \frac{{}^kS_k\, {}^{k-1}S_{p+r-1}}{a^{(k-1)(r+p)}} A_r x^k \right\} \right].$$

$$1 \leqslant k \leqslant n$$

Proof.: By (27), Note I and by (4) Note II [2]), we have

$$\sum_{p=0}^{l} {}^{l}O_p^- A_p u_x = e^{a^l(d/dx)} u_0 \tag{16}$$

If we put $l = 1, 2, \ldots n$ in (16), we have n equations from which by eliminating, we have

$$(-)^{k-1} \frac{x^k}{k!} \frac{d^k u_0}{dx^k} \lambda_n = \text{a determinant which develops into}$$

$$\{\sum_{p=0}^{n-1} (-)^p (a^{n-1})_{k,p+1}\}(u_x - u_0) + \sum_{r=1}^{n} \{\sum_{p=0}^{n-1} (-)^{p1+p} O_r^- (a^{n-1})_{k,p+1} A_r u_x\}$$

Hence by (2), (10) and (1),

$$\frac{d^k u_0}{dx^k} = (-)^{k-1} \frac{k!}{x^k} \left[\frac{{}^n S_k}{a^{nk}} (u_x - u_0) + \frac{{}^{n+1}S_{n+1}}{a^{nk+1}(a^{n-1})_{n-1}} \cdot \right. \tag{17}$$

$$\left. \cdot \sum_{r=1}^{n} \left\{ \sum_{p=0}^{n-r} (-)^{p-r+1} \, {}^{p+r}O_r^- \frac{1}{a^{n(p+r-1)}} \begin{pmatrix} 1, a^{n-1} \\ a^{n-r-p} \end{pmatrix}_{k-1} {}^{n-1}S_{p+r-1} A_r u_x \right\} \right]$$

The right side of (17) vanishes when $u_x = \alpha_1 x \; \alpha_2 x^2, \ldots \ldots \alpha_n x^n$ except when $u_x = \alpha_k x^k$. Hence putting $u_x = \alpha_k x^k$ and hence $n = k$ in (17), we get (15).

8. Theories of Higher Differences may also be applied to Interpolations and Statistics. One method is given below.

THEOREM: *If the values of u_x, a rational integral function of x of degree n in x, be given, corresponding to $x = 1, a, a^2, \ldots a^n$ then its approximate general expression is given by*

$$u_x = \sum_{p=0}^{n} \frac{(x)_p}{(a^p)_p} A_p u_1 \tag{18}$$

(Vide Art. 9, Note 1).

[2]) CHAKRABARTI S. C., *On Higher Defferences*, Note 1 and Note 11. Rend. Sem. Matem. Padova, XXIII, 1954, 255-76.

Notice here that the values of x are in G. P. and when $n+1$ values of u_x are given, u_x is assumed to be of degree n in x.

Ex.: Given

$$\log 3\cdot 1411 = \cdot 4970818\,, \quad \log 3\cdot 1412 = \cdot 4970956\,,$$

$$\log 3\cdot 1414 = \cdot 4971232 \quad \text{and} \quad \log 3\cdot 1418 = \cdot 4971785$$

find $\log 3\cdot 14159$ approximately.

Sol.: Omit the decimal points and take the number 31410 for the origin. The distances of the terms of the series 31411, 31412 etc from the origin are 1, 2, 4 and 8 which are in G. P. common ratio being 2 ie $a = 2$.

Let the logs of the terms of the given series be denoted by u_1, u_2, u_4 and u_8, u_x being the general expression. Here $x = 5\cdot 9$ which is the distance of 314159 from the origin. There are 4 terms given, so $n = 3$. The table for numbers and differences is as follows

	u_1	u_2	u_4	u_8
	4970818	4970956	4971232	4971785
A_1	138	276	553	
A_2	0	1		
A_3	1			

The first column gives the values of u_1 and its differences upto A_3u_1. Now by (18)

$$u_x = u_1 + \frac{x-1}{a-1}A_1u_1 + \frac{(x-1)\left(\frac{x}{a}-1\right)}{(a^2-1)(a-1)}A_2u_1 +$$

$$+ \frac{(x-1)\left(\frac{x}{a}-1\right)\left(\frac{x}{a^2}-1\right)}{(a^3-1)(a^2-1)(a-1)}A_3u_1 = 4971495$$

$\therefore \log 3 \cdot 14159 = \cdot 4971495$ which is correct to the last place of decimals.

Note: If the number sought lies between the r th and $r+1$ th of the given numbers, u_x is of the r th degree in x.