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ON A SERIES OF CARLITZ INVOLVING ULTRASPHERICAL POLYNOMIALS

Nota () di S. K. CHATTERJEA (a Calcutta)*

1. - In a recent paper¹⁾, Bhonsle obtained some relations involving Legendre polynomials with the help of a finite summation formula²⁾, given by Rainville. The main purpose of this paper is also to deduce certain relations involving ultraspherical polynomials with the help of the corresponding finite summation formula³⁾, of ultraspherical polynomials, given by Carlitz. Incidentally two new finite summation formulae involving the polynomials encountered by Karle and the Laguerre polynomials of order zero, and a few consequences of these formulae are pointed out.

2. From the formula (3, p. 132) of Carlitz, we derive⁴⁾;

$$(2.1) \quad 2^{n/2}(1+x)^{n/2}P_n^{\lambda+1/2}\left(\sqrt{\frac{1+x}{2}}\right) = \sum_{k=0}^n \binom{n+2\lambda}{n-k} P_k^{\lambda+1/2}(x)$$

(*) Pervenuta in redazione il 10 maggio 1961.

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¹⁾ BHONSLE B. R.: Proc. Amer. Math. Soc., 8 (1957), 10-14.

²⁾ RAINVILLE E. D.: Bull. Amer. Math. Soc., 51 (1945), 268-271.

³⁾ CARLITZ L.: Bull. Cal. Math. Soc., 51 (1959), 132-133.

⁴⁾ CHATTERJEA S. K.: *Notes on a formula of Carlitz*. Rend. Sem. Mat. Univ. Padova, 31 (1961), 243-248.

From (2.1) and from the orthogonal property we get (4, p. 247):

$$\begin{aligned}
 & \int_{-1}^1 (1+x)^{n/2} (1-x^2)^\lambda P_n^{\lambda+1/2} \left(\sqrt{\frac{1+x}{2}} \right) P_r^{\lambda+1/2}(x) dx \\
 (2.2) \quad & = 0; \quad (r > n) \\
 & = 2^{-n/2} \binom{n+2\lambda}{n-r} \frac{(2\lambda+1)_r \Gamma\left(\frac{1}{2}\right) \Gamma(\lambda+1)}{r! \left(r+\lambda+\frac{1}{2}\right) \Gamma\left(\lambda+\frac{1}{2}\right)}; \\
 & \hspace{15em} (0 \leq r \leq n) \quad (\lambda > -1)
 \end{aligned}$$

Using $\lambda = 0$ in (2.2) we obtain the formula of Bhonsle (1, p. 10):

$$\begin{aligned}
 & \int_{-1}^1 (1+x)^{n/2} P_n \left(\sqrt{\frac{1+x}{2}} \right) P_r(x) dx \\
 (2.3) \quad & = 0; \quad (r > n) \\
 & = 2^{1-n/2} \binom{n}{r} / (2r+1); \hspace{10em} (0 \leq r \leq n)
 \end{aligned}$$

Following the method of Carlitz, we have recently proved (4, p. 243) that

$$\begin{aligned}
 (2.4) \quad & (\cot \alpha \sin \beta)^n \Phi_n(\tan \alpha) = \\
 & = \sum_{r=0}^n \binom{n}{r} \left[\frac{\sin(\beta-\alpha)}{\sin \alpha} \right]^r \cos^{n-r} \beta \Phi_{n-r}(\tan \beta)
 \end{aligned}$$

$$\begin{aligned}
 (2.5) \quad & (\cot \alpha \sin \beta)^n L_n(\tan \alpha) = \\
 & = \sum_{r=0}^n \binom{n}{r} \left[\frac{\sin(\beta-\alpha)}{\sin \alpha} \right]^r \cos^{n-r} \beta L_{n-r}(\tan \beta)
 \end{aligned}$$

$$\alpha \neq (2n+1) \frac{\pi}{2}, \quad \beta \neq (2n+1) \frac{\pi}{2}.$$

where $\Phi_n(x)$ denotes the polynomial of degree n , encountered by Karle and $L_n(x)$ denotes the Laguerre polynomial of degree n and of order zero.

Putting $\beta = 2\alpha$, we derive

$$(2.6) \quad (1 + \cos 2\alpha)^n \Phi_n(\tan \alpha) = \sum_{r=0}^n \binom{n}{r} (\cos 2\alpha)^{n-r} \Phi_{n-r}(\tan 2\alpha)$$

$$(2.7) \quad (1 + \cos 2\alpha)^n L_n(\tan \alpha) = \sum_{r=0}^n \binom{n}{r} (\cos 2\alpha)^{n-r} L_{n-r}(\tan 2\alpha)$$

But if we put $\alpha = 2\beta$, we can derive

$$(2.8) \quad (\cos 2\beta)^n \Phi_n(\tan 2\beta) = \sum_{r=0}^n (-)^r \binom{n}{r} (1 + \cos 2\beta)^{n-r} \Phi_{n-r}(\tan \beta)$$

$$(2.9) \quad (\cos 2\beta)^n L_n(\tan 2\beta) = \sum_{r=0}^n (-)^r \binom{n}{r} (1 + \cos 2\beta)^{n-r} L_{n-r}(\tan \beta)$$

Now if $\tan \beta = x$, we have from (2.8) and (2.9):

$$(2.10) \quad \left(\frac{1-x^2}{2}\right)^n \Phi_n\left(\frac{2x}{1-x^2}\right) = \sum_{r=0}^n (-)^r \binom{n}{r} \left(\frac{1+x^2}{2}\right)^r \Phi_{n-r}(x)$$

$$(2.11) \quad \left(\frac{1-x^2}{2}\right)^n L_n\left(\frac{2x}{1-x^2}\right) = \sum_{r=0}^n (-)^r \binom{n}{r} \left(\frac{1+x^2}{2}\right)^r L_{n-r}(x)$$

The last result (2.11) can be easily verified from the well-known result ⁵⁾, of Laguerre polynomials:

$$L_n^{(\alpha)}(\lambda x) = \sum_{m=0}^n \binom{n+\alpha}{m} \lambda^{n-m} (1-\lambda)^m L_{n-m}^{(\alpha)}(x)$$

⁵⁾ ERDELYI A.: Higher Transcendental Functions. Vol. 2 (1953), 192.

In (4, p 246) we have also shown that

$$(2.12) \quad (1+x)^n \Phi_n \left(\sqrt{\frac{1-x}{1+x}} \right) = \left(\frac{1}{\pi} \right) \int_0^\pi [(1+x) + \sqrt{1-x^2} \cos \theta]^n d\theta,$$

which has a striking resemblance to Laplace's first integral for $P_n(x)$.

Proceeding exactly in the same manner we derive from (2.10):

$$(2.13) \quad (1-x^2)^n \Phi_n \left(\frac{2x}{1-x^2} \right) = \frac{(-)^n}{\pi} \int_0^\pi (x^2 - 2x \cos \theta - 1)^n d\theta = \frac{1}{\pi} \int_0^\pi (1 + 2x \cos \theta - x^2)^n d\theta$$

3. Now we use certain generating functions for $P_n^{\lambda+1/2}(x)$ to evaluate some integrals involving ultraspherical polynomials. First we notice that *);

$$(3.1) \quad e^{xt} {}_0F_1 \left[\begin{matrix} - \\ \lambda + 1; \end{matrix} \frac{t^2(x^2 - 1)}{4} \right] = \sum_{n=0}^\infty \frac{P_n^{\lambda+1/2}(x) t^n}{(2\lambda + 1)_n}$$

Now we know that for $\lambda > -\frac{1}{2}$ and $-1 \leq x \leq 1$, $|P_n^{\lambda+1/2}(x)| \leq \frac{(2\lambda + 1)_n}{n!}$; and therefore the series in (3.1) is absolutely and uniformly convergent for x in $-1 \leq x \leq 1$. Thus we derive

$$(3.2) \quad \int_{-1}^1 (1+x)^{n/2} (1-x^2)^\lambda e^{xt} \cdot {}_0F_1 \left[\begin{matrix} - \\ \lambda + 1; \end{matrix} \frac{t^2(x^2 - 1)}{4} \right] P_n^{\lambda+1/2} \left(\sqrt{\frac{1+x}{2}} \right) dx$$

*) RAINVILLE E. D.: Special Functions (1960), 278.

$$= \sum_{k=0}^n \binom{n+2\lambda}{n-k} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(\lambda+1)}{2^{n/2} k! \left(k + \lambda + \frac{1}{2}\right) \Gamma\left(\lambda + \frac{1}{2}\right)} t^k; \quad \left(\lambda > -\frac{1}{2}\right)$$

Putting $\lambda = 0$ in (3.2) we obtain

$$(3.3) \quad \int_{-1}^1 (1+x)^{n/2} e^{xt} {}_0F_1 \left[\begin{matrix} -; \\ 1; \end{matrix} \frac{t^2(x^2-1)}{4} \right] P_n \left(\sqrt{\frac{1+x}{2}} \right) dx \\ = \sum_{k=0}^n \binom{n}{k} \frac{t^k}{2^{n/2-1} k! (2k+1)} = 2^{1-n/2} {}_2F_2 \left[\begin{matrix} -n, 1/2; \\ 1, 3/2; \end{matrix} -t \right]$$

Again since

$${}_0F_1 \left(-; 1; \frac{1}{4} t^2(x^2-1) \right) = J_0(t \sqrt{1-x^2}),$$

we have from (3.3):

$$(3.3') \quad \int_{-1}^1 (1+x)^{n/2} e^{xt} J_0(t \sqrt{1-x^2}) P_n \left(\sqrt{\frac{1+x}{2}} \right) dx \\ = 2^{1-n/2} {}_2F_2 \left[\begin{matrix} -n, 1/2; \\ 1, 3/2; \end{matrix} -t \right]$$

Next we use Bateman's generating function (6, p. 280):

$$(3.4) \quad {}_0F_1 \left[\begin{matrix} -; \\ \lambda+1; \end{matrix} \frac{t(x-1)}{2} \right] \cdot {}_0F_1 \left[\begin{matrix} -; \\ \lambda+1; \end{matrix} \frac{t(x+1)}{2} \right] \\ = \sum_{n=0}^{\infty} \frac{P_n^{\lambda+1/2}(x) t^n}{(2\lambda+1)_n (\lambda+1)_n}.$$

The series in (3.4) is absolutely and uniformly convergent for x in $-1 \leq x \leq 1$, when t is finite and $\lambda > -\frac{1}{2}$, and thus

we get

$$\begin{aligned}
 (3.5) \quad & \int_{-1}^1 (1+x)^{n/2} (1-x^2)^\lambda {}_0F_1 \left[\begin{matrix} - \\ \lambda+1 \end{matrix}; \frac{t(x-1)}{2} \right] \\
 & \cdot {}_0F_1 \left[\begin{matrix} - \\ \lambda+1 \end{matrix}; \frac{t(x+1)}{2} \right] P_n^{\lambda+1/2} \left(\sqrt{\frac{1+x}{2}} \right) dx \\
 & = \sum_{k=0}^n \binom{n+2\lambda}{n-k} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(\lambda+1)}{2^{n/2} k! (\lambda+1)_k \left(k+\lambda+\frac{1}{2}\right) \Gamma\left(\lambda+\frac{1}{2}\right)} t^k; \left(\lambda > -\frac{1}{2}\right).
 \end{aligned}$$

Using $\lambda = 0$ in (3.5) we obtain

$$\begin{aligned}
 (3.6) \quad & \int_{-1}^1 (1+x)^{n/2} {}_0F_1 \left[\begin{matrix} - \\ 1 \end{matrix}; \frac{t(x-1)}{2} \right] \\
 & \cdot {}_0F_1 \left[\begin{matrix} - \\ 1 \end{matrix}; \frac{t(x+1)}{2} \right] P_n \left(\sqrt{\frac{1+x}{2}} \right) dx \\
 & = \sum_{k=0}^n \binom{n}{k} \frac{t^k}{2^{n/2-1} (k!)^2 (2k+1)} = 2^{1-n/2} {}_2F_3 \left[\begin{matrix} -n, 1/2; \\ 1, 1, 3/2; \end{matrix} -t \right]
 \end{aligned}$$

4. We have also (6, p 284)

$$\begin{aligned}
 (4.1) \quad & P_n^\lambda (1-2y^2) = \\
 & = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\left(\lambda - \frac{1}{2}\right)_k (\lambda)_{n-k} (2n+1-4k)}{k! \left(\frac{3}{2}\right)_{n-k}} P_{n-2k} (1-2y^2)
 \end{aligned}$$

We again have⁷⁾;

$$(4.2) \quad \int_0^1 P_n (1-2y^2) J_0(yx) y dy = \frac{1}{x} J_{2n+1}(x)$$

⁷⁾ MITRA: Proc. Edin. Math. Soc. (2), vol. 4, Part. III, 111.

Thus using (4.1) with (4.2) we obtain

$$(4.3) \quad \int_0^1 P_n^\lambda(1 - 2y^2) J_0(yx) y dy = \frac{1}{x} \sum_{k=0}^{[n/2]} \frac{\left(\lambda - \frac{1}{2}\right)_k (\lambda)_{n-k} (2n + 1 - 4k)}{k! \left(\frac{3}{2}\right)_{n-k}} J_{2n+1-4k}(x)$$

Next returning to (2.1) we have, on using $\sqrt{\frac{1+x}{2}} = y$,

$$(4.4) \quad 2^n y^n P_n^\lambda(y) = \sum_{k=0}^n (-)^k \binom{n + 2\lambda - 1}{n - k} P_k^\lambda(1 - 2y^2)$$

Thus using (4.3) with (4.4) we derive

$$(4.5) \quad \int_0^1 y^{n+1} J_0(yx) P_n^\lambda(y) dy = \frac{1}{2^n x} \sum_{i=0}^n (-)^i \binom{n + 2\lambda - 1}{n - i} \sum_{k=0}^{[n/2]} \frac{\left(\lambda - \frac{1}{2}\right)_k (\lambda)_{i-k} (2i + 1 - 4k)}{k! \left(\frac{3}{2}\right)_{i-k}} J_{2i+1-4k}(x)$$

Putting $\lambda = \frac{1}{2}$ and defining that $(\alpha)_k = 1$, if $\alpha = k = 0$; and $(\alpha)_k = 0$, if $\alpha = 0, k \neq 0$; we get the formula of Bhonsle (1, p 14):

$$(4.6) \quad \int_0^1 y^{n+1} J_0(yx) P_n(y) dy = \frac{1}{2^n x} \sum_{k=0}^n (-)^k \binom{n}{k} J_{2k+1}(x).$$

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