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On extension of a given finitely additive field-valued, non negative measure, on a finitely additive Boolean tribe, to another tribe more ample

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ON EXTENSION OF A GIVEN FINITELY ADDITIVE FIELD - VALUED, NON NEGATIVE MEASURE, ON A FINITELY ADDITIVE BOOLEAN TRIBE, TO ANOTHER TRIBE MORE AMPLE [1])

Nota (*) di OTTON MARTIN NIKODÝM (*a Gambier, Ohio U.S.A.*)

1. - The theory of measure in a Boolean tribe enables us to investigate the structure of tribes, especially when the measure is real-number valued and efective, i. e. when it vanishes on the null-element only. Now, it is known that there are finitely additive tribes which do not admit any finitely additive number valued and effective measure [2]), so the idea of introducing measures whose values are taken from a general linearly ordered (algebraic) field, which may be not archimedean, seems to be promising. The present paper deals with such measures and gives a (positive) solution of the problem of extending a non-negative measure from a given tribe to another, wider one, containing it. There is a method. available in the literature, which is adequate to deal with the problem of extension, viz. the S. Banach's method for extension linear functionals (1). This method was applied with success by Banach himself and by other authors. We shall also apply it in our problem.

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[1]) Composed under the grant from the National Science Foundation (U. S. A).

[2]) An example of such a tribe was kindly communicated by MR. J. DIXMIER in a letter to me.

2. - Though the idea is known and rather simple, nevertheless to apply it in our problem many auxiliary items are needed, because the behaviour of general fields differs from that of the field of real or complex numbers. General smaller or larger auxiliary theories shall be developped to master the situation, and since our arguments must have a rather subtle character, clarification of some known basic notions will be also needed. Some of the mentioned theories will serve as a tool not only to solve our problem, but they will be also useful for the subsequent papers by the author. Therefore the auxiliaries are exposed in a more general way, than needed for the proof of the result the present paper is aiming at, and with more details, just for purpose of future references. Some of these auxiliary topics imply new methods which, we believe, will be useful in the capacity of a new technique suitable for various branches of mathematics, some ones clarifying known but confused topics.

3. - An abstract of the present paper has been published in four C. R.-notes (2). However, the present paper contains arguments which differ slightly from those in these notes, and in addition to that, it constitutes a much more explicite setting. The auxiliaries, mentioned above, are exposed in six first chapters entitled: § I. Endings and their ordering (there are many sections labelled § I A, B, C, D, E, F, G, H, K). Their purpose is to overcome the difficulty in linearly ordered non-archimedian fields which consists of the fact that a bounded set may not admit a supremum and infimum. § 2 gives a precise setting of partitions in a Boolean tribe. § 3 deals with « aggregates » and constitutes a particular case (adapted to the main purpose of the present paper) of the general theory sketched in the C. R. notes by the author on « functionoids » (3). § 4 introduces the exterior and interior ending-valued Jordan measure in Boolean tribes. § 5 deals with a kind of linear functionals and their ending-valued norm. The aim of § 6 is to prove the possibility of extension of linearly ordered fields (which may be not archimedian), by placing a new element in a

gap which may be a true or not true Dedekindian section. The proof of the corresponding existence–theorem is relied on the known theory of reel-closed fields (4), which in turn is based on the Steinitz–theorem (5) on the existence of the algebraic closure of a field, (see also (6)).

Now, the last theorem has not yet been proved correctly in the spirit of theory of types (7), and it is known that mere considering of a class whose elements have not the some logical type invariably leads to contradiction [3]).

Our § 6 supplies a correct proof which is even simplier than the original proof by Steinitz. The necessity of discrimination of logical types does not allow any « identification » of isomorphic elements, neither a merely formalistic approach to polynomials and operations on them. In addition to that the notion of single element extension of a field, which is rather confused in the literature, needed clarification. Having this all in mind, it seemed necessary to the author, to give a thorough refoundation of the algebra of fields. Especially new definitions were necessary which separated some similar notions from one another. Concerning ordinals and cordinals. we refer to the abstract (8) by the author and Mrs. Stanislawa Nikodým, where these notions are defined in accordance with the necessity of discrimination of types. The definitions in the auxiliary chapter, are stated carefully, but many theorems, whose proofs do not require special technique, are stated without proof, in the belief that the reader will be able to supply them. Especially § 1 on « endings » contains a great quantity of « small » theorems, whose proofs, sometimes not too short. if given explicitely (as they are in the manuscript of the author), would increase many times the volume of the present article. Also proofs of the known theorems in the theory of fields are omitted.

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[3]) Eg. If we shall take into consideration the set whose only elements are the number 1 and the class composed of the numbers 1 and 2, a contradiction will result. It will however not result, if the class is composed of the set composed of 1 and of the set composed of 1 and 2.

5. Preliminaries and notations. - Since our topic is rather subtle, and since il is generally true, that in abstract things nothing is « obvious », the following preliminaries will be in order — to avoid misunderstanding. Indeed, mathematical understanding of a topic differs from the intuitive common place-understanding.

We shall deal with various theories, as e. g. with chains and tribes. In each theory there is notion of *equality of its elements*, with respect to which the operations and relations considered are invariant, and which conditions the notion of sets of elements. that of correspondences and the notion of uniqueness of the element, satisfying a given condition. If the theory is abstract, the notion of equality is axiomatized or defined, and if the theory is constructed, the notion of equality is defined or taken from another theory. We shall call it *equality governing the theory.* If A denotes the theory, its equality will be denoted by $\stackrel{A}{=}$, though the letter may be omitted if no misunderstanding is to be feared of.

A set E of elements of the theory A must be $\stackrel{A}{=}$-invariant, which means that if $a \in E$ and $a' \stackrel{A}{=} a$, then $a' \in E$. E. g. in the theory of measurable subsets of $(0, 1)$ the notion of their governing equality is «equality almost everywhere», «$\doteq$»; hence a class of sets must be $\doteq$-invariant, and so must be the operations on them. e. g. if $a \doteq a'$, $b \doteq b'$, $c \doteq c'$, and $a \cup b \doteq c$, then $a' \cup b' \doteq c'$. We say that a is the *$\doteq$-unique element satisfying a condition* $\varphi(x)$ (invariant with respect to $\doteq$), whenever $\varphi(a)$, and if $\varphi(a)$ and $\varphi(a')$, then $a \doteq a'$.

Let M be a not empty subset of a set N, (taken from a theory) on which « $=$ » is the governing equatity. If we do not analyse this equality. it may be termed « identity ». Suppose that. for some purpose, we change the notion of equality, on M, into another one $\stackrel{M}{=}$, satisfying, of course, the conditions of symmetry, transitivity and reflexiveness [4]. This is e. g. the case when the elements of N are sets of

[4]) New equalities are termed « equivalences », though, really, there is no logical difference between them and equalities.

real numbers and M constitutes the measurable subsets of $\langle 0, 1 \rangle$, $\overset{M}{=}$ being the equality « almost everywhere ». We must discriminate between N-sets and M-sets. Instead of the equality $\overset{M}{=}$ we can consider equivalence classes which are N-sets and operate on them through representatives. However, many times it is more comfortable to consider the $\overset{M}{=}$-equality, than identity of the corresponding equivalence classes; e. g. in abstract algebra, in the theory of Hilbert space whose vectors are square-summable functions, and in the theory of ordinary fractions.

In the literature the notion of equality is usually, with few exceptions, not taken into account. Its importance in the theory of Boolean tribes is, however, emphasized by the author (10), (11). E. g. the difference between free vectors, gliding vectors and bound vectors in the euclidean geometry derives from different kinds of equalities.

6. - The « relations » of Russell and Whitehead (7) will be termed *correspondences* or mappings. We consider correspondences, in accordance with these authors, as belonging to propositional functions of two variables, and not, as is now fashionable, as classes of ordered couples. A couple (a, A) will be understood as the correspondence $\{x, y \mid x =' a, y ='' A\}$. The domain of the correspondence R will be denoted by $(\!| R$ and the range by $|\!) R$. They may be of different types. If the type is the same, $(\!| R \cup |\!) R$ is meaningful: it will be termed *campus* of R and denoted by $(\!) R$. The correspondence R must be equality invariant in its domain and equality invariant in its range (where these equalities may have a different character). This means that if aRA, $a =' a_1$, $A ='' A_1$, then $a_1 R A_1$. We shall say R is $=' =''$-*invariant*. Instead of aRA we shall also write $a \overset{R}{\rightarrow} A$. Isomorphisms, as correspondences shall fit this requirement. By a *function* we shall understand a pluri-one correspondence. If $E \subseteq (\!| R$, by $E \upharpoonright R$ we shall understand the correspondence $\{x, y \mid x \in E, xRy\}$, i. e. the correspondence R restricted to A, (7).

If speaking of isomorphisms or homomorphisms we shall say which relations and operations should be maintained by them. E. g. We shall say « order-addition-multiplication -zero-and unit-isomorphism », if these items are preserved through it. Set-operations will be denoted by Bourbaki -symbols $\cup$, $\cap$, —, $\bigcup$, $\bigcap$ and the complementary by co. We avoid « empty » unions and intersections for reasons explained in (10). There will be, in § 1E, some other operations on sets: they will be defined there. If the elements of the sets E, F have the type α, β respectively, and R is a correspondence with $\mathrm{C} R = E$, $\mathrm{D} R = F$, the type of R will be denoted by $\{\alpha; \beta\}$. The type of a class of elements of the type α will be deneted by cl α.

7. - If a set A is organized into a structure by introducing some operations or relations, the structure will be denoted by (A) or $[A]$, or even $[(A)]$, for more clarity, if needed. The following structures will be considered:

Orderings (usually referred to as « partial ordering » (12), (13). An *ordering* is defined as a not empty correspondence R, (abstract or constructed) mainly denoted by « $\leq$ », such that 1) if $a \in \mathrm{CD} R$, then aRa, 2) if « aRb, bRc » then aRc, 3) if a, $b \in \mathrm{CD} R$ then « $a = b$ » is equivalent to « aRb and bRa ». By a *chain* (linear ordering) we understand an ordering R which satisfies the additional condition: if a, $b \in \mathrm{CD} R$, then either aRb or bRa. By a *tribe* [5] (Boolean tribe, Boolean lattice, Boolean algebra), we shall understand an ordering which is a distributive and complementary lattice, (12), (13). A tribe B is said to be *not trivial* whenever its unit 1_B differs from its zero 0_B. The elements of a tribe will be termed *elements* or *somata* [6]. *Somatic operations* will be denoted by $+$, $\cdot$, co, Σ, Π, —, the algebraic addition $(a - b) + (b - a)$ which organizes (together with the multiplication) the tribe into a Stone's-ring (17) will be denoted by $\dot{+}$. A *measure on B* will be supposed to be equality-invariant. If the values of a measure are

[5]) Term borrowed from (16).

[6]) Term borrowed from (15).

taken from a linearly ordered field, and are non negative, the measure will be termed *effective* whenever it vanishes only on 0_B.

8. - The term *ring* and *field* are understood in the usual way with the exception that if the structure is confined to a single element (which must be the zero), we shall call the ring or field *trivial*. We shall consider abelian groups and semigroups only. Their « identity » wil be denoted by 0 (with an index, if needed). and the group-operation by $+$. A structure A with operations $\odot, \ldots$ and correspondences $\backsim, \ldots$ will be said to be $(\odot, \ldots, \backsim, \ldots)$*-genuine substructure of* B whenever, for elements $a, b, c, \ldots$ of A the following are equivalent: (10)

1) $a \odot b \overset{A}{=} c$ and $a \odot b \overset{B}{=} c$, ...

2) $a \backsim b$ and $a \backsim' b$.

If in addition to that the equality $\overset{A}{=}$ is identical with $(A \upharpoonright \overset{B}{=})$ i.e. with the restriction of $\overset{A}{=}$ to B, we shall call A $(\odot, \ldots \backsim, \ldots)$*-genuine strict substructure of* A.

9. - References to former theorems and subsections will be stated as e.g. [§ 3; 4], but in references to the actual section the number, e.g. § 3, will be omitted, e.g. [4]. The numbers in bold parentheses, as **(2)**, refer to the literature at the end of the article.

10. - The main theorem on extension of measure, proved in this paper, is stated at the end of § 8.

§ 1. - Endings and their ordering.

1A. Left endings.

1. - Let (M) be a chain, denoted « $\leq$ ». It will be kept fixed through the whole present § 1. We shall consider only not empty subsets $E, F, \ldots$ of M.

Given E, F, we define $E \cdot\!\leq F$ (also written $F \cdot\!\geq E$) as « for every $y \in F$ there exists $x \in E$ such that $x \leq y$ ». $E \leq\!\cdot F$

(also written $F >\cdot E$) will mean « for every $x \in E$ there existe $y \in F$ such that $x < y$ ».

The following properties hold true: $E \cdot\leq E$. If $E \cdot\leq F$, $F \cdot\leq G$, then $E \cdot\leq G$. For any two not empty sets E, F we have either $E \cdot\leq F$ or $F \cdot\leq E$.

2. - The correspondence $\cdot\leq$ induces a notion of *equality* $E \cdot= F$ defined as « $E \cdot\leq F$ and $F \cdot\leq E$ ». It possesses the formal properties of identity. The notion $\cdot\leq$ is invariant with respect to the equality $\cdot=$; thus the notion $\cdot\leq$ organizes the class of all not empty subsets of M into a chain $(\cdot\leq)$ with $(\cdot=)$ as governing equality. The notion $(\cdot=)$ coincides, for chains, with the notion by J. W. Tukey of coinitial similarity (18).

3. - If $E \subseteq F$, then $F \cdot\leq E$.

4. - If $\{a\}$ is the set composed of the single element a, where $a \in M$, let us agree to write a instead of $\{a\}$, when no ambiguity will be feared of. We have: if $x \in E$. then $E \cdot\leq x$.

5. - *By left ending of* E, $\mathfrak{LE}(E)$, we shall understand the class of all sets F such that $F \cdot= E$. By a *left ending* [7]) we shall understand any $\mathfrak{LE}(E)$ where $E \subseteq M$, $E \neq 0$. Left endings will be denoted by greek letters provided with a star: $^*\alpha$, $^*\beta$, ... If $\mathfrak{LE}(E)$ is denoted by $^*\alpha$, we call E *representative of* $^*\alpha$. Every set belonging to the class $^*\alpha$ is a representative of $^*\alpha$. The set E is a representative of $\mathfrak{LE}(E)$.

6. - If $^*\alpha$, $^*\beta$ are left endings, we define the correspondence $^*\alpha \leq {}^*\beta$ by « there exist representatives E. F of $^*\alpha$, $^*\beta$ respectively, such that $E \cdot\leq F$ ». The notion of $(\leq)$ for left endings does not depend on the choice of representa-

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[7]) This notion is a particular case of J. W. Tukey's « coinitial type » (18).

tives. The following are equivalent: I. $*\alpha \leq *\beta$, II. for all representatives E, F of $*\alpha$, $*\beta$ respectively we have $E \cdot\leq F$.

7. - If $*\alpha$, $*\beta$ are left endings, we define « $*\alpha = *\beta$ » by « $*\alpha \leq *\beta$ and $*\beta \leq *\alpha$ ». The notion of equality of left endings is the identity of classes of sets.

The following are equivalent: I. $*\alpha = *\beta$, II. there exist representatives E, F of $*\alpha$, $*\beta$ respectively, such that $E \cdot= F$, III. for all representatives E, F of $*\alpha$, $*\beta$ respectively we have $E \cdot= F$. IV. The classes $*\alpha$, $*\beta$ of sets coincide.

The notion of equality ($=$) of left endings satisfies all conditions of identity, and the notion ($\leq$) for left endings is invariant with respect to it.

8. - The following properties are true: $*\alpha \leq *\alpha$; if $*\alpha \leq *\beta$ and $*\beta \leq *\gamma$, then $*\alpha \leq *\gamma$. The class of all left endings in (M) is organized into a chain ($\leq$) with ($=$) as governing equality.

9. - If E possesses the minimum a, then $E \cdot= a$. If $a \in M$ and $E \cdot= a$, then a is a minimum in E.

A left ending whose one of the representatives is the set $\{a\}$ composed of a single element a of M, will be termed *point left ending*. To simplify writing we agree to denote such an ending by a.

If E is a representative of $*\alpha$ and E has the minimum a, then $\{a\}$ is a representative of $*\alpha$ and $*\alpha = a$.

If $*\alpha = a$, and E is a representative of $*\alpha$, then E has the minimum a and a is a representative of $*\alpha$.

10. - If $E \neq 0$ is a subset of M, and $a \in M$, then the following are equivalent: I. $a \cdot\leq E$, II. for every $y \in E$ we have $y \leq a$.

If E is a representative of $*\alpha$ and $a \in M$, then the following are equivalent: I. $a \leq *\alpha$, II. for every $x \in E$ we have $a \leq x$.

11. - If $F \neq 0$ is a subset of M, $a \in M$, then the following are equivalent: I. $E \cdot\leq a$, II. there exists $y \in E$ such that

$y \leq a$. If E is a representative of $^*\alpha$, $a \in M$, then the following are equivalent: I. $^*\alpha \leq a$, II. $E \cdot\!\leq a$, III. there exists $e \in E$ such that $e \leq a$, IV. there exists an element p of M such that 1) there exists $e \in E$ with $e \leq p$, and 2) for every $e \in E$ with $e \leq p$ we have $e \leq a$.

12. - If $a, b \in M$, then the following are equivalent: I. $a \cdot\!\leq b$, II. $a \leq b$, and the following are equivalent: I. $a \cdot\!= a$, II. $a = a$.

If E has at least two different points and a is not the minimum of E, $a \in E$, then $E \cdot\!= \{x \mid x \in E,\ x < a\}$.

If E is a representative of $^*a$, $a \in E$, a is not the minimum of E, then the set $E_1 \underset{df}{=} \{x \mid x \in E,\ x < a\}$ is also a representative of $^*\alpha$.

If $E \subseteq M$, $E \neq 0$, $a \in E$, then $E \cdot\!= \{x \mid x \in E,\ x < a\}$.

If E is a representative of $^*\alpha$, $a \in E$, then $E_1 \underset{df}{=} \{x \mid x \in E,\ x < a\}$ is also a representative of $^*\alpha$.

13. - If $^*\alpha$ has a representative composed of the single point a and F is any set, ($F \subseteq M$) with minimum a, then F is a representative of $^*\alpha$.

14. - The following are equivalent:

I. $\mathfrak{LE}(E) \leq \mathfrak{LE}(F)$, II. either there exists $a \in E$ such that for every $b \in F$ we have $a \leq b$, or $\mathfrak{LE}(E) = \mathfrak{LE}(F)$.

If $^*\alpha$, $^*\beta$ have representatives E, F respectively, then the following are equivalent: I. $^*\alpha \leq {}^*\beta$, II. either $^*\alpha = {}^*\beta$ or there exists $a \in E$ such that for every $b \in F$ we have $a \leq b$.

1B. Right endings.

1. - Given E, F, we define $E \leq\!\cdot F$ as « for every $x \in E$ there exists $y \in F$ such that $x \leq y$ ». We have: $E \leq\!\cdot E$; if $E \leq\!\cdot F$, $F \leq\!\cdot G$, then $E \leq\!\cdot G$.

For any two sets E, F (not empty) we have either $E \leq\!\cdot F$ or $F \leq\!\cdot E$.

2. - The correspondence $\leq\!\cdot$ induces a notion of equality $E =\!\cdot F$, defined as « $E \leq\!\cdot F$ and $F \leq\!\cdot E$ ». The notion $\leq\!\cdot$

is invariant with respect to the equality $=\cdot$. Consequently the notion $\leq\cdot$ organizes the class of all not empty subsets of M into a chain $(\leq\cdot)$ with $(=\cdot)$ as governing equality.

3. - If $E \subseteq F$, then $E \leq\cdot F$.

4. - If $x \in E$, then $x \leq\cdot E$.

5. - By *the right ending of* E, $\mathfrak{RE}(E)$ we shall understand the class of all sets F such that $F =\cdot E$.

By a *right ending* we shall understand any $\mathfrak{RE}(E)$ where $E \subseteq M$, $E \neq 0$.

Right ending will be denoted by greek letters provided with a star α^*, β^*, ...

If $\mathfrak{RE}(E)$ is denoted by α^*, we call E *representative of* α^*.

Every set belonging to the class α^* is a representative of α^*.

6. - If α^*, β^* are right endings, we define the correspondence $\alpha^* \leq \beta^*$ by « there exist representatives E, F of α^*, β^* respectively, such that $E \leq\cdot F$ ».

The notion of $\leq$ for right endings does not depend on the choice of representatives.

The following are equivalent: I. $\alpha^* \leq \beta^*$. II. for all representatives E, F of α^*, β^* respectively we have $E \leq\cdot F$.

If α^*, β^* are right endings, we define $\alpha^* = \beta^*$ by « $\alpha^* \leq \beta^*$ and $\beta^* \leq \alpha^*$ ».

Thus, this notion is the identity of classes of sets. There are properties similar to those of left endings, stated in [§ 1A, 7 and 8].

7. - If E possesses the maximum a, then $E =\cdot a$. If $a \in M$ and $E =\cdot a$, then a is the maximum in E.

A right ending whose one of the representatives is the set $\{a\}$ composed of a single element a of M, will be termed *point-right ending*. To simplify writing we agree to denote this ending by a.

8. - The notions related to right endings have properties analogous to those, stated in [§ 1A, 9, 10, 11, 12, 13, 14].

1C. Inequalities in endings.

1. - The relation

$$^*\alpha \leq \beta^*, \ (\beta^* \geq {}^*\alpha)$$

will mean that there exist representatives E, F of $^*\alpha$, β^* respectively such that for every $x \in E$ and every $y \in F$ we have $x \leq y$.

2. - The relation

$$\alpha^* \leq {}^*\beta$$

will mean that there exist representatives E, F of α^*, $^*\beta$ respectively such that for every $x \in E$ and every $y \in F$ we have $x \leq y$.

3. - The following are equivalent: I. $\alpha^* \leq {}^*\beta$, II. for every representatives E, F of α^*, $^*\beta$ respectively we have: for all $x \in E$ and $y \in F$, we have $x \leq y$.

4. - For every $E \subseteq M$, $E \neq 0$ we have

$$\mathfrak{L}\mathfrak{E}(E) \leq \mathfrak{R}\mathfrak{E}(E).$$

If $\mathfrak{R}\mathfrak{E}(E) \leq \mathfrak{L}\mathfrak{E}(E)$, then E is composed of a single point.

If $\alpha^* \leq {}^*\beta$ and $^*\beta \leq \alpha^*$, then there exists a point $a \in M$, such that $\{a\}$ is a representative of both α^* and $^*\beta$. If α^*, $^*\beta$ have both a representative $\{a\}$, then $\alpha^* \leq {}^*\beta$ and $^*\beta \leq \alpha^*$.

If a is the maximum of E and, at the same time, the minimum of F, then $\mathfrak{R}\mathfrak{E}(E) \leq \mathfrak{L}\mathfrak{E}(F)$ and $\mathfrak{L}\mathfrak{E}(F) \leq \mathfrak{R}\mathfrak{E}(E)$.

5. - The class off all endings, right and left makes up a chain. This can be established by the following steps:

If $\alpha_1^* \leq \alpha^*$, $\alpha^* \leq {}^*\beta$, $^*\beta \leq {}^*\beta_1$, then $\alpha_1^* \leq {}^*\beta_1$,

If $^*\alpha_1 \leq {}^*\alpha$, $^*\alpha \leq \beta^*$, $\beta^* \leq \beta_1^*$, then $^*\alpha_1 \leq \beta_1^*$.

If $^*\alpha \leq \beta^*$, $\beta^* \leq {}^*\gamma$, then $^*\alpha \leq {}^*\gamma$.

If $\alpha^* \leq {}^*\beta$, $^*\beta \leq \gamma^*$, then $\alpha^* \leq \gamma^*$.

These properties prove that the correspondance $\leq$, which is defined for all endings, is transitive. This correspondence is also reflexive.

The equalities $\alpha^* = \beta^*$ and $^*\alpha = {}^*\beta$ were already defined in [§ 1A and § 1B].

Now define $\alpha^* = {}^*\beta$ as « $\alpha^* \leq {}^*\beta$ and ${}^*\beta \leq \alpha^*$ » and define ${}^*\beta = \alpha^*$ as « ${}^*\beta \leq \alpha^*$ and $\alpha^* \leq {}^*\beta$ ».

The following are equivalent: I. $\alpha^* = {}^*\beta$, II. If E and F are representative of α^* and ${}^*\beta$, then E has a maximum and F has a minimum. (Both coincide).

The notion of equality of endings is reflexive, symmetric and transitive. The notion of « $\leq$ » for endings is invariant with respect to the notion « $=$ » of equality of endings. Given any two endings φ, ψ, we have either $\varphi \leq \psi$ or $\psi \leq \varphi$.

The above properties prove that the notion of « $\leq$ » organizes the class of all endings, right and left, into a chain on which the equality « $=$ » of endings is the governing equality.

Concerning point-endings, there is no need to make discrimination between right point endings and left point endings. We shall call them shortly *point-endings* and denote, as before, by the quantities of M, determining them.

Thus if $a \in M$, a can be conceived as a right point ending or as a left point ending.

1D. Additional properties of endings.

1. - The following properties of endings can be proved without any requirement of special tenchnique:

We define for endings φ, ψ the inequality $\varphi < \psi$, $(\psi > \varphi)$ as $\varphi < \psi$ but $\varphi \neq \psi$.

2. - The following are equivalent: I. $a < \mathfrak{L}\mathfrak{S}(F)$, II. for every $y \in F$ we have $a < y$.

The following are equivalent: I. $\mathfrak{L}\mathfrak{S}(E) < b$, II. there exists $x \in E$ with $x < b$.

3. - The following are equivalent: I. $\mathfrak{L}\mathfrak{S}(E) < \mathfrak{L}\mathfrak{S}(F)$, II. there exists $x \in E$ whit $x < \mathfrak{L}\mathfrak{S}(F)$, III. there exists $x \in E$ such that for every $y \in F$ we have $x < y$.

If $\mathfrak{L}\mathfrak{S}(E) < \mathfrak{L}\mathfrak{S}(F)$, then for every $y \in F$ we have $\mathfrak{L}\mathfrak{S}(E) < y$, (but not conversely).

The following are equivalent: I. for every $y \in F$ we have $\mathfrak{L}\mathfrak{S}(E) < y$, II. for every $y \in F$ there exists $x \in E$ with $x < y$.

4. - The following are equivalent: I. $a \leq \mathfrak{L}\mathfrak{S}(F)$, II. for every $y \in F$ we have $a \leq y$.

The following are equivalent: I. $\mathfrak{L}\mathfrak{S}(E) \leq b$, II. there exists $x \in E$ with $x \leq b$.

5. - Consider the statements: I. $\mathfrak{L}\mathfrak{S}(E) \leq \mathfrak{L}\mathfrak{S}(F)$, II. there exists $x \in E$ with $x \leq \mathfrak{L}\mathfrak{S}(F)$, III. there exists $x \in E$ such that for every $y \in F$ we have $x \leq y$.

From II follows I, but from I does not follow II. II and III are equivalent.

All three following statements are equivalent: I. $\mathfrak{L}\mathfrak{S}(E) \leq \leq \mathfrak{L}\mathfrak{S}(F)$, II. for every $y \in F$ we have $\mathfrak{L}\mathfrak{S}(E) \leq y$, III. for every $y \in F$ there exists $x \in E$ such that $x \leq y$.

6. - The following are equivalent: I. $a < {}^*\beta$, II. for every representative F of ${}^*\beta$ and for every $y \in F$ we have $a < y$, III. there exists a representative F of ${}^*\beta$ such that for every $y \in F$ we have $a < y$.

The following are equivalent: I. ${}^*\alpha < \beta$, II. for every representative E of ${}^*\alpha$ there exists $x \in E$ with $x < b$, III. there exists a representative E of ${}^*\alpha$ such that there exists $x \in E$ with $x < b$.

7. - The following are equivalent: I. ${}^*\alpha < {}^*\beta$, II. for every representative E of ${}^*\alpha$ there exists $x \in E$ such that $x < {}^*\beta$, III. there exists a representative E of ${}^*\alpha$ such that there exists $x \in E$ with $x < {}^*\beta$, IV. for every representative E of ${}^*\alpha$ and every representative F of ${}^*\beta$ there exists $x \in E$ such that for every $y \in F$ we have $x < y$, V. there exist representatives E, F of ${}^*\alpha$ and ${}^*\beta$ respectively such that there exist $x \in E$ such that for every $y \in F$ we have $x < y$.

If ${}^*\alpha < {}^*\beta$, then for every representative F of ${}^*\beta$ and every $y \in F$ we have ${}^*\alpha < y$, (but not conversely).

8. - If there exists a representative E of ${}^*\alpha$ such that there exists $x \in E$ with $x \leq {}^*\beta$, then ${}^*\alpha \leq {}^*\beta$, (but not conversely).

9. - The following are equivalent: I. $^*\alpha \leq \beta$, II. there exists a representative E of $^*\alpha$ such that there exists $x \in E$ with $x \leq b$, III. for every representative E of $^*\alpha$ there exists $x \in E$ such that $x \leq b$.

The following are equivalent: I. $a \leq {}^*\beta$, II. there exists a representative F of $^*\beta$ such that for every $y \in F$ we have $a \leq y$, III. for every representative F of $^*\beta$ we have: for every $y \in F$ we have $a \leq y$.

10. - The following are equivalent: I. $^*\alpha \leq {}^*\beta$, II. there exists a representative F of $^*\beta$ such that for every $y \in F$ we have $^*\alpha \leq y$, III. for every representative F of $^*\beta$ and for every $y \in F$ we have $^*\alpha \leq y$, IV. there exist representatives E, F of $^*\alpha$, $^*\beta$ respectively such that for every $y \in F$ there exists $x \in E$ with $x \leq y$, V. for every representatives E, F of $^*\alpha$, $^*\beta$ respectively and for every $y \in F$ there exists $x \in E$ with $x \leq y$.

11. - The following are equivalent: I. $^*\alpha < {}^*\beta$, II. there exists a representative E of $^*\alpha$ such that for every $x \in E$ we have $x < {}^*\beta$.

The following are equivalent: $^*\alpha \leq {}^*\beta$, II. for every representative F of $^*\beta$ there exists $y \in F$ with $^*\alpha \leq y$.

12. - The properties [2 - 11] of left endings have their analogy in the corresponding properties of right endings.

13. - **Remark.** The endings cannot be identified with Dedekind sections in the ordering M. Indeed let R be the chain of all ordinary real numbers, and let $a \in R$.

If we define

$$R_a^+ \underset{df}{=} \{ x \mid x \in R,\ a < x \},\quad R_a^- \underset{df}{=} \{ x \mid x \in R,\ x < a \},$$

then we have

$$\mathfrak{R}\mathfrak{E}(R_a^-) < a < \mathfrak{L}\mathfrak{E}(R_a^+),$$

where a denotes the point ending whose representative is $\{ a \}$.

The smallest ending is $\mathfrak{L}\mathfrak{E}(R)$, the greatest $\mathfrak{R}\mathfrak{E}(R)$. Every real number a is replaced by three different endings. This circumstance may be useful e. g. in the theory of ordinary functions of bounded variation.

1E. Operations on sets in a lineary ordered semigroup.

1. - This will be an auxiliary topic for operations on endings. To introduce these operations we shall suppose that the ordering M is a kind of *linearly ordered abelian semigroup* G, which will be axiomatized as follows: Let F be a non trivial linearly ordered commutative field. There is an addition $a + b$, for $a, b \in G$, which is always performable and yielding an element of G. It is supposed to be $\overset{G}{=}$- invariant, where $\overset{G}{=}$ is the equality governing on G, (supposed to satisfay the conditions of identity). There is a multiplication $\lambda \cdot a$ performable for every $a \in G$ and every $\lambda \in F$ where $\lambda \geq 0$. We shall write $a\lambda$ or λa.

This operation is supposed to be invariant with respect to $\overset{G}{=}$ and to the equality $\overset{F}{=}$ governing in F.

In stating the following admitted axioms we shall drop superscripts over the signs of equality.

$$(1)\ldots\left\{\begin{array}{l} a+b=b+a\,;\;\; a+(b+c)=(a+b)+c\,;\;\; \text{for } \lambda\geq 0,\\ \mu\geq 0 \text{ we have } (\lambda+\mu)\,a=\lambda a+\mu a\,;\;\; \text{if } \lambda\geq 0, \text{ then}\\ \lambda(a+b)=\lambda a+\lambda b\,;\;\; \text{if } \lambda\geq 0\;\; \lambda\geq 0, \text{ then } \lambda(\mu a)=(\lambda\mu)a\,;\\ 1_F\cdot a=a. \end{array}\right.$$

To these axioms, which look like the usual axioms for a modulus (with exception of the cancellation law), we add the following:

There exists an element θ of G such that for every $a \in G$ we have $a + \theta = a$.

[We prove that θ is $\overset{G}{=}$- unique]. We shall denote it by 0_G.

We also admit that for every $a \in G$, we have $0_F \cdot a \overset{G}{=} 0_G$. Thus G is a kind of semigroup with zero.

We suppose that there is a correspondence $\overset{G}{\leq}$ with domains G and range G, which makes up a chain, and is $\overset{G}{=}$-invariant.

We admit the axioms: $a = b$ is equivalents to $a \leq b$, $b \leq a$;

if $a \leq b$, then $a + c \leq b + c$;

if $a \leq b$, $\lambda \geq 0$, then $\lambda a \leq \lambda b$;

if $a \geq 0_G$, $0 \leq \lambda \leq \mu$, then $\lambda a \leq \mu a$;

We prove that if $a \leq b$, $a' \leq b'$, then $a + a' \leq b + b'$.

We may call G linearly ordered semigroup with non negative multipliers taken from a linearly ordered commutative field.

2. - G will be a linearly ordered abelian group over F, when 1) $\lambda \cdot a$ is performable for any $\lambda \in F$, and the condition in (1) of nonnegativeness of factors is omitted, 2) the cancellation law: « if $a + b = a + b'$, then $b = b'$ » is admitted.

If G is supposed to be a semigroup only, we shall write $G^{(s)}$.

3. - We shall consider subsets of $G^{(s)}$ (or G) but only not empty ones. They will be denoted by $E, F, \ldots$

We define:

$$E + F \underset{df}{=} \{ x + y \mid x \in E,\ y \in F \}$$

and

$$\lambda \cdot E \underset{df}{=} \{ \lambda x \mid x \in E \},$$

(for $G^{(s)}$ we suppose that $\lambda \geq 0$), $\lambda \in F$.

4. - The following are equivalent: I. $z \in E + F$, II. there exist $x \in E$ and $y \in E$, such that $z = x + y$.

The following are equivalent: (for any $\lambda \in F$ for G, and for $\lambda \geq 0$ for $G^{(s)}$): I. $z \in \lambda E$, II. there exists $x \in E$, such that $z = \lambda x$.

5. - We have $E + F = F + E$, $E + (F + G) = (E + F) + G$, $1_F \cdot E = E$.

6. - $(\lambda + \mu)E \subseteq \lambda E + \mu E$ (for any λ, μ for G, and for $\lambda \geq 0$, $\mu \geq 0$ for $G^{(s)}$.

Remark. Notice that, even for $\lambda > 0$, $\mu > 0$ and G, the relation $\lambda E + \mu E \subseteq (\lambda + \mu)E$ may be not true.

If for G and for every $\lambda, \mu \in F$ we have $(\lambda + \mu)E = \lambda E + \mu E$, then E is composed of a single point-and conversely.

7. - $\lambda(E+F) = \lambda E + \lambda F$ (for any λ for G and for $\lambda \geq 0$ for $G^{(s)}$).

8. - $\lambda(\mu E) = (\lambda\mu)E$ for any λ, μ for G and for $\lambda \geq 0$, $\mu \geq 0$ for $G^{(s)}$.

9. - For the linearly ordered groups G we define $E - F \underset{df}{=} \{x - y \mid x \in E,\ y \in F\}$, $-E \underset{df}{=} \{0\} - E$, $+E \underset{df}{=} \{0\} + E$, (The last also for $G^{(s)}$).

10. - For G the following are equivalent: I. $a \in E - F$, II. there exist $x \in E$ and $y \in F$ such that $a = x - y$, and the following are equivalent: I. $a \in -E$, II. there exists $x \in E$ with $a = -x$.

11. - We have for $G^{(s)}$ and G: $+E = E$.

12. - For G we have $E - F = E + (-F)$, $-E = (-1_F) \cdot E$, $-(\lambda E) = (-\lambda) \cdot E$, $-(E + F) = (-E) + (-F) = -E - F$. $\lambda(-E) = -\lambda E$, $(\lambda - \mu)E \subseteq \lambda E - \mu E$. $\lambda(E - F) = \lambda E - \lambda F$, $-(-E) = E$.

13. - We define for G and $G^{(s)}$, $E \leq F$ as « for every $x \in E$ and every $y \in F$ we have $x \leq y$ », and $E < F$ as « for every $x \in E$ and every $y \in F$, $x < y$ ».

We define $a \pm E$ as $\{a\} \pm E$ and $E \pm a$ as $E \pm \{a\}$.

1F. Operations on endings.

1. - We suppose that the chain M in which endings will be considered, is a linearly ordered abelian semi-group or group. Multipliers will be taken from a linearly ordere non trivial commutative field F.

2. - If $E \cdot \leq E'$, $F \cdot \leq F'$, then $E + F \cdot \leq E' + F'$.
If $E \cdot = E'$, $F \cdot = F'$, then $E + F \cdot = E' + F'$.

3. - If $^*\alpha$, $^*\beta$ are left endings, E, E' are representatives of $^*\alpha$ and F, F' are representatives of $^*\beta$, then $E+F\cdot=$ $=E'+F'$ and $\mathfrak{LE}(E+F)=\mathfrak{LE}(E'+F')$.

Definition. If $^*\alpha$, $^*\beta$ are left endings, then by $^*\alpha+{}^*\beta$ we shall understand the left ending $\mathfrak{LE}(E+F)$, where E, F are representatives of $^*\alpha$ and $^*\beta$ respectively. This notion of *addition of left endings* does not depend on the choice of representatives.

If $^*\gamma={}^*\alpha+{}^*\beta$. G, E, F are representatives of $^*\gamma$, $^*\alpha$, $^*\beta$ respectively, then $G\cdot=E+F$. Conversely, if $G\cdot=E+F$ and G, E, F are representatives of $^*\gamma$, $^*\alpha$, $^*\beta$ respectively, then $^*\gamma={}^*\alpha+{}^*\beta$.

The addition $^*\alpha+{}^*\beta$ is invariant with respect to the equality of all endings.

4. - If $E\cdot\leq F$, $\lambda\geq 0$, then $\lambda E\cdot<\lambda F$. If $E\cdot=F$, $\lambda\geq 0$, then $\lambda E\cdot=\lambda F$.

5. - Definition. If $\lambda\geq 0$, then by $\lambda\cdot{}^*\alpha$ we shall understand the left ending $\mathfrak{LE}(\lambda\cdot E)$ where E is a representative of $^*\alpha$. This notion of multiplication does not depend on the choice of the representative E of $^*\alpha$.

If E, F are representatives of $^*\alpha$, $^*\beta$ respectively and $\lambda\geq 0$, then the following are equivalent: I. $\lambda E\cdot=F$, II. $\lambda\cdot{}^*\alpha={}^*\beta$.

The product $\lambda\cdot{}^*\alpha$ is invariant with respect to the equality of all endings.

6. - We have $^*\alpha+{}^*\beta={}^*\beta+{}^*\alpha$, and $(^*\alpha+{}^*\beta)+{}^*\gamma=$ $={}^*\alpha+(^*\beta+{}^*\gamma)$.

7. - If $\lambda\geq 0$, then $\lambda\cdot(^*\alpha+{}^*\beta)=\lambda\cdot{}^*\alpha+\lambda\cdot{}^*\beta$, $1.\ {}^*\alpha=$ $={}^*\alpha$; if $\lambda\geq 0$, $\mu\geq 0$, then $\lambda(\mu\cdot{}^*\alpha)=(\lambda\mu)\cdot{}^*\alpha$.

8. - If $\lambda\geq 0$, $\mu\geq 0$, then $(\lambda+\mu)\cdot{}^*\alpha=\lambda\cdot{}^*\alpha+\mu\cdot{}^*\alpha$.

Proof. Let E be a representative of $^*\alpha$. Then λE, μE, $(\lambda+\mu)E$ are representatives of $\lambda\cdot{}^*\alpha$, $\mu\cdot{}^*\alpha$ and $(\lambda+\mu)\cdot{}^*\alpha$ respectively. By [§ 1E; 6], we have $(\lambda+\mu)E\subseteq\lambda E+\mu E$. Hence, by [§ 1A; 3], $\lambda E+\mu E\cdot\leq(\lambda+\mu)E$, and then, by

[Def. § 1A; 6], $\mathfrak{L}\mathfrak{E}(\lambda E + \mu E) \cdot \leq (\lambda + \mu) \cdot {}^*\alpha$, i. e., [Def. § 1F; 3] and [Def. § 1F; 5], $\lambda \cdot {}^*\alpha + \mu \cdot {}^*\alpha \leq (\lambda + \mu) \cdot {}^*\alpha$ ..... (1).

Now, let $y \in \lambda E + \mu E$. There exist $x', x'' \in E$ such that $y = \lambda x' + \mu x''$.

We shall prove that there exists $y' \in (\lambda + \mu)E$ such that $y' \leq y$. We may suppose that $x' \leq x''$. We get $\mu x' \leq \mu x''$ and then $\lambda x' + \mu x' \leq \lambda x' + \mu x'' = y$.

Hence $(\lambda + \mu)x' \leq y$. We put $y' \underset{df}{=} (\lambda + \mu)x'$.

Consequently, for every $y \in \lambda E + \mu E$ there exists $y' \in (\lambda + \mu)E$ such that $y' \leq y$.

Hence $(\lambda + \mu)E \cdot \leq \lambda E + \mu E$, which gives $(\lambda + \mu) \cdot {}^*\alpha \leq \lambda \cdot {}^*\alpha + \mu \cdot {}^*\alpha$ ..... (2).

From (1) and (2) the theorem follows.

9. - $0_F \cdot {}^*\alpha = 0$, i. e. the point-left ending whose representative is the set composed of the single element $0 \in \boldsymbol{M}$. We also have for every ${}^*\alpha$ the equality ${}^*\alpha + 0 = {}^*\alpha$.

10. - If ${}^*\alpha \leq {}^*\beta$, then ${}^*\alpha + {}^*\gamma \leq {}^*\beta + {}^*\gamma$;
If ${}^*\alpha \leq {}^*\beta$, $\lambda \geq 0$, then $\lambda \cdot {}^*\alpha \leq \lambda \cdot {}^*\beta$,
If ${}^*\alpha \geq 0$, $0 \leq \lambda \leq \mu$, then $\lambda \cdot {}^*\alpha \leq \mu \cdot {}^*\alpha$.

11. - The properties [1F, 6-10] show that the left endings in a linearly ordered abelian semigroup with zero or in a linearly ordered abelian group, make up another linearly ordered abelian semigroup with zero and with nonnegative multipliers taken from F. This allow to consider endings and operations on them in the chain of left endings. A similar behaviour show the right endings in $\boldsymbol{M}$.

12. - Till the end of this subsection § 1F, we shall suppose that the chain $\boldsymbol{M}$, under consideration, is a linearly ordered abelian group with arbitrary multipliers taken from a linearly ordered commutative, non trivial field F.

13. - The following are equivalent: I. $E \cdot \leq F$, II. $(-F) \leq \cdot (-E)$.

The following are equivalent: I. $E \leq \cdot F$, II. $(-F) \cdot \leq (-E)$.

If $E \cdot \leq F$, $\lambda \leq 0$, then $\lambda \cdot F \leq \cdot \lambda \cdot E$.

If $E \leq \cdot F$, $E' \leq \cdot F'$, then $E + E' \leq \cdot F + F'$.

If $E \leq \cdot F$, $\lambda \geq 0$, then $\lambda E \leq \cdot \lambda F$, and if $E \leq \cdot F$, $\lambda \leq 0$, then $\lambda F \cdot \leq \lambda E$.

If $E = \cdot F$, $E' = \cdot F'$, then $E + E' = \cdot F + F'$,

If $E = \cdot F$, $\lambda \leq 0$, then $\lambda E = \cdot \lambda F$, and if $E = \cdot F$, $\lambda \leq 0$. then $\lambda F \cdot = \lambda E$.

If $E \cdot = F$, $\lambda \leq 0$, then $\lambda F = \cdot \lambda E$.

The following are equivalent: I. $E \cdot = F$, II. $(-F) = \cdot (-E)$, and the following are equivalent: I. $E = \cdot F$, II. $(-F) \cdot \cdot = (-E)$.

14. - Definition. We define the *addition of right endings* as follows: By $\alpha^* + \beta^*$ we shall understand the rigt ending $\mathfrak{RE}(E + F)$, where E, F are representatives of α^*, β^* respectively.

This notion does not depend on the choice of representatives.

15. - Definition. We define the *multiplication of right endings by non negative quantities* as follows: If $\lambda \geq 0$, then by $\lambda \cdot \alpha^*$ we shall understand the right ending $\mathfrak{RE}(\lambda E)$, where E is a representative of α^*. This notion does not depend on the choice of the representative E of α^*.

The above operations [§ 1F; 14, 15] are invariant with respect to the equality of all endings.

16. - The addition and multiplication of right endings by non negative factors behaves exactly as the same operations performed on left endings. Thus laws similar to [§ 1F, 2-10] are valid for right endings.

17. - Now we are going to introduce further operations on endings.

Definition. If $\lambda \leq 0$, we define $\lambda \cdot {}^*\alpha$ as the right ending $\mathfrak{RE}(\lambda \cdot E)$, where E is a representative of ${}^*\alpha$. If $\lambda \leq 0$, we define $\lambda \cdot \alpha^*$ as the left ending $\mathfrak{LE}(\lambda E)$ where E is a representative of α^*.

These notions are independent of the choice of representatives.

18. - If $\lambda = 0$ we can apply the definition [17] as well as [5] for $\lambda \cdot {}^*\alpha$. Both yield the same, namely the point -ending 0. The same holds for $\lambda \cdot \alpha^*$ if $\lambda = 0$.

19. - The notions [17] are invariant with respect to the equality of all endings.

20. - Definition. By $-{}^*\alpha$ we shall understand the right ending $\mathfrak{RE}(-E)$ where E is a representative of ${}^*\alpha$.

By $-\alpha^*$ we shall understand the teft ending $\mathfrak{LE}(-E)$ where E is a representative of α^*.

These notions do not depend on the choice of the representative E. They are invariant with respect to the equality of all endings.

21. - For any $E \neq 0$ we have

$$\mathfrak{RE}(E) = -\mathfrak{LE}(-E); \qquad \mathfrak{LE}(E) = -\mathfrak{RE}(-E)$$
$$-\mathfrak{RE}(E) = \mathfrak{LE}(-E): \qquad -\mathfrak{LE}(E) = \mathfrak{RE}(-E).$$

22. - The following are aquivalent: I. ${}^*\alpha = -\beta^*$, II. $\beta^* = -{}^*\alpha$.

We have $-(-{}^*\alpha) = {}^*\alpha$, $-(-\alpha^*) = \alpha^*$, $(-1) \cdot {}^*\alpha = -{}^*\alpha$; $(-1) \cdot \alpha^* = -\alpha^*$.

23. - $-(\alpha^* + \beta^*) = (-\alpha^*) + (-\beta^*)$; $-({}^*\alpha + {}^*\beta) = (-{}^*\alpha) + (-{}^*\beta)$.

24. - For any $\lambda \in F$ we have: $-(\lambda \cdot {}^*\alpha) = (-\lambda) \cdot {}^*\alpha$; $-(\lambda \cdot \alpha^*) = (-\lambda) \cdot \alpha^*$.

25. - For any λ, $\mu \in F$ we have: $\lambda(\mu \cdot {}^*\alpha) = (\lambda\mu) \cdot {}^*\alpha$; $\lambda(\mu \cdot \alpha^*) = (\lambda\mu) \cdot \alpha^*$.

For any $\lambda \in F$ we have $\lambda(\alpha^* + \beta^*) = \lambda \cdot \alpha^* + \lambda \cdot \beta^*$; $\lambda({}^*\alpha + {}^*\beta) = \lambda \cdot {}^*\alpha + \lambda \cdot {}^*\beta$.

26. - **Remark**. The equalities $(\lambda + \mu) \cdot {}^*\alpha = \lambda \cdot {}^*\alpha + \mu \cdot {}^*\alpha$; $(\lambda + \mu) \cdot \alpha^* = \lambda \cdot \alpha^* + \mu \cdot \alpha^*$, hold true for $\lambda \geq 0$, $\mu \geq 0$, and also for $\lambda \leq 0$, $\mu \leq 0$, but not for $\lambda > 0$, $\mu < 0$ or $\lambda < 0$, $\mu > 0$.

27. - We have defined the addition for homogenuous endings only; we do not add heterogenuous endings, but we define subtraction for heterogenuour endings only.

Definition. We define $*\alpha - \beta*$ as $*\alpha + (-\beta*)$ and $\alpha* - *\beta$ as $\alpha* + (-*\beta)$.

If one of two given endings is a point–ending, say a, the expressions $*\alpha \pm a$, $\alpha* \pm a$, $a \pm *\alpha$, $a \pm \alpha*$ are meaningful, since a point–ending can be considered as a right ending, as well as a left ending.

We have $*\alpha + 0 = *\alpha$, $\alpha* + 0 = \alpha*$.

28. - We have some theorems concerning inequalities in endings:

If $*\alpha \leq *\beta$, $\lambda \leq 0$, then $\lambda \cdot *\beta \leq \lambda \cdot *\alpha$. If $\alpha* \leq \beta*$, $\lambda \leq 0$, then $\lambda \cdot \beta* \leq \lambda \cdot \alpha*$.

29. - If $\lambda \leq \mu$, and λ, μ are any quantities of F, and $*\alpha \geq 0$, then $\lambda \cdot *\alpha \leq \mu \cdot *\alpha$.

30. - If $\alpha* \geq$, $0 \leq \lambda \leq \mu$, then $\lambda \cdot \alpha* \leq \mu\alpha*$.

31. - The following are equivalent: I. $*\alpha \leq *\beta$, II. $-*\beta \leq -*\alpha$, and the following are equivalent: I. $\alpha* \leq \beta*$, II. $-\beta* \leq -\alpha*$.

1G. Supplementary theorems on endings.

1. - The following rules have a lemmatic character, since they will be only used in the discussion of extending measure.

2. - The statements I. $*\alpha \leq a + *\beta$, II. $*\alpha - a \leq *\beta$ are equivalent, and so are I. $\alpha* \leq a + \beta*$, II. $\alpha* - a \leq \beta*$.

3. - If $a \leq *\alpha + *\beta$, then $a - *\alpha \leq *\beta$.

4. - If $a - *\alpha \leq b + *\beta$, then $a - b \leq *\alpha + *\beta$, and conversely.

5. - If $*\alpha < *\beta$, then $*\alpha + c < *\beta + c$.

We notice that if $*\alpha < *\beta$ and $*\gamma$ is any, we cannot say that $*\alpha + *\gamma < *\beta + *\gamma$.

The cancellation law: « if $*\alpha + *\gamma = *\alpha + *\gamma_1$ then $*\gamma = *\gamma_1$ » is not valid.

6. - We have for any chain M: if $E \leq F$ then $\mathfrak{RE}(E) \leq \leq \mathfrak{LE}(F)$.

7. - For any chain M the following statements are valid: If for every $x \in E$ we have $x \leq *\alpha$, then $\mathfrak{RE}(E) \leq *\alpha$. If for every $y \in F$ we have $\alpha^* \leq y$, then $\alpha^* \leq \mathfrak{LE}(F)$.

1H. Endings in a superchain of a given chain.

1. - Let M, M' be not empty $\overset{N}{=}$-subsets of N and suppose that M, M' are organized into chains $\overset{M}{\leq}, \overset{M'}{\leq}$, denoted by (M), (M') respectively (see [Preface 5]). Let $\overset{M}{=}, \overset{M'}{=}$ be the corresponding equalities governing on them. These equalities may differ from $\overset{N}{=}$ and from one another.

2. - Definition. According to [Preface 83] we say that (M) is an *order-genuine subchain of* (M') whenever for N-sets we have

1) $M \subseteq M'$ in N

2) if a, b belong to the $\overset{N}{=}$-set M, then the following are equivalent: I. $a \overset{M}{\leq} b$, II. $a \overset{M'}{\leq} b$.

It follows that also the following are equivalent for $a, b \in M$: I. $a \overset{M}{=} b$, II. $a \overset{M'}{=} b$. A further conseguence is this:

If the « $\overset{N}{=}$ »-subset E of M is also an « $\overset{M}{=}$ »-subset of M, there exists a « $\overset{M'}{=}$ »-subset E' of M' such that $E \subseteq E'$ in N. Thus the $\overset{M}{=}$-equivalence classes A are N-contained in the corresponding $\overset{M'}{=}$-equivalence classes A'.

Denote by s the correspondence which attaches to every $\overset{M}{=}$-equivalence class (which is an « $\overset{N}{=}$ »-set) the $\overset{M'}{=}$-equivalence class A' (which is also an « $\overset{N}{=}$ »-set) such that $A \subseteq A'$ in N. We call s ***natural correspondence***. $\mathrm{C}| s$ is the class of all $\overset{M}{=}$-equivalence classes and $|\mathrm{D}\, s$ is contained in the class of all $\overset{M'}{=}$-equivalence classes. s is one-to-one. If E is an

$\overset{M}{=}$–subset of M, the correspondence s generates the corresponding $\overset{M'}{=}$–subset E' of M'.

We shall write $E' \overset{N}{=} sE$. For sake of simplicity we shall sometimes use the same symbol E for both E and E', but, if needed, we shall emphasize the discrimination by saying E in (M) and E in (M').

3. - Definition. Let (M) be an order–genuine subchain of (M'). If the equality $\overset{M}{=}$ coincides with the equality $\overset{M'}{=}$ if restricted to M, i. e. if $a \in M$, then the two statements $a \overset{M}{=} b$, $a \overset{M'}{=} b$ are equivalent, we say that (M) is an *order–genuine strict subchain* of (M'), [Preface; 8]. In this case s is identity of N–sets, (restricted to $\overset{M}{=}$–equivalence classes).

If (M) is an order–genuine subchain of (M'), (M) can be modified so as to obtain an order genuine strict subchain of (M').

Indeed, let us replace M by the N–set M_1 of all elements y, such that there exists $x \in M$ with $x \overset{M'}{=} y$, i. e. instead of $\overset{M}{=}$–equivalence classes we consider the s–corresponding $\overset{M'}{=}$–equivalence classes. We organize M_1 into a chain (M_1), by defining $a_1 \overset{M_1}{\leq} b_1$ by « there exist a, $b \in M$ such that $a \overset{M'}{=} a_1$, $b \overset{M'}{=} b_1$ and $a \overset{M}{\leq} b$ ». The ordering (M_1) is a chain which is order–isomorphic with (M) and, at the same time, (M_1) is an order–genuine strict subchain of (M'). (M_1) is order–isomorphic with (M) through the isomorphism s.

4. - Definition. If (M), (M') are chains, then, [Preface; 8], (M) is an *order genuine subchain of* (M') *through the isomorphism* t, whenever:

1) t is a one-to-one correspondence, with domain M, and is $\overset{M}{=}$–invariant in it.

2) t transforms (M) into a chain (M_2) which is an order–genuine subchain of (M'). (t may be not $\overset{M'}{=}$–invariant in its region). t is invariant, in its region, with respect to $\overset{M_2}{=}$.

5. - We have devoted quite much room to the above discussion, because of the subtlety of subsequent topics which, otherwise, could be confused.

If (M) is an order-genuine subchain of M', and $E, F \overset{M}{\supseteq} M$. then the following are equivalent:

I. $E \cdot \overset{M}{\leq} F$ in (M), II. $E \cdot \overset{M'}{\leq} F$ in (M').

(Here E, F denote in I. the $\overset{M}{=}$-subsets of M and in II., the s-corresponding $\overset{M'}{=}$-subsets of M').

The following are also equivalent:

I. $E \cdot \overset{M}{=} F$ in (M), and II. $E \cdot \overset{M'}{=} F$ in (M'),

I. $E \overset{M}{\leq} \cdot F$ in (M), and II. $E \overset{M'}{\leq} \cdot F$ in (M'),

I. $E \overset{M}{=} \cdot F$ in (M), and II. $E \overset{M}{=} \cdot F$ in (M').

6. - The left ending $\mathfrak{L}\mathfrak{E}_M(E)$ is the class of all M-sets $F \overset{M}{\subseteq} M$ such that $F \cdot \overset{M}{=} E$, and the left ending $\mathfrak{L}\mathfrak{E}_{M'}(E)$ is the class of all M'-sets $F' \overset{M'}{\subseteq} M$ such that $F' \cdot \overset{M'}{=} E$.

It may happen that these classes are different: the class of all those F' may be « larger » than that of the sets F–even if (M) is a strict subchain of (M'). Nevertheless, we can define a *natural order isomorphism* N_l between the M-left endings and the corresponding M'-left endings, as follows: Let $*\alpha$ be an M-left ending and E_M its representative. Let $E_{M'}$ be the s-corresponding subset of M'. By $n_l(*\alpha)$ we shall understand the left ending $\mathfrak{L}\mathfrak{E}_{M'}(E_{M'})$.

The correspondence n_l, thus defined, does not depend on the choice of the representative E_M of $*\alpha$. The domain of n_l is the class of all M-left endings, and its region is contained in the class of all M'-left endings. n_l is invariant, in its domain, with respect to the equality $\overset{M}{=}$ of left endings in M, and is invariant, in its range, with respect to the equality $\overset{M'}{=}$ of left endings in M'.

The correspondence n_l preserves the ordering, i. e. if $*\alpha$, $*\beta$ are left endings in M, then the following are equivalent: I. $*\alpha \overset{M}{\leq} *\beta$, II. $n_l(*\alpha) \overset{M'}{\leq} n_l(*\beta)$.

$n_l(*\alpha)$ will be termed *natural image of* $*\alpha$ *in* M'. Many times we can identify, without any harm, the left ending $*\alpha$ with *its natural image*. However, we shall denote them differently if discrimination will be relevant.

7. - We proceed in a similar way with right endings,

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defining their ***natural images in M′*** by means of the analogous correspondence, denoted by n_r. The joint correspondence n of n_l and n_r transforms M-endings into M'-endings in a one-to-one way, preserving the ordering of endings.

Indeed we have the following properties:

If E_M is a representative of the ending γ in M, then $s(E_M) = E_{M'}$ is also a representative of $n(\gamma)$ in M'. If γ is a point-ending in M, then $n(\gamma)$ in also a point-ending in M', and conversely if γ is an ending in M, and $n(\gamma)$ is a point-ending in M', then γ is a point-ending in M.

The point ending in M, generated by the element a may be denoted by the same letter a, which will be also used to denote its n-corresponding point-ending.

The above can be proved by transforming (M) into a strict subchain of (M'), as indicated in [§ 1H; 3]. We have the properties:

If ${}^*\alpha_1$, ${}^*\beta_1$, α_1^*, β_1^* denote n-images of the M-endings ${}^*\alpha$, ${}^*\beta$, α^*, β^* respectively, the following are equivalent:

$$(1)\quad \begin{matrix} {}^*\alpha \overset{M}{\leq} {}^*\beta \text{ and } {}^*\alpha_1 \overset{M'}{\leq} {}^*\beta_1, \\ {}^*\alpha \overset{M}{=} {}^*\beta \text{ and } {}^*\alpha_1 \overset{M'}{=} {}^*\beta_1, \end{matrix} \qquad (2)\quad \begin{matrix} \alpha^* \overset{M}{\leq} \beta^* \text{ and } \alpha_1^* \overset{M'}{\leq} \beta_1^*, \\ \alpha^* \overset{M}{=} \beta^* \text{ and } \alpha_1^* \overset{M'}{=} \beta_1^*, \end{matrix}$$

$$(3)\quad \begin{matrix} {}^*\alpha \overset{M}{\leq} \beta^* \text{ and } {}^*\alpha_1 \overset{M'}{\leq} \beta_1^*, \\ {}^*\alpha \overset{M}{=} \beta^* \text{ and } {}^*\alpha_1 \overset{M'}{=} \beta_1^*, \end{matrix} \qquad (4)\quad \begin{matrix} \alpha^* \overset{M}{\leq} \beta^* \text{ and } \alpha_1^* \overset{M'}{\leq} {}^*\beta_1, \\ \alpha^* \overset{M}{=} {}^*\beta \text{ and } \alpha_1^* \overset{M'}{=} {}^*\beta_1, \end{matrix}$$

Thus to the chain of all M-endings there corresponds, through n, a chain of M'-endings, which is an order-genuine strict subchain of the chain of all M'-endings.

8. The natural isomorphism of endings in (M) and (M') preserves the operations on left endings and those on right endings.

1K. The chain of endings.

1. - The given chain (M) is an order-genuine subchain of the chain $(\mathcal{M})$ of all endings through the order-isomor-

phism which makes corespond to the element $a \in M$ the point-ending a.

If A is a non empty set of M-endings, then by the ***supremum of A in M*** we shall understand an ending β such that

1) for every $\alpha \in A$ we have $\alpha \overset{M}{\leq} \beta$, 2) if for every $\alpha \in A$ we have $\alpha \overset{M}{<} \beta'$, then $\beta \overset{M}{\leq} \beta'$.

In a similar way we define the *infimum of A in M*. If we consider the chain M_r of right endings, we define analogously the supremum and infinum in M_r of a set of right endings.

2. - Theorem. If $A \neq 0$ is a set of right endings, then A admits in M a supremum. This supremum is a right ending α^*. This ending α^* is also a supremum of A in M_r. If for every $\xi^* \in A$, we have $\xi^* \leq \beta'$, then $\alpha^* \leq {}^*\beta'$.

Proof. To prove that we shall rely on the ***lemma***:

If $\lambda^* < \mu^*$, then there exists $y \in M$ such that $\lambda^* < y \leq \mu^*$ in the ordering of M-endings.

The theorem is true if we suppose that A admits a maximum.

Suppose that A does not admit any maximum.

The following general theorem holds true [8]) for any chain P which does not admit a maximal element: There exists a distinguished well ordering $p_1 < p_2 < \dots < p_\alpha < \dots$ of elements of P which is cofinal with P, i. e. for every $q \in P$ there exists α such that $q \overset{P}{\leq} p_\alpha$.

2a. - Let

$$e_1^* < e_2^* < \dots < e_\alpha^* < \dots \tag{1}$$

be a distinguisched well ordering cofinal with A, where $e_\alpha^* \in A$. To apply the lemma, let us well order the set M in

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[8]) See (8). An ordinal is termed ***distinguished*** when it is infinite and the smallest among all ordinals with the same power.

the form of a distinguisched well order-sequence.

(2) $$a_1, a_2, \ldots, a_\beta, \ldots$$

Given an index α from (1), find in (2) the element with the smallest index — denote it by y_α — such that $e^*_\alpha < y_\alpha \leq$ $\leq e^*_{\alpha+1}$ in M. The sequence

(3) $$y_1 < y_2 < \ldots < y_\alpha < \ldots$$

is a well ordering whose ordinal is that of (1).

Denote by Y the set off all elements of (3) and put

(4) $$\beta^* \underset{df}{=} \mathfrak{RE}(Y) \text{ in } M.$$

If we put $Y(\alpha) \underset{df}{=} \{ y_{\alpha'} \mid \alpha' > \alpha \}$ for $\alpha = 1, 2, \ldots,$ the set $Y(\alpha)$ is for every α a representative of β^*, by [§ 1A; 12].

Applying [§ 1D; 11], we get $e^*_\alpha < \beta^*$ for every α, which proves that β^* is an upper bound of Y, and then also for A, because (1) is cofinal with A.

Now let γ^* be a right ending in M such that $\gamma^* < \beta^*$. We shall prove that γ^* is not an upper bound of A. Let C be a representative of γ^*. On account of [§ 1D; 7] there exists y_α such that for every $z \in C$ we have $z < y_\alpha$. We have $z < e^*_{\alpha+1}$ for every $z \in C$, and hence, by [§ 1D; 10] we get $\gamma^* \leq e^*_{\alpha+1} < e^*_{\alpha+2}$, which proves that γ^* is not an upper bound of Y, and then neither of A.

Consequently, if β^*_1 has the property that for every α we have $e^*_\alpha < \beta^*_1$, then we cannot have $\beta^*_1 < \beta^*$. Consequently $\beta^* \leq \beta^*_1$, which proves that β^* is the supremum of A in M_r.

2b. - Suppose that for every $\xi^* \in A$ we have $\xi^* \leq {}^*\beta'$. Hence we have $y_\alpha \leq {}^*\beta'$ for every α.

Let B be a representative of ${}^*\beta'$. By [§ 1D; 9], for every $z \in B$ we have $y_\alpha \leq z$ for all α. Since Y is a representative of β^*, it follows

$$\beta^* \leq {}^*\beta'.$$

2c. - It remains to prove that β^*, defined in (4) is the supremum of A with respect to M. This, however, follows from what has been proved. Indeed, if $A \leq \gamma$ where γ is a right ending, we have $\beta^* \leq \gamma$, and if γ is a left ending, we also have $\beta^* \leq \gamma$. The theorem is proved.

3. - **Theorem.** If $A \neq 0$ is a set of left endings, then A admits in M an infimum. This infimum is a left ending ${}^*\alpha$. It is also an infimum of A in M_l.

If, for every ${}^*\xi \in A$, we have ${}^*\xi \geq \beta^*$, then $\beta^* \leq {}^*\alpha$. Proof similar to the preceding one.

4. - **Theorem.** If $A \neq 0$ is a set of right endings in M, $B \neq 0$ is a set of left endings in M, and for every $\alpha^* \in A$ and every $\beta^* \in B$ we have $\alpha^* \leq {}^*\beta$, then

$$\sup A \leq \sup B.$$

5. - **Theorem.** If $A \neq 0$ is a set of right endings and does not admit a minimum, then A admits an infimum in M, which is a left ending ${}^*\beta$. We also have:

1) If, for every $\xi^* \in A$, we have $\gamma^* \leq \xi^*$, then $\gamma^* < {}^*\beta$.
2) If, for every $\xi^* \in A$, we have ${}^*\delta \leq \xi^*$, then ${}^*\delta \leq {}^*\beta$.
3) We have $\beta_1^* \underset{d}{=} \sup \{ \gamma^* \mid \gamma^* \leq A \} < {}^*\beta$.
4) β_1^* is the infimum of A in M_r.

Proof. We shall rely on the following *lemma*:

If $y \leq \alpha^*$, and E is a representative of α^*, then the set $\{ x \mid x \in E, \ x \geq y \}$ is also a representative of α^*.

5a. - There exists a distinguished well ordering

$$e^* > e_2^* > \dots > e_\alpha^* > \dots \tag{1}$$

coinitial with A, where $e_\alpha^* \in A$ for all α. By lemma [§ 1K; 2] for every α there exists $y \in M$ such that

$$e_{\alpha+1}^* < y \leq e_\alpha^*. \tag{2}$$

Let

$$a_1 \ a_2, \dots, a_\beta, \dots$$

be a distinguished well ordering with domain M. For every

α take, in (3), the element with the smallest index satisfying (2), and denote it by y_α, so that

$$(4) \qquad e^*_{\alpha+1} < y_\alpha \leq e^*_\alpha .$$

Denote by Y the set (1), and put

$$^*\beta \underset{df}{=} \mathfrak{L}\mathfrak{G}(Y) \text{ in } M.$$

Let $F_1, F_2, \dots, F_\alpha, \dots$ be representatives of the endings (1) resppectively, and put

$$F'_\alpha \underset{df}{=} \{ z \mid z \in F_\alpha,\ y_\alpha \leq z \}.$$

By the lemma (in this proof) F'_α is a representative of e^*_α. We get

(5) $\quad {}^*\beta < e^*_\alpha$ for all α, so ${}^*\beta$ is a lower bound of A in M.

If we suppose that ${}^*\beta < {}^*\gamma$ holds true, we can prove, by [§ 1D; 7] that there exists α such that $e^*_{\alpha+1} < {}^*\gamma$. Hence ${}^*\beta$ is among all left endings the greatest lower bound of A.

5b. - Suppose that for a right ending γ^* we have $\gamma^* \leq A$. Hence $\gamma^* \leq e^*_\alpha$ for all α.

It follows that $\gamma^* < y_\alpha$ for all α. let C be a representative of γ^*.

We have: for every $x \in C$, $x < y_\alpha$ for all y. Since Y is a representative of ${}^*\beta$, it follows that $\gamma^* < {}^*\beta$, [§ 1C; 3]. I say that the equality

$$(6) \qquad \gamma^* = {}^*\beta$$

is impossible. Indeed, if (6) were true, ${}^*\beta$ would be a point ending, and then the set Y would have a minimum, which is excluded. Thus

$$\gamma^* < {}^*\beta.$$

5c. - Consider the set of all right endings γ^* such that $\gamma^* \leq A$. This set admits a supremum, by [§ 1K; 2], which is a right ending, say β_1^*. We have

$$\beta_1^* \leq A;$$

hence, by what has been proved in [5b],

$$\beta_1^* < {}^*\beta.$$

Thus we have proved the thesis 3).

5d. - We have proved that if γ is a right ending with $\gamma \leq A$, then $\gamma \leq {}^*\beta$, and if γ is a left ending with $\gamma < A$, then $\gamma \leq {}^*\beta$. It follows that ${}^*\beta$ is the infimum of A in $\boldsymbol{M}$. Now, $\sup\{\gamma^* \mid \gamma^* \leq A\}$ is the infimum of A in $\boldsymbol{M}_r$, so 4) is proved. The theorem is established.

6. - **Theorem.** If $A \neq 0$ is a set of left endings and A does not admit a maximum, then A admits the supremum in $\boldsymbol{M}$ which is a right ending β^*. We have also:

1) If, for every ${}^*\xi \in A$, we have ${}^*\gamma \leq {}^*\xi$, then ${}^*\gamma > \beta^*$;
2) if, for every ${}^*\xi \in A$, we have $\delta^* \geq {}^*\xi$, then $\delta^* \geq \beta^*$;
3) we have ${}^*\beta_1 \underset{df}{=} \inf\{\gamma \mid {}^*\gamma \geq A\} > \beta^*$;
4) ${}^*\beta_1$ is the supremum of A in $\boldsymbol{M}_l$.

Proof analogous to the preceding one.

7. - **Theorem.** If $A \neq 0$ is a set of left endings then A admits a supremum in $\boldsymbol{M}$ which is a left ending whenever A admits a maximum, and which is a right ending in the opposite case. An analogous theorem holds true for non empty set of right endings.

8. - **Theorem.** If $A \neq 0$ is a set of any endings, then A admits a supremum in $\boldsymbol{M}$ and an infimum in $\boldsymbol{M}$.

Proof. We shall confine us to the proof of the existance of the supremum in the case where A does not admit any maximum. There exists a distinguished well ordering

$$(1) \qquad \gamma_1 < \gamma_2 < \ldots < \gamma_\alpha < \ldots$$

of endings, cofinal with A, where $\gamma_\alpha \in A$. Let

$$(2) \qquad \alpha_1^* < \alpha_2^* < \ldots < \alpha_\lambda^* < \ldots$$

be the subsequence of (1) composed of all right endings. (point endings included) and

(3) $$^*\beta_1 < {}^*\beta_2 < \dots < {}^*\beta_\mu < \dots$$

the subsequence of (1) composed of all left endings (point endings excluded).

Of course there may exist only one of these sequences. or one of them may be finite. In the case one of these sequences is finite or has a maximum, the theorem follows from the foregoing theorem. Suppose that both sequences (2), (3) are infinite, both without any maximum. It suffices to suppose that both sequences (3) and (2) are cofinal with (1).

The set (2) admits in $\mathfrak{M}$ a supremum ξ^* and the set (3) admits in $\mathfrak{M}$ a supremum η^*. We have $A \leq \xi^*$, $A \leq \eta^*$.

By [§ 1K; 2] we have $\xi^* \leq \eta^*$ and, by theor. [§ 1K; 6], $\eta^* \leq \xi^*$.

Consequently $\xi^* = \eta^*$. The theorem is proved.

9. - The above theorems are apt to overcome the main difficulty in dealing with non-archimedian linearly ordered fields, where a non empty set of quantities, even if bounded, may not admit any supremum or infimum.

10. - If M is a chain, then its right endings make up a new chain $\mathfrak{M}_r$ and the elements of $\mathfrak{M}_r$ satisfy the axioms of a linearly ordered semigroup, given in [§ 1E; 1], (see [§ 1F; 11]). Hence we can consider right and left endings in $\mathfrak{M}_r$. To these « double » endings we can apply the operations of additions and multiplication.

A similar behaviour shows $\mathfrak{M}_l$. Each M-ending $^*\alpha$ generates an $\mathfrak{M}$-point ending whose one of $\mathfrak{M}$-representatives is the set $\{ {}^*\alpha \}$ composed of the single element $^*\alpha$.

If Φ is a set of M-endings α^*, and Φ' the set of corresponding $\mathfrak{M}$-point endings $\mathfrak{RE}$ $(\{ \alpha^* \})$, then the supremum of Φ' in the ordering of $\mathfrak{M}$-endings can be smaller than the supremum of Φ in the ordering of M-endings. Indeed, the chain $\mathfrak{M}$ is « larger » than M.

§ 2. - Partitions in a tribe.

1. - Partitions are usually considered as an easy topic which does not require any detailed setting. Since the present paper requires precision, we shall devote some care to correct and not ambiguos definitions.

Let B be a finitely additive (abstract or constructed) non trivial tribe. By *partition in B* we shall understand any finite sequence

$a_1, a_2, \ldots, a_n$, $(n \geq 1)$, $a_i \in B$, such that $a_i \cdot a_j = 0_B$ for $i \neq j$, and $a_1 + \ldots + a_n = 1_B$.

It will be denoted by $\{a_i\}$.

Expl. The sequences $\{0, 1\}$; $\{0, 1, 0, 0, 0\}$; $\{\mathrm{co}\, a, a\}$, $\{1\}$ are partitions.

2. - If $\{a_i\}$ is a partition, there exists j such that $a_j \neq 0$. A partition $\{a_i\}$ is said to be *equal to the partition* $\{b_j\}$,

$$\{a_i\} \stackrel{\pi}{=} \{b_j\},$$

whenever for every i with $a_i \neq 0$ there exists j such that $a_i \stackrel{B}{=} b_j$ and for every j with $b_j \neq 0$ there exists i such that $b_j \stackrel{B}{=} a_i$.

The notion $\stackrel{\pi}{=}$ of equality of partitions is invariant with respect to the equality $\stackrel{B}{=}$, governing in B. The equality $\stackrel{\pi}{=}$ obeys the usual formal rules of identity.

3. - The following are equivalent: I. $\{a_i\} \stackrel{\pi}{=} \{b_j\}$, II. $\{a_i \mid a_i \neq 0\} \equiv \{b_j \mid b_j \neq 0\}$.

Let E be a non empty, finite set of mutually disjoint somata, such that

$$\Sigma_{a \in E}\, a = 1,$$

let $n \geq 1$ be the number of elements of E, and let $a(i)$ be a function with $\mathrm{C}\!\!\mid a$ equal to the set $(1, 2, \ldots, n)$ and with $\mathrm{D}\, a = E$. Then $\{a(i)\}$ is a partition in B. Let $a_1, \ldots, a_n$ be a partition, and T a one-to-one mapping of the set $1, 2, \ldots, n$ into itself, (permutation). If we put $b_j = a_{T(j)}$ for $j = 1, 2, \ldots, n$, then $\{b_j\}$ is also a partition, and we have $\{a_i\} \overset{\pi}{=} \{b_j\}$. If $a_1, \ldots, a$ is a partition, then $a_1, \ldots, a_n$, 0 is also a partition which is $\overset{\pi}{=}$-equal to the former one. If $a_1, \ldots, a_n$, 0 is a partition, so is $a_1, \ldots, a_n$, and the last is equal to the former one. To every partition $\{a_i\}$ there exists a partition $\{b_j\}$ such that all $b_j \neq 0$ and $\{a_i\} \overset{\pi}{=} \{b_j\}$. The set $\{b_j\}$ is unique.

4. - The above properties show that the partition is, up to equality, well determined by the set of its somata which differ from the null-soma. The presence or absence of a finite number of null-somata in the sequence $\{a_1, \ldots, a_n\}$ does not matter. The permutation of elements of the sequence does neither. Thus, from the point of view of formal logic, a partition is, with respect to $\overset{\pi}{=}$, neither a set of somata nor a sequence. But, with respect to $\overset{B}{=}$, a partition could be defined [9] as a suitable class of sequences of somata of B.

5. - A partition $\{a_j\}$ is said to be *a subpartition of* $\{b_j\}$, $\{a_i\} \overset{\pi}{\leq} \{b_j\}$, whenever for every i there exists j such that $a_i \overset{B}{\leq} b_j$.

If $\{a_i\} \overset{\pi}{\leq} \{b_j\}$, $\{b_j\} \overset{\pi}{\leq} \{c_k\}$, then $\{a_i\} \overset{\pi}{\leq} \{c_k\}$. We have $\{a_i\} \overset{\pi}{\leq} \{a_i\}$.

The following are equivalent: I. $\{a_i\} \overset{\pi}{=} \{b_j\}$. II. $\{a_i\} \overset{\pi}{\leq} \{b_j\}$ and $\{b_j\} \overset{\pi}{\leq} \{a_i\}$.

The notion $\overset{\pi}{\leq}$ is invariant with respect to the equality $\overset{\pi}{=}$ of partitions, and to the equality $\overset{B}{=}$, governing in B. If $\{a_i\}$, $\{b_k\}$ are partitions in B, then the double sequence $\{a_i, b_k\}$, if ordered in any way into a single sequence, is a subpartition of $\{a_i\}$ and $\{b_k\}$. If $\{b_k\}$ is a subpartition of $\{a_i\}$, then there exists a partition $\{a_{ik}\}$ such that 1) $\Sigma_k a_{ik} = a_i$ for all i, 2) $\{b_k\} \overset{\pi}{=} \{a_{ik}\}$.

6. - Given two partitions $\{a_i\}$, $\{b_j\}$, there exists a $\overset{\pi}{=}$-unique partition $\{c_k\}$, such that 1) $\{c_k\} \overset{\pi}{\leq} \{a_i\}$, $\{c_k\} \overset{\pi}{\leq} \{b_j\}$; 2) if $\{d_t\} \overset{\pi}{\leq} \{a_i\}$ and $\{d_t\} \overset{\pi}{\leq} \{b_j\}$, then $\{d_t\} \overset{\pi}{\leq} \{c_k\}$.

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[9]) differently from our setting.

This partition $\{c_k\}$ equals $\{p_{ij}\}$, where $p_{ij} \underset{df}{=} a_i \cdot b_j$ for all i and j.

We call $\{a_i b_j\}$, i. e. $\{c_k\}$, *product of* $\{a_i\}$ *and* $\{b_j\}$:

$$\{a_i\} \cdot \{b_j\}.$$

The product of two partitions is invariant with respect to $\overset{\pi}{=}$ and $\overset{B}{=}$. We have $\{a_i\} \cdot \{a_i\} \overset{\pi}{=} \{a_i\}$; the product is commutative and associative. The relation $\{a_i\} \overset{\pi}{\leq} \{b_j\}$ is equivalent to $\{a_i\} \cdot \{b_j\} \overset{\pi}{=} \{a_i\}$.

The *maximal partition is* the sequence $\{1_B\}$ composed of the single element 1_B.

It contains every partition. Of course, for any $a \in B$ the sequence $\{a, \text{co}\, a\}$ is a partition.

If $\{a_i \overset{\pi}{\leq} \{b_j\}$, then there exists a partition $\{c_k\}$ such that $\{a_i\} \overset{\pi}{=} \{b_i\} \cdot \{c_k\}$.

However, we notice that if $\{a_i\} \cdot \{b_k\} \overset{\pi}{=} \{a_i\} \cdot \{b'_t\}$, this does not imply the equality $\{b_k\} \overset{\pi}{=} \{b'_t\}$, so the cancellation law does not hold true.

7. - The above properties show that the set of all partitions constitutes an ordering which admits finite meets and a unit. It makes up a *down-stream* (directed set).

Remark. In a similar way we can define partition of a given, fixed soma $a \in B$, where $a \neq 0$.

§ 3. - Aggregates.

1. - Let B be a finitely additive, non trivial tribe and F a non trivial, commutative ring with unit. By a BF-aggregate we shall understand a finite sequence of ordered couples

$$(a_1, \lambda_1), (a_2, \lambda_2), \ldots, (a_n, \lambda_n), \; (n \geq 1),$$

where $a_1, \ldots, a_n$ is a partition in B (see [§ 2]) and $\lambda_i \in F$.

The aggregates will be written

$$[\lambda_1 a_1 \dot{+} \lambda_2 a_2 \dot{+} \ldots \dot{+} \lambda_n a_n] \text{ or } [\sum_{i=1}^{n}{}^{\boldsymbol\cdot} \lambda_i a_i], \; [\Sigma^{\boldsymbol\cdot} \lambda_i a_i] \text{ or } [\Sigma^{\boldsymbol\cdot} a_i \lambda_i]$$

where $\dot{+}$, $\Sigma^{\cdot}$ and [] are mere symbols (unqualified). The notion is $\overset{B}{=}$ and $\overset{F}{=}$ invariant. All a_i are disjoint and $\sum_i a_i = 1_B$.

2. - The aggregate $[\Sigma_i^{\cdot} \lambda_i a_i]$ is said *to be equal to the aggregate* $[\Sigma_j^{\cdot} \mu_j b_j]$,

$$[\Sigma^{\cdot} \lambda_i a_i] \overset{A}{=} [\Sigma^{\cdot} \mu_j b_j],$$

whenever for every i and j with $a_i \cdot b_j \neq 0_B$ we have $\lambda_i \overset{F}{=} \mu_j$.

The notion of equality of aggregates is invariant with respect to $\overset{B}{=}$ and $\overset{F}{=}$.

If $\{a_i\} \overset{\pi}{=} \{b_j\}$, [see § 2], and if for every i, j with $a_i = b_j$, $a_i \neq 0$ we have $\lambda_i = \mu_j$, then

$$[\Sigma_i^{\cdot} \lambda_i a_i] \overset{A}{=} [\Sigma_j^{\cdot} \mu_j b_j].$$

This theorem expresses a kind of invariance of $\overset{A}{=}$ with respect to equality $\overset{\pi}{=}$ of partitions.

If $\{a_i\}$ is a partition in B, $(i = 1, 2, \dots, n)$, $(n \geq 1)$, T is one-to-one correspondence with domain and range 1, 2, ..., n, and $\lambda_1, \dots, \lambda_n \in F$, then $[\lambda_1 a_1 \dot{+} \dots \dot{+} \lambda_n a_n] = [\lambda_{T(1)} a_{T(1)} \dot{+} \dots \dot{+} \lambda_{T(n)} a_{T(n)}]$.

It follows that for any aggregate we have

$$[\Sigma_i^{\cdot} \lambda_i a_i] = [\Sigma_i^{\cdot} \lambda_i a_i],$$

and if

$$[\Sigma_i^{\cdot} \lambda_i a_i] = \Sigma_j^{\cdot} \mu_j b_j], \text{ then } [\Sigma_j^{\cdot} \mu_j b_j] = [\Sigma_i^{\cdot} \lambda_i a_i].$$

If $a_1 = a_1' \dot{+} a_1''$, $a_1' \cdot a_1'' = 0$, then $[\lambda_1 a_1 \dot{+} \dots \dot{+} \lambda_n a_n] = [\lambda_1 a_1' \dot{+} \lambda_1 a_1'' \dot{+} \lambda_2 a_2 \dot{+} \dots \dot{+} \lambda_n a_n]$.

We have for any $\lambda_0 \in F$:

$$[\lambda_1 a_1 \dot{+} \dots \dot{+} \lambda_n a_n] = [\lambda_0 \cdot 0_B \dot{+} \lambda_1 a_1 \dot{+} \dots \dot{+} \lambda_n a_n].$$

Thus we see that permutation of terms and absence or presence of a term, involving a null-soma, does not change the aggregate up to $\overset{A}{=}$.

3. - $\{a_i\}$, $\{b_j\}$, $\{c_k\}$ are partitions in B, and

$$[\Sigma_i^{\cdot} \lambda_i a_i] = [\Sigma_j^{\cdot} \mu_j b_j], \text{ then } [\sum_{ijk}^{\cdot} \lambda_i (a_i b_j c_k)] = [\sum_{ijk}^{\cdot} \mu_j (a_i b_j c_k)]$$

If $\{a_i\}$, $\{b_j\}$, $\{c_k\}$ are partitions in B, and

$$[\sum_{ijk}^{\cdot} \lambda_i(a_i b_j c_k)] = [\sum_{ijk}^{\cdot} \mu_j(a_i b_j c_k)], \quad \text{then} \quad [\sum_i^{\cdot} \lambda_i a_i] = [\sum_j^{\cdot} \mu_j b_j].$$

This allows to prove the transitivity of the equality $\overset{A}{=}$. Thus the equality $\overset{A}{=}$ of aggregates satisfies the formal conditions of identity. The above theorems imply that the presence or absence of a finite numbers of terms with null-soma, permutation of terms, and splitting of somata into a finite number of disjoint parts do not affect the aggregates-up to $\overset{A}{=}$.

4. - If $\{b_j\}$ is a partition in B, then $[\Sigma_i^{\cdot} \lambda_i a_i] = [\sum_{i}^{\cdot}{}_j \lambda_i(a_i b_j)$.

Any two aggregates $[\Sigma_i^{\cdot} \lambda_i a_i]$, $[\Sigma_j^{\cdot} \mu_j b_j)]$ can be represented in the form involving the same partition: $\sum_{i}^{\cdot}{}_j \lambda_i(a_i b_j)$, $\sum_{i}^{\cdot}{}_j \mu_j(a_i b_j)$.

5. - By the ***sum of two aggregates***, $[\Sigma_i^{\cdot} \lambda_i a_i] + [\Sigma_j^{\cdot} \mu_j b_j]$ we shall understand the aggregate $\Sigma_{ij}^{\cdot} (\lambda_i + \mu_j) \cdot (a_i b_j)$.

The notion of sum is invariant with respect to $\overset{F}{=}$, $\overset{B}{=}$ and to the equality $\overset{A}{=}$ of aggregates. We have

$$[\Sigma_i^{\cdot} \lambda_i a_i] + [\Sigma_i^{\cdot} \mu_i a_i] \overset{A}{=} [\Sigma_i^{\cdot} (\lambda_i + \mu_i) a_i].$$

6. - If $\lambda \in F$, then by ***the product*** λ. $[\Sigma_i^{\cdot} \lambda_i a_i]$ we shall understand the aggregate $[\Sigma_i (\lambda\lambda_i) a_i]$.

This notion of multiplication of an aggregate by a « scalar » is invariant with respect to $\overset{F}{=}$, $\overset{B}{=}$ and $\overset{A}{=}$.

7. - The addition of aggregates is commutative and associative. If X, Y, Z are aggregates and $X + Y \overset{A}{=} X + Z$, then $Y \overset{A}{=} Z$.

We have $\lambda(X + Y) \overset{A}{=} \lambda X + \lambda Y$ for any aggregates X, Y and any $\lambda \in F$.

The following holds true:

$$\lambda(\mu X) \overset{A}{=} (\lambda\mu)X, \quad (\lambda + \mu)X \overset{A}{=} \lambda X + \lambda Y, \quad 1_F \cdot X \overset{A}{=} X.$$

Consequently the class of all BF-aggregates is organized by $\overset{A}{=}$, addition and multiplication into a linear vector-space with multipliers taken from F.

The aggregate $[0_F \cdot 1_B]$ is the *zero-vector* 0_A of the space. It is $\overset{A}{=}$-unique.

The subtraction can be defined, and the *difference* $X - Y$ of two aggregates is $\overset{A}{=}$ invariant. We define $+X \underset{df}{=} 0_A + X$, $-X \underset{df}{=} 0_A - X$.

We have $X - Y \overset{A}{=} X + (-1)Y$, where $(-1) \in F$.

8. - We have for any aggregate

$$[\Sigma_i^{\cdot} \lambda_i a_i] \overset{A}{=} \lambda_1[1_F \cdot a_1 \dotplus 0_F \cdot \mathrm{co}\, a_1] + \lambda_2[1_F \cdot a_2 \dotplus 0_F \cdot \mathrm{co}\, a_2] + \ldots$$

If we put, in general. for $b \in B$: $Sb \underset{df}{=} [1_F \cdot b \dotplus 0_F \cdot \mathrm{co}\, b]$, we get $[\Sigma_i^{\cdot} \lambda_i a_i] \overset{A}{=} \Sigma_i \lambda_i S a_i$.

Now, we can simplify orthography by writing a_i instead of Sa_i, getting the following manner of writing aggregates $\Sigma_i \lambda_i a_i$, where the sum means summation of aggregates. In the sequel we shall use only this manner of writing, because no ambiguity will be caused by this licence.

9. - If the ring F is linearly ordered, so that the absolute value $|\lambda|$ of $\lambda \in F$ is available, we define the *absolute value of* $X \underset{df}{=} \Sigma_i \lambda_i a_i$ as

$$|X| \underset{df}{=} \sum_i |\lambda_i| a_i.$$

The notion of $|X|$ is $\overset{A}{=}$-invariant, hence it does not depend one the representation of X as sum.

§ 4. - Exterior and interior Jordan-measure.

1. - Let $\mathfrak{b}$ be a finitely additive, non trivial tribe. $\mu(a) > 0$ a finitely additive measure on $\mathfrak{b}$, with values taken from a linearly ordered not trivial field f. We suppose that $\mu(1_b) > 0$.

Let $\mathfrak{b}'$ be a *finite operations-zero — and unit — genuine strict supertribute of* $\mathfrak{b}$. This means that 1) the equality $\overset{b}{=}$ is the restriction of $\overset{b}{=}$ is the restriction of $\overset{b'}{=}$ to $\mathfrak{b}$, i. e. if $a \in \mathfrak{b}$, then the following are equivalent: I. $a \overset{b'}{=} b$, II. $a \overset{b}{=} b$,

2) if a, b, $c \in \boldsymbol{b}$, then the relations $a \overset{b}{+} b = c$ and $a \overset{b'}{+} b = c$ are equivalent, and so are $a \overset{b}{\cdot} a = c$ and $a \overset{b'}{\cdot} b = c$; 3) $0_b = 0_{b'}$, 4) $1_b = 1_{b'}$. (Compare **(11)** and [Preface 8]). We have $\boldsymbol{b} \subseteq \boldsymbol{b}'$. The operations, zero, unit and the equality governing on $\boldsymbol{b}$ are just taken from $\boldsymbol{b}'$. The somata of $\boldsymbol{b}$ will be denoted by small latin letters and those of $\boldsymbol{b}'$ by latin capitals. $\mu(a)$ is supposed to be equality invariant.

2\. - *By a $\boldsymbol{b}$-(Jordan) covering of $A \in \boldsymbol{b}$* we shall understand any non empty set φ composed of a finite number of somata of $\boldsymbol{b}$: $a_1, a_2, \ldots, a_n$, $(n \geq 1)$, such that $A \subseteq a_1 + \ldots + a_n$. We put $J(\varphi) \underset{df}{=} \mu(a_1) + \ldots + \mu(a_n)$ and $E(A) \underset{df}{=} \{ J(\varphi) \mid \varphi$ is a $\boldsymbol{b}$-Jordan covering of $A \}$.

By $\boldsymbol{b}$-exterior measure of A with respect to μ and f we shall understand the left-ending

$$^*\mu_{bf}(A) \underset{df}{=} \mathfrak{L}\mathfrak{G}(E(A)).$$

By $\boldsymbol{b}$-interior measure of A with respect to μ and f we shall understand the right-f-ending:

$$\mu^*_{bf}(A) \underset{df}{=} \mu_{bf}(1_b) - {}^*\mu_{bf}(\text{co } A).$$

If no misunderstanding is feared, we shall write simplier $^*\mu_b(A)$, $\mu_f(A)$ or even $^*\mu(A)$ and a similar agreement is admitted for μ^*.

3\. - Concerning μ, we have $\mu(0_b) = 0_f$, and for any a, $b \in \boldsymbol{b}$ we have $\mu(a + b) \leq \mu(a) + \mu(b)$. If $a \subseteq b$, then $\mu(a) \leq \mu(b)$.

Theorem. The following are equivalent:

I. $A \in \boldsymbol{b}$; II. $^*\mu_{bf}(A) = \mu^*_{bf}(A) = \mu(A)$.

Proof. Let. I., i.e. $A \in \boldsymbol{b}$. Put $a \underset{df}{=} A$. Let $a_1, \ldots, a_n$, $(n \geq 1)$ be a covering of A. We have: $\mu(a) \leq \mu(a_1) + \ldots + \mu(a_n)$.

Hence the set $E(A)$ has the minimum $\mu(a)$; indeed, a is a covering of A.

It follows [§ 1A; 9]

$$(1) \qquad {}^*\mu(A) = \mu(a).$$

We have $\operatorname{co} A = \operatorname{co} a \in \boldsymbol{b}$. Hence, by what has been proved for somata of $\boldsymbol{b}$, ${}^*\mu(\operatorname{co} A) = \mu(\operatorname{co} a)$; indeed $\operatorname{co} A \in \boldsymbol{b}$.

Since $a + \operatorname{co} a = 1$, $a \cdot \operatorname{co} a = 0$, it follows, by additivity of μ, $\mu(1) = \mu(a) + \mu(\operatorname{co} a)$. Hence

$$\mu^*(A) = \mu(1) - {}^*\mu(\operatorname{co} A) = \mu(a). \tag{2}$$

From (1) and (2) follows ${}^*\mu(A) = \mu^*(A)$, i. e. II.

Let II, i. e. $\mu^*(A) = {}^*\mu(A)$. By [§ 1C; 4], there exists a quantity $\lambda \in \boldsymbol{f}$ such that $\{\lambda\}$ is a representative of both these endings. Since $E(A)$ is a representative of ${}^*\mu(A)$, it has the minimum λ, [§ 1A; 9]. Hence there exists a covering $a_1, \ldots, a_n$ of A such that $\lambda = \mu(a_1) + \ldots + \mu(a_n)$, $(n \geq 1)$.

Now, the set, composed of the single soma $b \underset{df}{=} a_1 + \ldots + a_n$, is also a covering of A, and we have

$$\mu(b) \leq \mu(a_1) + \ldots + \mu(a_n).$$

Hence $\mu(b) \leq \lambda$. Since λ is a minimum, we get $\mu(b) = \lambda$. We also have

$$A \subseteq b. \tag{3}$$

The set $\{\lambda\}$ is a representative of $\mu^*(A)$, i. e. of [§ 1F; 27],

$$\mu(1) - {}^*\mu(\operatorname{co} A) = \mu(1) + (-{}^*\mu(\operatorname{co} A)).$$

Hence $\lambda = \mu(1) + (-{}^*\mu(\operatorname{co} A))$, and then, by [§ 1G; 2], $\lambda - \mu(1) = -{}^*\mu(\operatorname{co} A)$.

Hence [§ 1F; 22] $-(\lambda - \mu(1)) = {}^*\mu(\operatorname{co} A)$,

$$-\lambda + \mu(1) = {}^*\mu(\operatorname{co} A).$$

Hence ${}^*\mu(\operatorname{co} A)$ is a point-ending. By similar argument we get a soma b' such that

$$\operatorname{co} A \subseteq b'. \tag{3.1}$$

Hence, by (3), $\operatorname{co} b' \subseteq A \subseteq b$. It follows

$$b \cdot b' = 0. \tag{4}$$

Now, as $A \subseteq b$, $\operatorname{co} A \subseteq b'$, it follows that $A + \operatorname{co} A \subseteq b + b' \subseteq 1$, and then

$$b + b' = 1. \tag{5}$$

From (4) and (5) we obtain

$$b' = \operatorname{co} b. \tag{6}$$

We have, from (3) $A \subseteq b$ and, from (3.1) and (6), $\operatorname{co} A \subseteq \subseteq \operatorname{co} b$, i. e. $b \subseteq A$.

Thus we get $A = b$ and, consequently $A \in \boldsymbol{b}$, i. e. we get I. The theorem is proved.

4. - **Theorem.** For any $A \in \boldsymbol{b}'$ we have

$$^*\mu_{bf}(A) = \mathfrak{L}\mathfrak{E}\,\{\,\mu(a) \mid A \subseteq a\,\}, \quad \mu^*_{bf}(A) = \mathfrak{R}\mathfrak{E}\,\{\,\mu(a) \mid a \subseteq A\,\}.$$

Proof. Let $A \subseteq a$. Since $\{a\}$ is a covering of A, we have $\mu(a) \in E(A)$; hence, by [§ 1A ; 4], $E(A) \cdot\leq \mu(a)$. Since $E(A)$ is a representative of $^*\mu(A)$, it follows, [§ 1A ; 11],

$$\mu^*(A) \leq \mu(a). \tag{1}$$

The inequality (1) holds for every a with $A \subseteq a$. Hence [§ 1G ; 7]

$$^*\mu(A) \leq \mathfrak{L}\mathfrak{E}\,\{\,\mu(a) \mid A \subseteq a\,\}. \tag{2}$$

The inequality (2) can be inverted. Indeed, let $y \in \{\,J(\varphi) \mid \varphi$ is a $\boldsymbol{b}$–Jordan covering of $A\,\} = E(A)$.

There exists a covering $\{a_k\}$ of A such that $y = \Sigma_k\, \mu(a_k]$. Put $a \underset{df}{=} \Sigma_k\, a_k$. We have $\mu(a) \leq \Sigma_k\, \mu(a_k)$, $x \underset{df}{=} \mu(a) \in \{\,\mu(b) \mid A \subseteq b\,\}$, $x \leq y$.

Hence, for every y belonging to $E(A)$ there exists $x \in \{\,\mu(b) \mid A \subseteq b\,\}$ with $x \leq y$.

This gives, [§ 1A ; 1], $\{\,\mu(b) \mid A \subseteq b\,\} \cdot\leq E(A)$, and then, [§ 1A ; 6],

$$\mathfrak{L}\mathfrak{E}\,\{\,\mu(b) \mid A \subseteq b\,\} \leq {}^*\mu(A). \tag{3}$$

From (2) and (3) it follows

$$^*\mu(A) = \mathfrak{L}\mathfrak{E}\,\{\,\mu(a) \mid A \subseteq a\,\}. \tag{4}$$

To prove the second formula of the thesis, notice that

$\mu^*(A) = \mu(1) - {}^*\mu(\operatorname{co}\, A) = \mu(1) - \mathfrak{L}\mathfrak{E}\,\{\,\mu(b) \mid \operatorname{co} A \subseteq b\,\}$, by (4).

Hence $\mu^*(A) = \mu(1) + [-\mathfrak{LE}\{\mu(b) \mid \mathrm{co}\, A \subseteq b\}]$, and then, by [§ 1G; 2], $\mu^*(A) - \mu(1) = -\mathfrak{LE}\{\mu(b) \mid \mathrm{co}\, A \subseteq b\}$, [§ 1F; 21], $= \mathfrak{RE}\{-\mu(b) \mid \mathrm{co}\, A \subseteq b\}$. Hence, by [§ 1G; 2],

(5) $$\mu^*(A) = \mu(1) + \mathfrak{RE}\{-\mu(b) \mid \mathrm{co}\, A \subseteq b\}.$$

$\{\mu(1)\}$ is a representative of the right ending $\mu(1)$ and $\{-\mu(b) \mid \mathrm{co}\, A \subseteq b\}$ is a representative of its right ending. Hence

(6) $$\{\mu(1)\} + \{-\mu(b) \mid \mathrm{co}\, A \subseteq b\} = \{\mu(1) - \mu(b) \mid \mathrm{co}\, A \subseteq b\}$$

is, [§ 1F; 3], a representative of the ending (5), on the right side of the equation.

Now (6) is equal to $\{\mu(\mathrm{co}\, b) \mid \mathrm{co}\, A \subseteq b\}$. Since the inclusion $\mathrm{co}\, A \subseteq b$ is equivalent to $\mathrm{co}\, b \subseteq A$, we have

$$\{\mu(\mathrm{co}\, b) \mid \mathrm{co}\, A \subseteq b\} = \{\mu(\mathrm{co}\, b) \mid \mathrm{co}\, b \subseteq A\} = \{\mu(c) \mid c \subseteq A\}.$$

Hence

$$\mu^*(A) = \mathfrak{RE}\{\mu(c) \mid c \subseteq A\},$$

which proves the second part of the theorem.

5. - **Theorem**. For every $A \in \mathfrak{b}'$ we have $\mu^*_{bf}(A) \leq {}^*\mu_{bf}(A)$.
Proof. Take a, b with $a \subseteq A \subseteq b$. We have $\mu(a) \leq \mu(b)$.
If we vary a and b, we get by [§ 1E, 13]

$$\{\mu(a) \mid a \subseteq A\} \leq \{\mu(b) \mid A \subseteq b\};$$

hence, by [1F; 3],

$$\mathfrak{RE}\{\mu(a) \mid a \subseteq A\} \leq \mathfrak{LE}\{\mu(b) \mid A \subseteq b\}.$$

Applying theor. [4] we obtain

$$\mu^*(A) \leq {}^*\mu(A). \qquad \text{Q. E. D.}$$

6. - **Theorem**. If $A \in \mathfrak{b}'$, $\{a_i\}$ is a partition in $\mathfrak{b}$ (see [§ 2], $\{a_{ik}\}$ is a subpartition of $\{a_i\}$, [§ 2; 3], α is the set of all i for which $a_i \cdot A \neq 0$ and β is the set of all couples (i, k) for which $a_{ik} \cdot A \neq 0$, then

$${}^*\mu_{bf}(A) \leq \sum_{(i,k)\in\beta} \mu(a_{ik}) \leq \sum_{i\in\alpha} \mu(a_i).$$

If $A \in \boldsymbol{b}'$, $\{a_i\}$ is a partition in $\boldsymbol{b}$, $\{a_{ik}\}$ a subpartition of $\{a_i\}$, α' is the set of all i for which $a_i \subseteq A$ and β' is the of all (i, k) for which $a_{ik} \subseteq A$, then

$$\mu^*_{bf}(A) \geq \sum_{(i,k)\in\beta'} \mu(a_{ik}) \geq \sum_{i\in\alpha'} \mu(a_i)$$

7. - **Theorem.** If $A, B \in \boldsymbol{b}'$, then

$${}^*\mu_{bf}(A + B) \leq {}^*\mu_{bf}(A) + {}^*\mu_{bf}(B).$$

Proof. The sets $E \underset{df}{=} \{\mu(a) \mid A \subseteq a\}$, $F \underset{df}{=} \{\mu(b) \mid B \subseteq b\}$, $G \underset{df}{=} \{\mu(c) \mid A + B \subseteq c\}$ are representatives of ${}^*\mu(A)$, ${}^*\mu(B)$ and ${}^*\mu(A + B)$ respectively. Hence, by [§ 1F; 4] $E + F$ is a representative of ${}^*\mu(A) + {}^*\mu(B)$.

Let $u \in E + F$. There exist $x \in E$, $y \in F$ with $u = x + y$. There exist a, b with $A \subseteq a$, $B \subseteq b$ such that $x = \mu(a)$, $y = \mu(b)$; hence $u = \mu(a) + \mu(b)$. We have $v \underset{df}{=} \mu(a + b) \leq \mu(a) + {} + \mu(b) = x + y = u$. We also have $v \in G$, because $A + B \subseteq {} \subseteq a + b$.

Consequently for every $u \in E + F$ there exists $v \in G$ such that $v \leq u$.

Hence, by [§ 1A; 1] $G \cdot\!\leq E + F$, and then

$${}^*\mu(A + B) \leq {}^*\mu(A) + {}^*\mu(B). \qquad \text{Q. E. D.}$$

§ 5. - Linear functionals of aggregates and the norm of an aggregate.

1. - **Hypothesis**: $\boldsymbol{b}'$ is a non trivial finitely additive tribe, $\boldsymbol{b}$ is its finite operations-zero and unit strict subtribe, (see [§4; 1] and [Preface 8]), i. e. with the same zero, unit and finite operations, and with the same equality, restricted to $\boldsymbol{b}$. We have $\boldsymbol{b} \subseteq \boldsymbol{b}'$.

The somata of $\boldsymbol{b}$ will be denoted by $a, b, c, \ldots$, and those of $\boldsymbol{b}'$ by $A, B, C, \ldots$, $M(a)$ is a finitely additive, $\overset{B}{=}$ invariant measure on $\boldsymbol{b}$; $M(a) \geq 0$; $M(1_b) > 0$; the values of M are taken from a given, not trivial, linearly ordered field $\boldsymbol{f}$. We shall consider $\boldsymbol{b}'\boldsymbol{f}$-aggregates, (see [§ 3]).

2. - LEMMA. If $\Sigma_i \lambda_i A_i \overset{A}{=} \Sigma_i \mu_i A_i$ are $b'f$-aggregates, and if $a \in b$, then

$$\{ \lambda_i \mid aA_i \neq 0 \} = \{ \mu_i \mid aA_i \neq 0 \}.$$

LEMMA. If 1. $\Sigma_i \lambda_i A_i \overset{A}{=} \sum_{kl} \mu_{kl} B_{kl}$ are $b'f$-aggregates, where $B_{kl} = A_k B_l$.

2. $\{ B_l \}$ is a partition in b', (see [§ 2]).

3. $a \in b$,

then $$\{ \lambda_i \mid aA_i \neq 0 \} \equiv \{ \mu_{kl} \mid aB_{kl} \neq 0 \}.$$

These two lemmas yield the following

LEMMA. If $\Sigma_i \lambda_i A_i \overset{A}{=} \Sigma_j \mu_j B_j$ are $b'f$-aggregates, and $a \in b$, then

$$\{ \lambda_i \mid aA_i \neq 0 \} \equiv \{ \mu_j \mid aB_j \neq 0 \},$$

and then

$$\max \{ | \lambda_\alpha | \mid a \cdot A_\alpha \neq 0 \} = \max \{ | \mu_\beta | \mid aB_\beta \neq 0 \}.$$

3. - Let $\{ a_1, \ldots, a_m \}$, $(m \geq 1)$ be a partition in b and $X \underset{df}{=} \Sigma_i \lambda_i A_i \overset{A}{=} \Sigma_j \mu_j B_j$ be $b'f$-aggregates. We have

$$\sum_{k=1}^{m} M(a_m) \max \{ |\lambda_\alpha| \mid a_k \cdot A_\alpha \neq 0 \} = \sum_{k=1}^{m} M(a_k) \max \{ |\mu_\beta| \mid a_k B_\beta \neq 0 \}.$$

This quantity of f, depending only on X and on $\{ a_k \}$, will be denoted by

$$\mathcal{G}[\{ a_k \}, X]$$

This quantity is invariant with respect to $\overset{b'}{=}$, $\overset{b}{=}$, $\overset{\pi}{=}$, $\overset{f}{=}$, and $\overset{A}{=}$.

If $\{ b_j \}$ is a subpartition of $\{ a_k \}$, and both are partitions in b, and X is a $b'f$-aggregate, then

$$\mathcal{G}[\{ a_i \}, X] \geq \mathcal{G}[b_j \}, X].$$

4. - **Definition.** Given a $b'f$-aggregate X, consider the set $\mathcal{G}_X$ of all quantities $\mathcal{G}[\{ b_k \}, X]$ where $\{ b_k \}$ are all partitions in b (see [§ 2]). Its left ending in f will be termed *bM-norm of X* and denoted by $^*p_{bM}(X)$. We have

$$^*p_{bM}(X) = \mathfrak{L}\mathfrak{E}\mathcal{G}_X .$$

The norm is defined for all $b'f$-aggregates X.

5. - If $X = \Sigma_i \lambda_i a_i$ is a bf-aggregate, then its bM-norm, ${}^*p(X)$, is a point-ending, and we have ${}^*p(X) = \Sigma_i M(a_i) \cdot |\lambda_i|$.

To prove that, we first show that for every subpartition $\{b_j\}$ in b of $\{a_i\}$ we have

$$\mathcal{G}[\{b_j\}, X] = \text{const} = \Sigma_i M(a_i) \cdot |\lambda_i|,$$

and then we go over to general partitions, proving a similar equality.

Remark. The converse is not true. ${}^*p(X)$ can be a point-ending, even if X is not a bf-aggregate.

6. - **Theorem.** If X, Y are $b'f$-aggregates, then ${}^*p_{bM}(X+Y) \leq {}^*p_{bM}(X) + {}^*p_{bM}(Y)$.

Proof. We shall prove it by proving that $\mathcal{G}_{X+Y} \cdot \leq \mathcal{G}_X + \mathcal{G}_Y$, (see [§ 1A; 6], [§ 1F; 3]).

Let $\zeta \in \mathcal{G}_X + \mathcal{G}_Y$, [§ 1E]. There exist

$$\xi \in \mathcal{G}_Y, \ \eta \in \mathcal{G}_Y \ \text{with} \ \zeta = \xi + \eta. \tag{1}$$

There exist pertitions $\{a_i\}$, $\{b_k\}$ in b such that, [3],

$$\xi = \mathcal{G}[\{a_i\}, X], \quad \eta = \mathcal{G}[\{b_k\}, Y]. \tag{2}$$

Take them. Since $\{a_i b_k\}$ is a subpartition of $\{a_i\}$ and $\{b_k\}$, we have

$$\mathcal{G}[\{a_i b_k\}, X] \leq \mathcal{G}[\{a_i\}, X], \quad \mathcal{G}[\{a_i b_k\}, Y] \leq \mathcal{G}[\{b_k\}, Y].$$

Hence, by (1) and (2)

$$\mathcal{G}[\{a_i b_k\}, X] + \mathcal{G}[\{a_i b_k\}, Y] \leq \xi + \eta = \zeta. \tag{3}$$

Now X, Y have representatives with a same partition in b':

$$X = \sum_\alpha \lambda_\alpha A_\alpha, \ Y = \sum_\alpha \mu_\alpha A_\alpha,$$

(see [§ 3; 4]), so

$$X + Y = \sum_\alpha (\lambda_\alpha + \mu_\alpha) A_\alpha, \ [\S\ 3;\ 5].$$

We have

$$\mathcal{G}[\{a_i b_j\}, X) = \sum_{ik} M(a_i b_k) \cdot \max |\lambda_\alpha|,$$

$$\mathcal{G}[\{a_i b_k\}, Y] = \sum_{ik} M(a_i b_k) \cdot \max |\mu_\alpha|.$$

$$\mathcal{G}[\{a_i b_k\}, X + Y] = \sum_{ik} M(a_i b_k) \max |\lambda_\alpha + \mu_\alpha|,$$

all these maxima being taken for all α for which

$$A_\alpha \cdot a_i \cdot b_k \neq 0.$$

Now we have for these α

$$\max |\lambda_\alpha + \mu_\alpha| \leq \max |\lambda_\alpha| + \max |\mu_\alpha|,$$

so, since $M \geq 0$, we get, taken (3) into account:

$$\delta \underset{df}{=} \mathcal{G}[\{a_i b_k\}, X + Y] \leq \mathcal{G}[\{a_i b_k\}, X] + \mathcal{G}[\{a_i b_k\}, Y] \leq \zeta.$$

As $\delta \in \mathcal{G}_{X+Y}$, we get: For every $\zeta \in \mathcal{G}_X + \mathcal{G}_Y$ there exists $\delta \in \mathcal{G}_{X+Y}$, such that $\delta \leq \zeta$, [§ 1A; 1]. Hence $\mathcal{G}_{X+Y} \cdot\!\leq \mathcal{G}_X + \mathcal{G}_Y$, which gives

$$*p(X + Y) \leq *p(X) + *p(Y).$$

7. - **Theorem.** If X is a $b'f$-aggregate, $\lambda \geq 0$, $\lambda \in f$, then for the bM-norm we have

$$*p_{bM}(\lambda X) = \lambda \cdot *p_{bM}(X).$$

Proof. Take a partion $\{a_i\}$ in b, and let $X = \Sigma_\alpha \lambda_\alpha A_\alpha$. We get $\mathcal{G}[\{a_i\}, \lambda X) = \lambda \cdot \mathcal{G}[\{a_i, X\}]$, and then $\mathfrak{L}\mathfrak{E}\mathcal{G}_{\lambda X} = \mathfrak{L}\mathfrak{E}(\lambda \cdot \mathcal{G}_X)$. Hence, by [1F; 5],

$$*p(\lambda X) = \lambda * p(X).$$

8. - **Theorem.** If X is a $b'f$-aggregate, then $*p_{bM}(-X) = *p_{bM}(X)$.

9. - If $X = 1_b \cdot 0_f$, then $*p(X) = 0_f$. We have for any $b'f$-aggregate X the equality $*p(X) = *(|X|)$.

10. - **Theorem.** If X, Y are $b'f$-aggregates and $|X| \leq |Y|$,

then for every partition $\{b_k\}$ in $\boldsymbol{b}$ we have $\mathcal{G}[\{b_k\}, X] \leq \leq \mathcal{G}[\{b_k\}, Y]$ and then

$$^*p(X) \leq {}^*p(Y).$$

We rely on [§ 1A; 1] and [§ 1A; 6].

11. - By the *characteristic aggregate of a soma* $A \in \boldsymbol{b}'$ we understand the aggregate Ω_A defined by $\Omega_A \underset{df}{=} 1_f \cdot A + 0_f \cdot \text{co}\, A$.

Theorem. If $A \in \boldsymbol{b}'$, then $^*p_{bM}(\Omega_A) = {}^*M_{bf}(A)$, where $^*M(A)$ is the exterior measure of A with respect to $\boldsymbol{b}$ and f (see [§ 4]).

Proof. Put $X \underset{df}{=} \Omega_A = 1_f \cdot A + 0_f \cdot \text{co}\, A$ and $\lambda_1 \underset{df}{=} 1$, $\lambda_2 \underset{df}{=} 0$, $A_1 \underset{df}{=} A$, $A_2 \underset{df}{=} \text{co}\, A$. We have

$$X = \lambda_1 A_1 + \lambda_2 A_2 .$$

Take a $\boldsymbol{b}$-partition $\{b_k\}$. We have

(1) $$\mathcal{G}[\{b_k\}, X] = \Sigma_k M(b_k) \cdot \max |\lambda_\alpha| \text{ for } b_k \cdot A_\alpha \neq 0.$$

Consider the sets of indices

$$K_1 \underset{df}{=} \{k \mid b_k \subseteq A\}, \quad K_2 \underset{df}{=} \{k \mid b_k \subseteq \text{co}\, A\},$$
$$K_3 \underset{df}{=} \{k \mid b_k \cdot A \neq 0 \text{ and } b_k \cdot \text{co}\, A \neq 0\}.$$

If $k \in K$, we have

$$\max_{b_k \cdot A_\alpha \neq 0} |\lambda_\alpha| = 1;$$

if $k \in K_2$, we have

$$\max_{b_k \cdot A_\alpha \neq 0} |\lambda_\alpha| = 0;$$

and if $k \in K_3$, we have

$$\max_{b_k \cdot A_\alpha \neq 0} |\lambda_\alpha| = 1.$$

Hence for all $k \in K_1 \cup K_3$, the coefficient by $M(b_k)$, in (1), is 1, and for all $k \in K_2$, the coeficient is $= 0$.

The sets K_1, K_2, K_3 are mutually disjoint and their union is the set of all indices k. Thus

(2) $$\mathcal{G}[\{b_k\}, X] = \Sigma_{k \in K_1 \cup K_3} M(b_k).$$

The soma $b \underset{df}{=} \Sigma_{k \in K_1 \cup K_3} b_k$ is the «smallest covering» of A taken from the partition $\{ b_k \}$, and we have $\mathcal{G}[\{ b_k \}, X] = M(b)$.

On the other hand we know, [§ 4; 4] that

$$*M(A) = \mathfrak{LE} \{ M(b) \mid A \subseteq \dot{b} \}.$$

Now, if $A \subseteq b$, the sequence b, co b makes up a b-partition. Applying the obtained result (2) to this partition, we see that

$$M(b) = \mathcal{G}[\{ b, \text{co}\, b \}, X].$$

It follows that $*M(A)$ is the left ending of the set of all $\mathcal{G}[\{ b_k \}, X]$; hence $\mathfrak{LE}\mathcal{G}_X = *M(A)$, i. e.

$$*p(\Omega_A) = *M(A).$$

12. - **Theorem.** If $M(a)$ is an effective, f-valued, non negative measure on $\boldsymbol{b}$, X is a $\boldsymbol{b}'\boldsymbol{f}$ aggregate, $*p_{bM}(X) = 0$, then $X = 0_f \cdot 1_b$.

Proof. By hypothesis, $*p(X)$ is a point ending. Hence [§ 1A; 9] there exists a partition $\{ b_k \}$ in $\boldsymbol{b}$ such that $\mathcal{G}[\{ b_k \}, X] = 0$. We have

(1) $$\mathcal{G}[\{ b_k \}, |X|] = \Sigma_k M(b_k) \cdot \max_{A_\alpha \cdot b_k \neq 0} |\lambda_\alpha| = 0.$$

Since all terms in (1) are non negative, we get for all k:

$$M(b_k) \cdot \max_{A_\alpha \cdot a_k \neq 0} |\lambda_\alpha| = 0.$$

Let $b_k \cdot A_\alpha \neq 0$. We have $b_k \neq 0$, and then, by the effectiveness of M, $M(b_k) \neq 0$. Hence $\max |\lambda_\alpha| = 0$.

Thus for every α with $b_k \cdot A_\alpha \neq 0$ we have $\lambda_\alpha = 0$. Since b_k is a partition, therefore, given α, there exists k' with $A_\alpha \cdot b_{k'} \neq 0$. Hence $\lambda_\alpha = 0$. Thus $X = \Sigma_\alpha 0 \cdot A_\alpha$, and then $X = 0_f \cdot 1_b$.

13. - **Remark.** It may happen that there exists a $\boldsymbol{b}'\boldsymbol{f}$-aggregate X such that

$$*p(X) = \mathfrak{LE} \{ a \mid a \in f,\ a > 0 \}.$$

14. - By [§ 3; 7] the $\boldsymbol{bf}$-aggregates make up a linear vector-space V with multipliers taken from f.

Define for all *bf*-aggregates X: $f(X) \underset{df}{=} \Sigma_i M(a_i) \cdot \lambda_i$, where $X = \Sigma_i \lambda_i \cdot a_i$.

The function $f(X)$ is invariant with respect to the $\overset{A}{=}$ equality *bf*-aggregates.

$\mathbb{C} f = V$, $\mathbb{D} f \subseteq f$. We have $f(\lambda X) = \lambda f(X)$ for all $X \in V$ and all $\lambda \in f$, and we also have

$$f(X + Y) = f(X) + f(Y) \quad \text{for all} \quad X,\ Y \in V.$$

Thus $f(X)$ is an f-valued linear functional in V with multipliers taken from f.

15. - Theorem. We have $f(X) \leq {}^*p_{bM}(X)$ for all *bf*-aggregates X.

In the sequel linear functionals, and especially this one will play an important role.

16. - Theorem. If Y is a *b'f*-aggregates, $f(X)$ is the linear functional defined in [14], then

$$^*p_{bM}(Y) = \mathfrak{LG}\,\{ f(Z) \mid Z \in V, \mid Y \mid \leq Z \}.$$

§ 6. - Fields.

Polynomials:

1. - Let Φ be a non empty set of elements which may be abstract or defined and endowed with a notion of equality $\overset{\Phi}{=}$. Let E be an *abelian half-group* [10]) which we define as a structure having an addition of its elements $a, b, \ldots$, always performable, yielding an element of E, and satisfying the conditions: 1) $a + b$ is invariant with respect to the equality $\overset{E}{=}$ governing on E, 2) it is associative and commutative, 3) possesses a neutral element 0_E, and 4) obeys the cancellation law i. e. if $a + b = a + b'$, then $b = b'$ [11]).

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[10]) It differs from the semigroup considered in [§ 1E].

[11]) The half-group E can be extended, through an addition-isomorphism, to an abelian group G by means of ordered couples (a, b), where $a, b \in E$. We define $(a, b) \overset{G}{=} (a', b')$ as $a + b' \overset{E}{=} a' + b$, and we define the sum so as to have $(a, b) + (c, d) \overset{G}{=} (a+c, b+d)$. The defined addition is $(\overset{G}{=})$-invariant. We have $0_G \overset{G}{=} (a, a)$.

Choose an element of Φ, denote it by 0_Φ and keep it fixed. By an *$E\Phi$-function* we shall understand any function $f[s]$ from E into Φ, where $f[s] \overset{\Phi}{=} 0_\Phi$ for all $s \in E$, excepting perhaps a finite number of values of s. The class of all $E\Phi$-functions will be denoted by P. We define $f \overset{P}{=} g$ as $f[s] \overset{\Phi}{=} g[s]$ for all $s \in E$,

2. - For farther purposes it is needed to admit that Φ is organized into a commutative ring or even field F with unit 1_F and zero 0_F where $1_F \neq 0_F$. The elements of F will be termed *quantities*, and these of E *exponents*. The elements of P will be termed *EF-polynomials* (9). We define the *null-polynomial* 0_P as such that $0_P[s] \overset{F}{=} 0_F$ for all $s \in E$. Given two EF-polynomials f, g we define their *sum* $f + g$ as the polynomial h such that $h[s] \overset{F}{=} f[s] + g[s]$ for all $s \in E$. This *addition* is $(\overset{P}{=})$-invariant, associative, commutative, and admitting subtraction, so it is an abelian group with 0_P as group-identity.

To define *multiplication* in P, we need some auxiliary notions. Let f be an EF-polynomial; by a *set fitting* f we shall understand a not empty, finite subset A of E such that if $s \in E - A$, then $f[s] \overset{F}{=} 0_F$. Given two polynomials f, g and sets A, B fitting them sespectively we define (A, B) as the set of ordered couples:

$$\{(s, t) \mid s \in A,\ t \in B\},$$

and we put

$$\overline{A, B} \underset{df}{=} \{u \mid \text{there exist } s \in A \text{ and } t \in B \text{ such that: } u \overset{E}{=} s + t\}.$$

Given two EF-polynomials f, g, we define their *product* f, g as the function $h[s]$ such that:

1) if $u \in \overline{A, B}$, we put $h[u] \overset{F}{=} \Sigma f[s] \cdot g[t]$, where the summation is extended over all couples (s, t) for which $s \in A$, $t \in B$, $s + t = u$, the letters A, B denoting sets fitting f, g respectively and where each couple (s, t) is taken $\overset{E}{=}$ once only;

2) if $u \,\overline{\in}\, \overline{A, B}$, we put $h[u] \overset{F}{=} 0_F$. The function $h[s]$ is an EF-polynomial.

One can prove that the above notion does not depend on the choice of the sets A, B.

The multiplication is equality invariant, it admits 0_P as annihilator, it is associative and commutative, and admits as neutral element the unit-polynomial 1_P, defined by $1_P(0_E) \underset{df}{=} 1_F$, $1_P(s) \underset{df}{=} 0_F$ for all $s \neq 0_E$. We have $0_P \overset{P}{\neq} 1_P$.

The distributive law holds true. Thus, under the operations of addition and multiplication, P is organized into a commutative ring with $0_P \neq 1_P$ as zero and unit.

If we define the correspondence b with $\complement\!\mid b = F$, $\mathsf{D} b \subseteq P$ as such one which is $\overset{F}{=}$, $\overset{P}{=}$-invariant and carries the quantity $a \in F$ into the polynomial $b(a) \underset{df}{=} a^*[s]$ defined by $a^*[0_E] = a$, $a^*[1] = 0_F$ for all $s \in E$, then b is an addition multiplication-zero-and-unit-isomorphism from the ring F into the ring P. The set $\{ a^* \mid a \in F \}$ is organized within P into a commutative ring F^* with 0_P and 1_P as zero 0_{F^*} and unit 1_{F^*}.

3. - Let us suppose that E is linearly ordered; this will mean that there exists a correspondence « $<$ » with domain and range E, which is $(\overset{E}{=})$-invariant, and obeys the conditions: 1) if $s < t$, $t < r$, then $s < r$, 2) if $s < t$, then $s + r < t + r$, 3) given any exponents s, t, we have disjointedly either $s < t$, or $t < s$, or $t \overset{E}{=} s$.

Theorem. If E is a linearly ordered abelian half group and F is a commutative ring with $0_F \neq 1$, then the ring P of EF-polynomials is an integrity-domain i. e. if $f \cdot g \overset{P}{=} 0_P$, then either $f \overset{P}{=} 0_P$ or $g \overset{P}{=} 0_P$.

4. - In what will follow we shall admit that F is a commutative ring with $0_F \neq 1_F$, and that E is the linearly ordered half group $\{0, 1, 2, \ldots\}$ of all non negative integers ordered as usually.

Then the ring P of EF-polynomials is a commutative integrity domain with $0_P \neq 1_P$. The ring P can be conceived as extension of F^* by adjunction of the single element $\xi = \xi[s]$ defined by $\xi[1] = \xi[1_E] = 1_F$ and $\xi[s] = 0_F$ for all $s \neq 1$.

Indeed let us define $\xi^0 \underset{df}{=} 1_P$, $\xi^{n+1} \underset{df}{=} \xi^n \cdot \xi$ for $n = 0, 1 \ldots$

We have for $n = 0, 1, 2, \dots$ $\xi^n[n] = 1_F$ and $\xi^n[s] = 0_F$ for $s \neq n$.

Hence, if f is an EF-polynomial and $A = [s_1 < s_2 < \dots < s_k]$ a set fitting f, we get

$$f = f[s_1]^* \cdot \xi^{s_1} + \dots + f[s_k]^* \cdot \xi^{s_k}.$$

Since $[0, 1, \dots, s_k]$ is also a set fitting f we can write

$$f = \sum_{i=0}^{s_k} \xi^i \cdot a_i^* \tag{1}$$

where $a_i^* = f[i]$, $(i = 0, \dots, s_k)$.

Given $f \neq 0$, we always have the representation of f in the shape (1):

$$f = \sum_{i=0}^{n} \xi^i a_i^* \quad \text{with } a_n \neq 0_F \text{ for some } n \geq 0.$$

For a given f this representation is $\overset{F}{=}$-unique, i. e. if

$$f \overset{P}{=} a_0^* \xi^0 + \dots + a_n^* \xi^n \overset{P}{=} b_0^* \xi^0 + \dots + b_m^* \xi^m,$$

then $n = m$ and $a_i \overset{F}{=} b_i$ for all $i = 0, \dots, n$.

5. - **Theorem.** The correspondence b which attaches to each polynomial

$$f[s] = f_0^* \cdot \xi^0 + \dots + f_n^* \cdot \xi^n, \qquad (n = 0, 1, \dots)$$

the function

$$f(x) \underset{df}{=} f_0 x^0 + \dots + f_n x^n,$$

where x is a variable with domain F and $x^0 \underset{df}{=} 1$ for all $x \in F$, in an operation-zero-and unit homomorphism.

It is from P onto the ring of all above polynomial-functions, where addition and multiplication of functions are the usual ones, the zero and unit-functions being constant functions with valus 0_F and 1_F respectively.

Remark. The above homomorphism may be not an isomorphism. E. g. If F is the field of all integers modulo 3, the polynomial function $\sum_{i=0}^{5} x^i$ vanishes for all $x \in F$.

This circumstance does not allow to define polynomials as functions of the variable quantity of the ring. On the other hand, a definition of polynomials as expressions $\Sigma\, a_i x^i$ where x is just a mere symbol or a variable with indetermined domain is logically not clear–hence incorrect.

6. - If $f \neq 0_P$ is a polynomial, we define its *order*, ord f, as the maximal s for which $f[s] \neq 0_F$. The null-polynomial does not admit any order. The polynomials of order 0 are the polynomials $a^* \overset{P}{=} a^* \xi^0$ where $a \neq 0_F$. Those polinomials and 0_P will carry the common name *constant polynomials*, sometimes denoted by c_p. We have $\operatorname{ord}(f \cdot g) = \operatorname{ord} f + \operatorname{ord} g$, and if $\operatorname{ord} f < \operatorname{ord} g$, then $\operatorname{ord}(f \pm g) = \operatorname{ord} g$. We have $\operatorname{ord} f = \operatorname{ord}(c_P \cdot f)$, where $c_P \neq 0_P$. The polynomials of order $n \geq 0$ can be represented in a $\overset{F}{=}$-unique way as $a_0^* \xi^0 + = \dots + a_n^* \xi^n$, where $a_n \neq 0_F$ [12]).

7. - We shall consider ideals in the ring of EF-polynomials. By an *ideal* we understand, as usually, any non empty subset J of E with the properties: 1) if $f, g \in J$, then $f - g \in J$, 2) if $f \in J$, $p \in P$, then $f \cdot p \in J$.

The sets P and $\{0_P\}$ are ideals. $0_P \in J$ for every ideal J. We define the *equivalence modulo* J, $f \overset{J}{=} g$ as $f - g \in J$. By the *equivalence class modulo J represented by the polynomial* f we shall understand the set

$$\{p \mid p \in P,\ p \overset{J}{=} f\}.$$

f is termed *representative of the equivalence class.*

The equivalence class modulo $\{0_P\}$ represented by f is $\{f\}$, and that modulo P is $= P$. Every element of an equivalence class is its representative. The intersection of any non empty collection of ideals is an ideal.

If we change the equality $\overset{P}{=}$ governing on P into $\overset{J}{=}$,

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[12]) The so called «identification» of isomorphic elements whose type is different can be applied only with caution; in a treatise where the logical structure is emphasized, the identification is logically disorderly and leads to contradiction.

then the ring operations, zero and unit will be invariant with respect to $\overset{J}{=}$. E. g. If $f \overset{J}{=} f'$, $g \overset{J}{=} g'$, then $f + g \overset{J}{=} f' + g'$. Thus an ideal organizes P into another commutative ring P_J, termed *equivalence ring modulo* J of P. If we consider the set of all equivalence classes modulo J and define, for equivalence classes, the operations of *addition* and *multiplication, zero* and *unit* in the usual way by means of representatives, we get — as is well known — the *quotient ring*, P/J, which is operation — zero — and unit isomorphic with P_J [18]). Given a non empty set M of polynomials, there exists the smallest (unique) ideal containing M; it will be termed the *ideal spanned on* M. By a *principal ideal* we shall understand an ideal which is spanned on the set composed of a single polynomial, say f. It will be termed the *principal ideal determined by* f. The principal ideal determined by 1_P is P. The principal ideal determined by 0_P is $\{0_P\}$. The principal ideal determined by f coincides with the set

$$\{f \cdot p \mid p \in P\}.$$

In what will follow we admit that F is a commutative field with $1_F \neq 0_F$ and $E = \{0, 1, 2 \dots\}$. If $c_P \neq 0_P$ is a constant polynomial, then the principal ideal determined by $f \cdot c_P$ coincides with the principal ideal determined by f. We have the theorem: If ord $f \geq 1$, J is the principal ideal determined by f, and S an equivalence class modulo J, $S \neq J$, then S contains a polynomial of order $< f$. By its use we prove that the division algorithm holds for polynomials, i. e.: If $1 \leq \operatorname{ord} q \leq \operatorname{ord} p$, then there exists a $\overset{P}{=}$-unique decomposition $p \overset{P}{=} A \cdot q + B$, where either $B = 0_P$ or ord $B <$ ord q. From that it follows that every ideal in P is a principal ideal. The following are equivalent: I. J is the ideal determined by f_1 and also the ideal determined by f_2, II. there exists a constant polynomial $c_P \neq 0_P$ such that $f_1 = c_P f_2$.

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18) PJ and P/J are usually « identified ». This identification will be avoided in what follows.

The following are equivalent: I. J is the ideal determined by f. II. f is a polynomial $\neq 0_P$ of the lowest order among those belonging to J.

8. - f, g being polynomials, both $\neq 0_P$, we say that f ***is divisible by*** g whenever $f \overset{P}{=} g \cdot h$ for some $h \in P$. Given two polynomials, f, g both $\neq 0_P$, there exists a ***greatest common divisor*** (i. e with the greatest order); it is unique up to a multiplicative constant polynomial $\neq 0_P$. Two polynomials f, g, both $\neq 0_P$ are said to ***be prime to one another*** if their greatest common divisor is a constant polynomial $\neq 0_P$ [14]).

The following are equivalent for non-null polynomials f, g: I. f, g are prime to one another, II. there exist polynomials p, q such that $1_P \overset{P}{=} p \cdot f + q \cdot g$.

If p, q are non-null polynomials, $p \cdot q$ is divisible by d, q and d are prime to one another, then p is divisible by d.

If $a \in F$, $b \in F$, $a \neq 0_F$, then a^* is prime to $\xi - b^*$.

If a, $b \in F$, $a \neq b$, then $\xi - a^*$ is prime to $\xi - b^*$.

9. - If

$$f \overset{P}{=} f_0^* \cdot \xi^0 + \dots + f_n^* \cdot \xi^n, \quad \text{ord } f \geq 1, \; a \in F,$$

$$f_0 \cdot a^0 + \dots + f_n a^n \overset{F}{=} 0_F \text{ (where } a^0 \underset{df}{=} 1_F),$$

then f is divisible by $\xi - a^*$.

If

$$\text{ord } \xi \geq 1, \; f \overset{P}{=} f_0^* \xi^0 + \dots + f_n^* \xi^n, \; a_1, a_2, \dots, a_s \in F, \; (s \geq 1)$$

are all different,

$$f_0 a_i^0 + \dots + f_n a_i^n \overset{F}{=} 0_F \quad \text{for} \quad i = 1, 2, \dots, s,$$

then f is divisible by

$$(\xi - a_1^*) \cdot (\xi - a_2^*) \dots (\xi - a_s^*).$$

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[14]) We do not define divisibility and indivisibility involving null -polynomials. The notion of «prime» or not «prime» will be not defined for null-polynomials.

If in addition to that we suppose that $s \geq n+1$, then $f_0 = \dots = f_n = 0_F$.

10. - A polynomial f with ord $f \geq 1$ is said to be *irreducible* whenever there do not exist $g, h \in P$ such that ord $g >$ ord f, ord $h <$ ord f, $f = g \cdot h$. A polynomial f with ord $f = 1$ is irreducible [15]).

If a polynomial f is irreducible, it is divisible only by $c_P \cdot f$ and by c_P, where $c_P \neq 0_P$ is a constant polynomial. Conversely, if f is divisible only by $c_P \neq 0_P$ and by $c_P \cdot f$ and if ord $f \geq 1$, then f is irreducible.

If f is irreducible, $f \overset{P}{=} f_0^* \xi^0 + \dots + f_n^* \xi^n$, $(n \geq 1)$ and there exists $a \in F$ such that $f_0 a^0 + \dots + f_n a^n \overset{F}{=} 0_F$, then ord $f = 1$.

If f is reducible, then $f \overset{P}{=} g_1 \cdot g_2 \cdot \dots g_s \cdot c_P$, $s \geq 2$, ord $g_i \geq 1$, where all polynomials g_i have the coefficient 1_P at the term with highest power of ξ and where all g_i are irreducible with ord $g_i \geq 1$. This decomposition is $\overset{P}{=}$ unique up to permutation of factors.

11. - The ideal J is termed *maximal* (indecomposable) whenever 1) $J \neq P$ and 2) when for every ideal J', such that $J \subseteq J' \subseteq P$, we have either $J = J'$ or $J' = P$.

The following are equivalent: I. J is a maximal ideal, II. J is the ideal generated by an irreducible polynomial.

Remark. Notice that a *prime ideal* J, i. e. such that P_J is an integrity domain, may not be a maximal ideal e. g. put $J = \{0_P\}$.

If J is a maximal ideal, then P_J (and P/J) are commutative fields, and conversely.

If J is determined by the irreducible polynomial $f = a_0^* \cdot \xi^0 + \dots + a_n^* \xi^n$, ord $f = n \geq 1$, then the elements of the field P_J have the form $b_0^* \xi^0 + \dots + b_{n-1}^* \xi^{n-1}$, $b_i \in F$. This follows from the theorem on the division-algorithm.

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[15]) We do not define irreducibility and reducibility for constant polynomials.

12. - Given a polynomial

(1) $$f = a_0^* \xi^0 + \dots + a_n^* \xi^n, \quad (n \geq 1),$$

by its *derivative* f' we shall understand the polynomial

$$1_F^* a_1^* \xi^0 + 2_F^* a_n^* \xi^1 + \dots + n_F^* a_n^* \xi^{n-1}$$

(where $2_F \underset{df}{=} 1_F + 1_F$, $(n+1)_F \underset{df}{=} n_F + 1_F$ etc.).

The notion of derivative does not depend on the choice of the representation (1) of f. For a constant polynomial f we have $f' = 0_P$. We define for $r = 0, 1, 2, \dots$ $f^{(0)} \underset{df}{=} f$, $f^{(s+1)} \underset{df}{=} (f^{(s)})'$, i. e. the « higher derivatives » of f.

The following formulas are valid:

$$(f \pm g)' = f' \pm g', \quad (c_P f)' = c_P \cdot f', \quad (f \cdot g)' = f'g + fg'.$$

If ord $f = n$, then

$$f^{(n+s)} = 0_P \quad \text{for} \quad s = 1, 2, \dots$$

If

$$f = a_0^* \xi^0 + \dots a_n^* \xi^n, \quad 1 \leq \operatorname{ord} f \leq n, \quad h \in F,$$

then

$$a_0^*(\xi + h^*)^0 + \dots + a_n^*(\xi + h^*)^n = f^{(0)} +$$

$$+ \frac{h^*}{1\,!} f' + \frac{(h^*)}{2\,!} f'' + \dots + \frac{(h^*)^n}{n\,!} f^{(n)}$$

and for ord $f = 0$ we have $a_0^*(\xi + h^*)^0 = f^{(0)}$. (Here $1\,!, 2\,! \dots$ stand for $(1_F\,!)^*$, $(2_F\,!)^*$, etc).

Extension of a non trivial commutative field.

13. - Let F, G be two non trivial fields. $(F) \subseteq (G)$ will signify that F is an all operations-genuine strict subfield of G. (see [Preface 8]). We shall also call F *partial field of* G, and G *superfield of* F.

If A, B are correspondences, their composition will be written BA or $A \mid B$, (7).

It is defined as $\{ x, z \mid$, there exists y with xAy and $yAz \}$. Both above manners of writing are useful.

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If E is a set, then card E will mean the power of E, i.e. the cardinal number of E.

We also shall use the sign $\overset{cl}{=}$ which will mean equality of classes, (sets).

In the devolopment of the theory of fields the field P_J is more convenient that P/J, though in some topics, P/J is more useful. The field P_J is an all operations - zero - and unit - genuine extension of F through isomorphism b^J which is $\overset{F}{=}\ \overset{J}{=}$-invariant and carries $a \in F$ into the polynomial a^*. We call b^J *standard.* b^J differs from b (defined in [2]), in that b is $\overset{F}{=}\ \overset{P}{=}$-invariant only. Denote by t the $\overset{J}{=}\ \overset{cl}{=}$-invariant isomorphism from P_J onto P/J, defined by $f t \bar{f}$ where $\bar{f}$ is the equivalence class whose representative is f. The correspondence t carries the polynomial $\bar{b}_0^* \xi^0 + \dots + b_m^* \xi^m$ into $\bar{b}_0^*(\bar{\xi})^0 + \dots + \bar{b}_m^*(\bar{\xi})^m$.

The correspondence tb^J carries $a \in F$ into $\bar{a}^*$. It is $\overset{F}{=}\ \overset{cl}{=}$ - invariant.

We call P_J *elementary algebraic (equivalence) extension of* F *and* P/J *elementary algebraic (quotient) extension of* F.

14. - The ring P being an integrity domain with unit, it generates another ring Q, whose elements are ordered couples of polynomials, in the following way:

We consider only the couples (f, g) where $g \overset{P}{\neq} 0_P$. We define the *equality* $\overset{Q}{=}$ *governing in* Q by «$(f_1, g_1) = (f_2, g_2)$ means $f_1 \cdot g_2 \overset{P}{=} f_2 \cdot g_1$», and the *operations* by:

$$(f_1, g_1) + (f_2, g_2) \underset{df}{=} (f_1 \cdot g_2 + g_1 \cdot f_2,\ g_1 \cdot g_2)$$

$$(f_1, g_1) \cdot (f_2, g_2) \underset{df}{=} (f_1 \cdot f_2,\ g_1 \cdot g_2).$$

These operations are $\overset{Q}{=}$ - invariant. If we define $0_Q \underset{df}{=} (0_P, 1_P)$, $1_Q \underset{df}{=} (1_P, 1_P)$, we get *zero* and the *unit* of the ring Q. The ring Q is a field; we call it *quotient field of* P *(couple form).* If we consider saturated classes of mutually $\overset{Q}{=}$-equal couples, (so called $\overset{Q}{=}$-*equivalence classes*), and define the operations of *addition* and *multiplication* by means of representatives, we get another field $\tilde{Q}$, called *quotient-field of* P *(class-form).* Q and $\tilde{Q}$, are isomorphic with one another through the correspondence s which is $\overset{Q}{=}\ \overset{cl}{=}$-invariant and which attaches to the couple (f, g) the equivalence class whose representative is (f, g). Denote by c the correspondence which attaches to the polynomial f the couple $f^x \underset{df}{=} (f, 1_P)$, and consider c as $\overset{P}{=}\ \overset{Q}{=}$-invariant. The correspondence $b \mid c = cb$ carries $a \in F$ into the couple $(a^*, 1_P)$ and is $\overset{F}{=}\ \overset{Q}{=}$-invariant.

The field Q is an all operations - zero - and unit genuine extension of F through the isomorphism cb. The field $\tilde{Q}$ is also so through the isomorphism scb. We call Q *elementary transcendental extension of F (couple form)*, and $\tilde{Q}$ *elementary transcendental extension of F (class form)*. We have

$$(a_0^* \xi^0 + \dots + a_n^* \xi^n,\ b_0^* \xi^0 + \dots + b_m^* \xi^m) \overset{Q}{=} \frac{a_0^{*x}(\xi^x)^0 + \dots + a_n^{*x}(\xi^x)^n}{b_0^{*x}(\xi^x)^0 + \dots + b_m^{*x}(\xi^x)^m},$$

$$n \geq 0,\ m \geq 0,\ b_m \neq 0,$$

where on the right the operations are those in Q.

15. - Let $(F) \subseteq (G)$ be two non trivial fields, and let $u \in G$. Denote by $[F, u]$ the smallest partial field of G containing F and u. It does not depend on the choice of the superfield G, provided that $(F) \subseteq (G)$ and $u \in G$. We call $[F, u]$ *single element extension of F in superfield.* (In a similar way we define $[F, u_1, u_2, \dots, u_s]$). Under these circumstances there can occur two and only two following cases:

1) *Algebraic case*: There exists an irreductible EF-polynomial

$$(1) \qquad a_0^* \xi^0 + \dots + a_n^* \xi^n,\quad n \geq 1,\quad a_n \neq 0,\quad a_i \in F,$$

such that $a_0 u^0 + \dots + a_n u^n \overset{G}{=} 0_G$, (where $u^0 \underset{df}{=} 1_F$).

This polynomial is unique up to a multiplicative constant polynomial $\neq 0_P$.

Let J be the maximal ideal determined by (1). Then the $\overset{J}{=}\ \overset{G}{=}$-invariant correspondence a^J which carries $b_0^* \xi^0 + \dots + b_{n-1}^* \xi^{n-1}$ into $b_0 u^0 + \dots + b_{n-1} u^{n-1}$ for $b_i \in F$ is isomorphism from P_J onto $[F, u]$. We have $\xi a^J u$. The correspondence $t^{-1}a^J$ carries the quantities of P/J onto those of $[F, u]$. The correspondences a^J and $t^{-1}a^J$ will be termed *standard*; they are well determined by u. We call $[F, u]$ *single element algebraic extension of F within superfields.*

Such an extension is called *apparent* if $[F, u] = F$; if $u \in G - F$, it is termed *proper*. The quantity u, in both cases is termed *algebraic with respect to F, in superfields.*

2) *Transcendental case:* There does not exist any EF-polynomial

$$a_0^* \xi^0 + \dots + a_n^* \xi^n, \quad a_n \neq 0_F, \quad n \geq 1, \quad a_i \in F,$$

such that

$$a_0 u^0 + \dots + a_n u^n \overset{G}{=} 0_G.$$

We term $[F, u]$ *single element transcendental extension of F within superfields.*

The quantity u of G is termed *transcendental with respect to F in superfields.*

It never happens that $[F, u] = F$, so the extension is always proper. The $\overset{Q}{=} \overset{G}{=}$-invariant mapping d, defined by

$$(b_0^* \xi^0 + \dots + b_n^* \xi^n;\ c_0^* \xi^0 + \dots + c_m^* \xi^m) \overset{d}{\to} \frac{b_0 u^0 + \dots + b_n u^n}{c_0 u^0 + \dots + c_m u^m},$$

$$\text{for } b_i,\ c_j \in F,\ c_m \neq 0_F,\ n \geq 0,\ m \geq 0,$$

is an isomorphism from Q onto $[F, u]$. The $\overset{cl}{=} \overset{G}{=}$-invariant correspondence ds^{-1} is an isomorphism from $\bar{Q}$ onto $[F, u]$.

They both are called *standard*, and are well determined by u.

16. - A field F can be isomorphic with its algebraic single element extension. Exple.

Let R be the field of all ordinary rational numbers. Put $F \underset{df}{=} [R, \pi^2]$. The field $[F, \pi]$ is an algebraic extension of F and is isomorphic to F.

A field F can be isomorphic with its transcendental single element extension. Expl. Let $\alpha_1, \alpha_2, \dots$ be an infinite sequence of ordinary transcendental numbers, such that there does not exist any algebraic relation between $\alpha_1 \dots \alpha_n$, whatever n may be. Put $F = [R, \alpha_2, \alpha_3, \dots]$. $[F, \alpha_1]$ is a transcendental extension of F and is isomorphic to F.

However, if $(F) \subseteq (G)$, $u, v \in G$, and we are only interested in isomorphism m for which $ma = a$ for all $a \in F$, (*F-rigid -isomorphism*), then F is never isomorphic to any of its proper single element extension. Under the above requirement $[F, u]$ and $[F, v]$ are isomorphic if and only if either u, v are both transcendental or are both algebraic with the same ideal J. This can be proved by considering various standard mappings.

Concerning algebraic extension of F in superfields G, given an irreductible polynomial of order $n > 2$, there can exist in G at most n different quantities $u_1, \dots, u_n$ such that $[F, u_1], \dots, [F, u_n]$ are mutually F-rigid isomorphic.

17. - If $(F) \subseteq (G) \subseteq (H)$ we say that (G) is *an extension of F within H (in superfield)*. G is termed *algebraic in superfield* whenever every quantity $u \in G$ is algebraic with respect to F. It is known that if G is an algebraic extension of F and $v \in H$ is algebraic with respect to G, then v is also algebraic with respect to F. If $(F) \subseteq (G)$ and f is an EF-polynomial, then G is termed *splitting field* of f, if f can be decomposed into linear EG-polynomials in the EG-ring. (G) is termed *algebraic closure of F* whenever 1) every quantity of G is algebraic with respect to F, 2) G is a splitting field of every EF-polynomial with order ≥ 2.

Our next purpose is to prove the existance of the algebraic closure. To do this we shall give a proof of the existence of a splitting field for a given EF-polynomial. The proof of the main theorem will need only some supplementary remarks. We shall need few preparations.

18. - We shall perform many elementary algebraic extension of F, applied one after another. Now, such an extension changes the logical type of quantities considered. Indeed, let us fix the type of the numbers 0, 1, 2, ..., denoting it by ν. Let α be the logical type

of quantities of the field F. An EF-polynomial, being a function, is a correspondence between the elements of type v and elements of type α. It has the type which we may denote by $\{v, \alpha\}$ (see [Preface 6]). The elements of P/J are classes of polynomials, so their type may be denoted by $\mathrm{cl}\{v; \alpha\}$. The standard isomorphism from F into P/J is tb^J and its type is $\{a; \mathrm{cl}\{v; \alpha\}\}$, so the inverse correspondence, $(b^J)^{-1}t^{-1}$, has the type $\{\mathrm{cl}\{v; \alpha\}; \alpha\}$.

19. - Let Φ be a set of any elements (abstract or constructed) provided with an equality $\overset{\Phi}{=}$. Suppose that card $\Phi \geq \mathrm{card}\, F$. There exists one - to - one $\overset{F}{=}\overset{\Phi}{=}$-correspondence which maps the quantities of F into Φ. Choose such a correspondence m and put $F_0 \underset{df}{=} m(F)$. The set F_0 can be organized through m into a commutative field by defining the operations:

$$p + q \underset{df}{=} m(m^{-1}d + m^{-1}q), \qquad p \cdot q \underset{df}{=} m(m^{-1}p \cdot m^{-1}q),$$

and by defining

$$1_{F_0} \underset{df}{=} m1_F,\ 0_{F_0} \underset{df}{=} m0_F.$$

Thus, the given field can be isomorphically transformed into another one with elements having a type given in advance, satisfying only a condition concerning the power. This device will be now applied.

20. - Let F be a non trivial commutative field, Φ a set as in [19], such that card $\Phi > \max[\mathrm{card}\, F, \aleph_0]$. If F is finite, the $P = P^{(F)}$ is denumerable, and if F is infinite, card $P^{(F)} =$ card F. Thus in any case card $P < \Phi$. Transport F through an isomorphism, as in [19] into a field F_0 such that $F_0 \subset \Phi$. We shall operate only on F_0. Let J be a maximal ideal in the EF-ring $P^{(F_0)}$ and put

$$G' \underset{df}{=} P^{(F_0)}/J.$$

Let

$$(1) \quad b^{(F_0)}a_0 \cdot \xi^0 + \ldots + b^{(F_0)}a_n \cdot \xi^n,\ a_n \neq 0_{F_0},\ a_i \in F_0,\ n \geq 1$$

be an irreducible EF_0-polynomial generating J. Since card $(G' - tb^J F_0) <$ card $(\Phi - F_0)$ we can choose a $1 \rightarrow 1$ correspondence n which carries G' into a subset $G \subseteq \Phi$ in such a way that $tb^J n = m$. We organise G into a field (G) by means of n as in [19]. We get $(F_0) \subseteq (G)$.

Let $u \underset{df}{=} nt\xi$. Then we have

$$a_0 u^0 + \ldots + a_n u^n \overset{G}{=} 0,$$

and the EG-polynomial (1) which can be written as

$$b^{(G)} a_0 \cdot \xi^0 + \ldots + b^{(G)} a_n \cdot \xi^n$$

is divisible by $\xi - b^{(G)} u$. Every quantity of G has the shape $b_0 u^0 + \ldots b_{n-1} u^{n-1}$, so $G = [F_0, u]$. Thus we have extended F through isomorphism to G. Of course the extension may be apparent. Now we can proceed with G in the same way as we did with F_0, and by the choice of an ideal in the ring of EG-polynomials, extend G, getting a field H where $(F) \subseteq (G) \subseteq (H)$, $H \subset \Phi$. We call the construction, sketched above, $[F_0, J, n]$-*construction*. It can be performed because the cardinal of Φ is sufficiently high.

21. - Now, we are going to prove the existance of the splitting field of a polynomial.

Let F, F_0, n, Φ be as before in [20] and let f be an EF_0-polynomial with order ≥ 1. Let δ be the type of elements of Φ, and v - that of the numbers 0, 1, 2, ... Consider the class of all $1 \rightarrow 1$ correspondences which carry a set of elements of the type cl $\{\{v;\ \delta\};\ \delta\}$ into Φ, and order well this class, getting the well ordering

(1) $$n_1,\ n_2,\ \ldots,\ n_\alpha,\ \ldots$$

Consider all functions defined on finite subsets of $E = \{0, 1, 2, \ldots\}$ and with values in Φ. Order them well

(2) $$\pi_1,\ \pi_2,\ \ldots,\ \pi_\beta,\ \ldots$$

It may happen that f has the shape

$$c_P(\xi - c_1^*) \ldots (\xi - c_k^*),\ k \geq 1$$

where c_P is a constant polynomial $\neq 0_P$ and $c_i \in F_0$. In this case we stop the process and put

$$H \underset{df}{=} F_0 .$$

H is the splitting field of f.

In the remaining case we have

$$(3) \qquad f = f_1^{s_1} \dots f_m^{s_m} \cdot (\xi - c_1^*) \dots (\xi - c_k^*) \cdot c_P$$

where $s_1 \geq 1, \dots, s_m \geq 1, m \geq 1, k \geq 0, c_P \neq 0_P$, and where f_i are different irreducible polinomials of order ≥ 2 with the coefficient $= 1^*$ at the highest power of ξ.

The representation (3) is unique up to permutation of factors. In general, if g is an EF-polynomial, say $b_0\xi^0 + \dots \dots + b_n\xi^n$, $b_n \neq 0$, $n \geq 1$, there corresponds to it a well defined function π written in (2) which attaches to $0, 1, \dots, n$ the quantities $b_0, b_1, \dots, b_n$ respectively. This having noticed, choose among $f_1, \dots, f_m$ the polynomial f_i for which the corresponding function π in (2) has the smallest index β. The polynomial f_i is irreducible. Let the corresponding ideal be J. Apply the construction $[F_0, J, n_\alpha]$ by using the correspondence n in (1) with the smallest index α.

21a. - We get a field $[F_0, u] \underset{df}{=} F_1$ in which the polynomial f_i is divisible by $\xi - b^{(F_1)}u_1$. Considering f as an EF_1-polynomial, we get the decomposition

$$(1) \qquad f = g_1^{t_1} \dots g^{t_p}(\xi - d_1^*) \dots (\xi - d_k^*) \cdot c'_P \text{ where } k_1 > k \geq 0.$$

If $p = 0$, the process stops and F_1 is a splitting field H of f.

This process can carried through by ordinary induction until it stops. It must stop, because $0 < k < k_1 < k_2 < \dots \leq n$.

Thus we get a very definite sequence of nested fields

$$(2) \qquad (F_0) \subseteq (F_1) \subseteq (F_2) \subseteq \dots,$$

where the last, called H, is a splitting field of f. The theorem on the existence of a splitting field is established. We underscore that given F, Φ, m, f and the well orderings (1) and (2), the constructed splitting field is well determined. We denote it by $H(F_0, f)$.

22. - Sketch of a proof of the existence of the algebraic closure of a given field F.

We take over the entities F, m, F_0, Φ from [21] and the well orderings $\{ n_\alpha \}$, and $\} \pi_\beta \}$. Let us order well all different ideals of the ring of EF-polynomials:

$$J_1, J_2, \ldots, J_\gamma \ldots \tag{3}$$

The power of (3) is $<$ card Φ. Let

$$f_1, f_2, \ldots, f_\gamma, \ldots \tag{4}$$

be the corresponding irreducible polynomials, which we suppose to have the coefficient 1_P at the highest power of ξ. Thus the sequence (4) is well determined. Determine the splitting fields $H_1 = H(F_0, f_1)$, $H = H(H_1, f_2)$, etc. as in [21]. They are well determined and we have

$$(F_0) \subseteq (H_1) \subseteq (H_2) \subseteq \ldots$$

Indeed, suppose we have already defined all H_γ for $\gamma < \alpha$. If $\alpha - 1$ exists, we define $H_\alpha \underset{df}{=} [H_{\alpha-1}, f_\alpha]$, and if α is a limit ordinal, we define

$$H_\alpha \underset{df}{=} \bigcup_{\gamma < \alpha} H_\gamma .$$

The construction can be carried through, because card Φ is enough high. We can prove that all H_α are algebraic extensions of F_0. We have: if $\alpha_1 \leq \alpha_2$, then $(H_{\alpha_1}) \subseteq (H_{\alpha_2})$. The union of all H_α is the required algebraic closure of F through the isomorphism m. The theorem is established.

23. - **Linearly ordered fields.** Let F be a non trivial field and $P \neq 0$ a subset of F, satisfying the following conditions: 1) if $a \in F$, then one and only one of the relations $a \in P$, $-a \in P$, $a = 0$ holds true; 2) if $a, b \in P$, then $a + b \in P$ and $a \cdot b \in P$. We call the quantities of P *positive quantities* of F. We define the correspondence $a \leq b$ as « $b - a \in P$

or $b - a = 0$ ». The correspondence « $\leq$ » is $\overset{F}{=}$-invariant and it organizes F into a chain. A non trivial field endowed with thus defined ordering is termed *linearly ordered field*.

We define $a < b$ as $b - a \in P$, $b > a$ as $a < b$, and $b \geq a$ as $a \leq b$. We define the *absolute value* $|a|$ of a as this one among a, $-a$ which is ≥ 0. All usual formal rules of ordinary algebra of real numbers, with « $\leq$ », hold true in linearly ordered fields.

24. - A linearly ordered field F has the characteristic 0 i. e. its smallest partial field is isomorphic with the field of ordinary rational numbers. The power of F is infinite, therefore a polynomial $\sum\limits_{i=0}^{n} a_1^* \xi^i$, such that $\sum a^i c^i = 0$ for all $c \in F$, is necessarily the null polynomial (compare [5]). It follows that there exists an all operations-zero-and-unit-isomorphism between the polynomials $\sum\limits_{i=0}^{n} a_i^* \xi^i$ and the polynomial-functions $\sum\limits_{i=0}^{n} a_i \dot{x}^i$ where $\dot{x}$ *is a variable with domain* F. This fact allows us to replace polynomials p by *polynomial-functions* $p(\dot{x})$.

In the sequal, dealing with linearly ordered fields, we shall use polynomial-functions $p(\dot{x})$ only. The polynomial ξ will be replaced by polynomial-function $\dot{x}$. The derivative of the polynomial p can be denoted by $\dfrac{dp(\dot{x})}{dx}$, defined not as the limit of the quotient of increments, but in the pure formal way as $\sum\limits_{i=1}^{n} i a_i \dot{x}^{i-1}$, with $\dfrac{da}{dx} = 0\,(\dot{x}) = 0$ (see [12]). The Taylor formula

$$p(x + h) = p(x) + \frac{h}{1\,!} p'(x) + \dots + \frac{h^n}{n\,!} p^{(n)}(x)$$

holds true for any x, $h \in F$, and polynomial p.

25. - If $a > 0$, $b > 0$, then there exists $h_0 > 0$ such that for every $|h| < h_0$ we have $|bh| < a$. Let $f(x) = \sum\limits_{i=0}^{n} a_i x^i$, $(n \geq 0)$; there exists $M > 0$ such that for all x with $|x| \leq 1$ we have $|f(x)| < M$. If $a_1, a_2, \dots, a_n \in F$, $(n \geq 1)$, $b > 0$, then there exists $h > 0$ such that if $|x| \leq h$, then $|a_1 x + ... + a_n x^n| < b$.

If $a_s \neq 0$, $n \geq 1$, $b > 0$, $s \geq 0$, $P(x) \underset{df}{=} a_s y^s + ... + a_{s+n} x^{s+n}$, then there exists $h > 0$ such that if $x \neq 0$, $|x| < h$, then $|a_{s+1} x^{s+1} + ... + a_{s+n} x^{s+n}| < |a_s x^s| \cdot b$ and « $\leq$ » even for $x = 0$.

If $P(x) \underset{df}{=} a_0 x^n + ... + a_n$, $P(x_0) > 0$, $a_0 \neq 0$, $n \geq 0$, then there exists $h > 0$, such that if $|x - x_0| \leq h$, then $P(x) > \frac{P(x_0)}{2}$. The following « continuity-property » holds true: If for a polynomial $P(x)$ we have $P(x_0) = 0$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| \leq \delta$, then $|P(x)| < \varepsilon$.

26. - If $P(x)$ is a polynomial and $P'(x_0) > 0$, then there exists $h > 0$ such that if $x_0 - h \leq x_1 < x_2 \leq x_2 + h$, then $P(x_1) < P(x_2)$. The proof uses the identity $h_2^r - h_1^r = (h_2 - h_1) \cdot (h_2^{r-1} + h_2^{r-2} h_1 + ... + h_1^{r-1})$.

If $P'(x_0) = 0$, $P''(x_0) > 0$, then there exists $h_0 > 0$ such that if $x_0 - h_0 \leq x_1 < x_2 \leq x_0$, then $P(x_1) > P(x_2)$, and if $x_0 \leq x_1 < x_2 < x_0 + h_0$ then $P(x_1) < P(x_2)$. The proof is based on the inequality $h_2^r + h_2^{r-1} h_1 + ... + h_1^r \leq (h_2 + h_1)^r$, valid for non negative h_1, h_2.

27. - If for polynomial functions the Rolle's theorem holds true, then the Lagrange's and Cauchy's mean value tiieorems are valid. Consequently we get also the Taylors theorem « with remainder »:

$$P(x_0 + h) = P(x_0) + \frac{h}{1!} P'(x_0) + ... + \frac{h^r}{r!} P^{(r)}(x_0) + \\ + \frac{h^{r+1}}{(r+1)!} P^{(r+1)}(x_0 + \theta h)$$

where $0 < \theta < 1$. Under this condition, if we suppose that

$$P(x_0) = P'(x_0) = ... = P^{(r)}(x_0) = 0, \ P^{(r+1)}(x_0) \gtrless 0,$$

then 1) if r is even, there exists $\delta > 0$ such that if $x_0 - \delta \leq \leq x_1 < x_2 \leq x_0 + \delta$, then $P(x_1) \gtrless P(x_2)$, and 2) if r is odd, then uhere exists $\delta > 0$ such that if $x_0 - \delta \leq x_1 < x_2 \leq x_0$, we have $P(x_1) \gtrless P(x_2)$, and if $x_0 \leq x_1 < x_2 \leq x_0 + \delta$, then $P(x_1) \lesseqgtr P(x_2)$.

If for polynomial-functions $P(x)$ the *Weierstrass Null -theorem* holds true, i. e. «if a, $b \in F$, $a < b$, $P(a) < 0$, $P(b) > 0$, then there exists $c \in F$ such that $a < c < b$, $P(c) = 0$», then the Rolle's theorem can be proved for polynomial-functions.

28. - Formally real fields of Artin and Schreier, (4), (6). Let F be a non trivial field. Il is termed *formally-real* whenever $(-1) \in F$ cannot be represented as the sum of a finite number of squares of quantities of F. The field F is termed *algebraically closed formally real (acfr)* if 1) it is formally real, 2) there does not exist any proper algebraic extension of F which would be formally-real. A linearly ordered field always is formally real.

An acrf can always be linearly ordered and in a unique way. If F is an acrf, then the Weierstrass Null -theorem holds true for polynomials, and consequently the Rolle's theorem too.

Let F be a linearly ordered field, and, G its algebraic closure through isomorphism m. Put $F_1 = m\ F \subseteq G$. Then there exists at least one linearly ordered superfield G_1 of F, within G, such that G_1 is an acfr and all operations, - zero - unit - and order - genuine strict extension of F, within G. G_1 will be termed *algebraic real closure of F through isomorphism.* If we suppose that $F \subseteq G$ and that G is the algebraic closure of F, then there may exist many real algebraic closure F_1 of F within G, such that $(F) \subseteq (G_1) \subseteq (G)$ with preservation of order, but they all are all operations -zero-unit - and order - F-rigid-isomorphic. In the seqnal we shall rely on theorems taken from the theory of Artin and Schreier.

Extension of linearly ordered fields.

29. - In what follows we shall suppose that the given linearly ordered field F is a subfield of its algebraic closure F^A. We select among all real algebraic closures of F within F^A a single one, we shall keep it fixed and denote it by $\bar{F}$. So we have $(F) \subseteq (\bar{F}) \subseteq (F^A)$.

By a *gap in* (F) we shall understand a couple (A, B)

of subsets of F such that 1) $A \neq 0$, $B \neq 0$, 2) $A \cap B = 0$, $A \cup B = F$, 3) $A \overset{(F)}{<} B$, i. e. for every $x \in A$ and every $y \in B$ we have $x < y$ in (F). A gap is a kind of Dedekind-section of (F).

If (A, B) is a gap in (F) then the following is true:

if $a \in A$, $a' \overset{F}{\leq} a$, then $a' \in A$; if $b \in B$, $b' \overset{F}{\geq} b$, then $b' \in B$.

If (A, B) is a gap in (F), then it cannot happen that A admits a maximum and, at the same time, B admits a minimum. The expressions $\mathfrak{RE}(A)$ and $\mathfrak{LE}(B)$ are meaningful. We put $\alpha^* \underset{df}{=} \mathfrak{RE}(A)$, $^*\beta \underset{df}{=} \mathfrak{LE}(B)$. We call α^*, $^*\beta$ *endings determined by the gap* (A, B). We have $\alpha^* < {}^*\beta$ in the ordering of F-endings.

29. - As we mentioned in [24], we shall consider polynomials-functions instead of polynomials. The symbol $p_F(\dot{x})_{F'}$, where F, F' are fields will denote a polynomial function whose coefficients belong to F but whose range of the variable $\dot{x}$ is F'. We shall drop the indices and the dot if no misunderstanding can occur, so e. g. $p(x)$ can be understood as $p_F(\dot{x})_{F'}$. We shall call $p_F(\dot{x})_{F'}$ *polynomial in F with domain F'*.

Let $M \subseteq F$. We define $p(M) \underset{df}{=} \{p(x) \mid x \in M\}$. We have $p(M) \subseteq F$. The following are equivalent: I. $M \neq 0$, II. $p(M) \neq 0$.

If $p = c$ is a constand polynomial in F, then p(M) is composed of the single quantity c, for all $M \neq 0$.

30. - We have supposed that F is embedded in its algebraic real closure $\bar{F}$ with preservation of all operations and order. A gap (A, B) in F will be termed *apparent* whenever either A admits a maximum, say a_0, or B admits a minimum, say b_0.

In this case the single elements extensions $[E, a_0]$ or $[F; b_0]$ within any supperfield of F coincide with F. These extensions are apparent, and so are the corresponding elementary extensions.

31. - In the remaining case (A, B) will be termed *proper gap* in F. Such a proper gap will be termed *algebraic* whenever there exists in $\bar{F}$ a quantity z with

$$A < z < B.$$

This is equivalent to $\alpha^* < z < {}^*\beta$ in the ordering of F-endings and also to $\bar{\alpha}^* < z < {}^*\bar{\beta}$ in the ordering of $\bar{F}$-endings, where $\bar{\alpha}^*$, ${}^*\bar{\beta}$ are the natural correspondents of α^*, ${}^*\beta$ in $\bar{F}$ (see [§ 1G; 2]). The single element extension $[F; z]$ of F in $\bar{F}$ is algebraic and so is the corresponding elementary extension of F. The notion of an algebraic proper gap does not depend on the choice of the real closed algebraic extension $\bar{F}$ of F.

32. - Suppose that the proper gap (A, B) is not algebraic. Hence there does not exist any quantity $z \in \bar{F}$ with $A < z < B$ in $\bar{F}$.

Suppose that (A', B') is a gap in $\bar{F}$, such that $A \subseteq A'$, $B \subseteq B'$. We shall prove that

$$\bar{\alpha}^* = \mathfrak{RE}(B'), \qquad {}^*\bar{\beta} = \mathfrak{LE}(B').$$

We have by [§ 1A; 3]

$$\bar{\alpha}^* \leq \mathfrak{RE}(A'), \qquad \mathfrak{LE}(B') \leq {}^*\bar{\beta}.$$

Suppose that $\bar{\alpha}^* < \mathfrak{RE}(A')$. Then there exists $a' \in A'$ such that $\bar{\alpha}^* < a'$ [§ 1D; 3].

Since $a' \in A'$, $A' < B'$ and hence $A' < B$, it follows that $a' < \mathfrak{LE}(B)$, i. e. $a' < {}^*\beta$. Hence $\bar{\alpha}^* < a' < {}^*\bar{\beta}$ and then $A < a' < B$, i. e. (A, B) is an algebraic gap in F, which is a contradiction. Consequently $\bar{\alpha}^* = \mathfrak{RE}(A')$. In a similar way we prove that ${}^*\bar{\beta} = \mathfrak{LE}(B')$.

Def. We call (A', B') *the gap in $\bar{F}$ corresponding to* (A, B).

33. - **Theorem.** If 1) (A, B) is proper but not a proper algebraic gap, 2) $p_F(\dot{x})_{\bar{F}}$ is a polynomial function $\neq 0$, then there exist $a' \in A'$, $b' \in B'$ and $s > 0$, $s \in \bar{F}$ such that either [16] $p(\langle a', b' \rangle_{\bar{F}}) > s$ or $p(\langle a', b' \rangle_{\bar{F}}) < -s$, where (A', B') is the gap

[16]) $\langle a', b' \rangle_{\bar{F}}$ means $\{ x \mid x \in \bar{F}, a' \leq x \leq b' \}$.

in $\overline{F}$ with $A \subseteq A'$, $B \subseteq B'$; (the gap corresponding to (A, B), [Def. 3 2].

Proof. The thesis is true for constant polynomials. Suppose that p is not constant. The polynomial-function $\dot{p}_F(x)_{\overline{F}}$ and its algebraic derivative $\left(\frac{d\dot{p}(x)}{dx}\right)_{\overline{F}}$ may have roots in $\overline{F}$ or not. Suppose that neither p nor $\frac{dp}{dx}$ has. Then by the Weierstrass null-teorem p is either > 0 in $\overline{F}$ or $p < 0$ in $\overline{F}$, and the same holds for $\frac{dp}{dx}$. It follows, by the mean value theorem [28], that p is in $\overline{F}$ either constantly decreasing or constantly increasing. Take any $a' \in A'$, $b' \in B'$. We have $a' < b'$ and p is in $\langle a', b'\rangle_{\overline{F}}$ decreasing or increasing and has the same sign.

Suppose $p > 0$. The polynomial-function p admits as minimum in $\langle a', b'\rangle_{\overline{F}}$ either the quantity $p(a')$ or $p(b')$. Thus we have, denoting by 2s this minimum, $p(x) > s$ in $\langle a', b'\rangle_{\overline{F}}$, $s \in \overline{F}$ so, in this case the theorem is proved. Similar proof is in the case where $p < 0$. We get an inequality

$$p(x) < -s' \text{ in } \langle a', b'\rangle_{\overline{F}}, \text{ where } s' > 0,\ s' \in \overline{F}.$$

Now suppose that the set of quantities of $\overline{F}$ which are either roots of p or of $\frac{dp}{dx}$ is not empty. Let $z_0 < z_2 < \dots < z_n$, $(n \geq 1)$ be all these roots.

Using the Weierstrass null-theorem and the mean-value theorem we prove that in each open interval:

$$(-\infty, z_1)_{\overline{F}}, (z_1, z_2)_{\overline{F}}, \dots, (z_n, +\infty)_{\overline{F}}$$

the polynomial $\dot{p}_F(x)_{\overline{F}}$ has a constant sign and is either decreasing or increasing, (being, of course, $\neq 0$). The inequality

$$\bar{\alpha}^* < z_k < {}^*\bar{\beta} \tag{1}$$

is not true, whatever k may be. Indeed $z_k \in \overline{F}$, and (A, B) is a proper gap which is not algebraic. The equalities $\bar{\alpha}^* = z_k$ and ${}^*\bar{\beta} = z_k$ are also not true. Indeed, if $\bar{\alpha}^* = z_k$, we would have $\alpha^* = z_k$ too, and then, A would have a maximum. In a similar way we prove that the equality $^*\bar{\beta} = z_k$ does not take place. Consequently one and only one of the following inequalities takes place:

$$\bar{\alpha}^* < {}^*\bar{\beta} < z_1, \quad z_i < \bar{\alpha}^* < {}^*\bar{\beta} < z_{i+1}, \quad z_n < \bar{\alpha}^* < {}^*\bar{\beta} \quad (i=1, \dots, n-1).$$

Suppose that

$$z_i < \bar{\alpha}^* < {}^*\bar{\beta} < z_{i+1}.$$

There exist a, $b \in \bar{F}$ such that

$$z_i < a < \bar{\alpha}^* < {}^*\bar{\beta} < b < z_{i+1}$$

in the ordering of $\bar{F}$-endings. Since, in $(z_i, z_{i+1})_{\bar{F}}$, the polynomial p is either decreasing or increasing and has a constant sign, so it is also in $\langle a, b\rangle_{\bar{F}}$, and p admits there its minimum and maximum at the endpoints. Using a similar argument, to that applied at the beginning of the proof, we establish the thesis.

34. - We resume the notations of [33]. A polynomial p is said *to be positive*, « p pos » with respect to (A, B) whenever there exist a, $b \in \bar{F}$ and $s > 0$ such that $a < b$, $a \in A'$, $b \in B'$, $s \in \bar{F}$ and such that for the corresponding polynomial-function $p_{\bar{F}}(\dot{x})_{\bar{F}}$ we have $p(\langle a, b\rangle_{\bar{F}}) > s$.

The notion of p being positive does not depend on the choice of the algebraic real closure $\bar{F}$ of F. Indeed, any two real-algebraic closures of F are operation, order F-rigid-isomorphic.

Given a polynomial p in F, we have disjointedly either p pos or $(-p)$ pos or $p = 0$. The notion « p pos » is invariant with respect to equality of polynomials. If p pos, q pos, then $(p + q)$ pos and $(p \cdot q)$ pos.

Having this, consider ordered couples of EF-polynomials (p, q) with $q \neq 0$, and define their equality $\overset{Q}{=}$ and operations as in [14]. The collection of couples, considered in thus organized field Q, constitutes an elementary transcendental extension of F through the isomorphism cb which carries $a \in F$ into the couple $(a^*, 1_P)$.

From [14] and [15] we see that Q is a single element transcendental extension of $F^* \underset{df}{=} cb(F)$ within Q, namely $[F^*; \xi^*]$. The quantities of Q are

$$\frac{a_0^{*x}(\xi^x)^0 + \dots + a_n^{*x}(\xi^x)^n}{b_0^{*x}(\xi^x)^0 + \dots + b_m^{*x}(\xi^x)^m} \tag{1}$$

where

$$a_i,\ b_j \in F,\ n > 0,\ m > 0,\ b_m \neq 0,\ \xi^x \underset{df}{=} (\xi,\ 1_P),$$
$$a_i^{*x} = cb(a_i) = (a_i^*;\ 1_P),\ b_j^{*x} = cb(b_j) = (b_j^*,\ 1_P).$$

Now we shall define an ordering on Q. If $p \neq 0$ is a polynomial, define «sign p» as $+1$ or -1 accordingly to whether p pos or $(-p)$ pos. Having the couple (p, q) of EF-polynomials with $q \neq 0$, we call (p, q) *positive*, «(p, q) pos», whenever sign $p \cdot$ sign $q = +1$. We prove that the notion «(p, q) pos» is $\overset{Q}{=}$-invariant, and does not depend on the choice of the real algebraic closure $\overline{F}$ of F. If (p, q) pos and (p', q') pos, then $[(p, q) + (p', q')]$ pos and $[(p, q) \cdot (p', q')]$ pos. If $q \neq 0$ and p is any EF-polynomial, then either (p, q) pos, or $[-(p, q)]$ pos or $(p, q) \overset{Q}{=} 0_Q$, these alternatives excluding one another.

It follows that Q is organized into a linearly ordered field (Q). The correspondence cb transforms $a \in F$ into the couple $(a^*; 1_P)$ of polynomials, and the following are equivalent: I. a^* is positive, II. $(a^*; 1_P)$ is positive. Consequently the correspondence cb transforms the linearly ordered field (F) into an order-genuine linear partial field of (Q), so (Q) is an extension of (F) with preservation of order-through the isomorphism cb.

If $a \in A$, $b \in B$, we have $a^{*x} < \xi^x < b^{*x}$, so if we denote by A^{*x}, B^{*x} the cb-correspondents of the sets A, B, we have $A^{*x} < \xi^x < B^{*x}$. Thus we get, through isomorphism, a linearly ordered transcendental extension of F, which introduces a now element into the gap (A, B) of F. The above discussion yields a proof of the following theorem.

34. - Theorem. If 2. F is a linearly ordered field, 2. (A, B) is a gap in F, [31], which is proper but not alge-

2 1

braic, 3. Q is the elementary transcendental extension of F, 4. We define on Q the linear ordering by considering the couples (p, q) of EF-polynomials with $q \neq 0$ as positive whenever for the corresponding polynomial functions $p_F(\dot{x})_{\bar{F}}$, $q_F(\dot{x})_{\bar{F}}$, where $\bar{F}$ is a real closed algebraic extension of F, there exist $a' \in A'$, $b' \in B'$, $a' < b'$ such that $p((a', b')_{\bar{F}})$, $q((a', b')_{\bar{F}})$ are both > 0 or both < 0. ((A', B') is the gap in $\bar{F}$ corresponding to (A, B), [32]), then the linearly ordered field (Q) is an operation and order-genuine linear extension of F through the isomorphism cb.

If we put for $a \in F$, $a \underset{df}{=} cb\, a = a^{**}$, where $f^{x} \underset{df}{=} cf \overset{Q}{=} (f; 1_P)$ for EF-polynomials f, the elements of Q are

$$\frac{\bar{a}_0(\xi^x)^0 + \dots + \bar{a}_n(\xi^x)^n}{\bar{b}_0(\xi^x)^0 + \dots + \bar{b}_m(\xi^x)^m}, \quad b_n \neq 0, \; n \geq 0, \; m \geq 0,$$

and in the linear ordering (Q) we have $cb(A) < \xi^x < cb(B)$, so (Q) is the single element ξ^x-transcendental extension of $cb(F)$ within Q, i. e. $[CB(F); \xi^x]$.

The gap spoken of in [theor. 34] yields a well determined linearly ordered extension of F. We call it *elementary linear trascendental extension of F determined by the gap (A, B).*

35. - The isomorphism s, in [14], transforms the couple-form field Q into the class - form - quotient field $\tilde{Q}$. If we define through s, the order in $\tilde{Q}$, we get a linearly ordered field $(\tilde{Q})$. Thus $(\tilde{Q})$ is an elementary extension of F through the isomorphism scb which preserves operations, zero, unit and order.

The type $\psi(\alpha)$ of quantities of $\tilde{Q}$ is that of a class of couples of polynomials. Now, by [18], the type of a polynomial is $\{v; \alpha\}$, where α is the type of elements of F. Hence

$$(1) \qquad \psi(\alpha) = \mathrm{cl}\,\{\{v;\, \alpha\};\, \{v;\, \alpha\}\}.$$

36. - In [30-35] we have proved the existence of elementary linear extensions of a linearly ordered field F, by placing a new element into any gap of F. We have obtained three categories of extensions: apparent, algebraic and transcendental. The apparent extension can be considered as an algebraic extension, so we have obtained two different kinds of extensions, if the type of elements is only considered: the algebraic, apparent or not, where the elements of the extended field have the type

(2) $$\varphi(a) = \mathrm{cl}\,\{ v;\ \alpha \},\ [18],$$

and the transcendental, where the type of elements is $\psi(\alpha)$, [35, (1)].

Now since we shall need to perform many times these extensions, we shall unify the types of elementary extension, as follows:

Denote, in general, by $O(\rho)$ the empty set whose elements have the type ρ. Let M, N, be sets whose elements have the type λ, μ respectively. Then the ordered couples

$$(M,\ O(\mu)),\ (O(\lambda),\ N)$$

have the same type $\{ \mathrm{cl}\,\lambda;\ \mathrm{cl}\,\mu \}$. Applying this remark to extensions of a linearly ordered field F, whose quantities have the type α, consider the elements x of an extension F/J and the elements y of the extension $\tilde{Q}$. Let us make correspond, to every x, the ordered couple

(3) (set composed of x only, $O(\psi)$), [35].

This one - to - one correspondence organizes the class of all couples (2) into a linearly ordered field, as in [19], which is isomorphic to F/J.

Analogously let us make correspond to every y the couple

(4) ($O(\varphi)$, set composed of y only), [36; (2)].

We get a field isomorphic to $\tilde{Q}$. Now the type of (3) and (4) is the same, viz,

(5) $$\sigma(\alpha) \underset{df}{=} \{ \mathrm{cl}\,\varphi,\ \mathrm{cl}\,\psi \} = \{ \mathrm{cl}\,\mathrm{cl}\,\{ v;\ \alpha \};\ \mathrm{cl}\,\mathrm{cl}\,\{ \{ v;\ \alpha \};\ \{ v;\ \alpha \} \} \}.$$

Thus we have obtained for any elementary gap–extension a field whose elements have the same type (5).

37. - Let us remark that the real algebraic closure $\overline{F}$ of of F had only the capacity of a tool for proving the existence of a gap–extension. The type of $\overline{F}$ does not influence the type of extensions, since is can be always performed through a suitable isomorphism.

38. - Thus we have proved the following

Theorem. If 1. F is a non trivial, linearly ordered field whose elements have the type α,

2. (A, B) is a gap in F,

then there exists an elementary linear extension G of F, (induced by the gap), trough isomorphism, such that

1) the type of elements of G is $\sigma(\alpha)$, [36 ; (5)],

2) if we denote by w this isomorphism, then there exists $\rho \in G$ such that

$$w(A) \leq \rho \leq w(B).$$

39. - Notice that, if F is given, and we need to perform many elementary linear gap–extension, step by step, (as we shall do in § 7), we always can arrange so as to have, for all these extension, the same type β of elements. To do this, we can use the method indicated in [19]. This is possible whenever the cardinal number of the set of all elements of type β is sufficiently great.

§ 7. - Construction of a well ordering of fields.

1. - All necessary preparations having been made, we shall proceed to the problem of extension of measure.

Let B be a non trivial finitely additive Boolean tribe and B' its all operations, zero and unit genuine strict extension. So we have $1_B = 1_{B'}$, $0_B = 0_{B'}$, $B \subseteq B'$.

We suppose that $B \neq B'$. Let φ_0 be a linearly ordered not trivial field, and $\mu(a)$ finitely additve measure on B, such that $\mu(a) \in \varphi_0$ and $\mu(a) \geq 0$ for every $a \in B$.

We shall prove that the measure can be extended, through isomorphism, to B'. Of course φ_0 may not suffice for the extension, so φ_0 should be extended to a sufficiently wide linearly ordered field. This will be made by stepwise extension of φ_0, so we shall construct a suitable well ordering of more and more wide linearly ordered fields. Each step will be a proper or apparent elementary linear extension. Since a proper elementary extension of an abstract field can be made only through isomorphism, therefore the elements of the amplier field will be of a logical type differing from that of the elements of the given field. Now, there may occur, in the general case, that an infinity of proper extensions will be needed, so we may obtain an infinity of fields of different types. Finally, we must construct a field which would contain them all through isomorphism; hence we shall make a union of these fields. Consequently we shall apply results and methods, shown in § 6, concerning unification of logical types of various elementary extensions.

2. - A set φ organized into an ordering, field or linearly field will be sometimes, for mor clarity denoted by (φ), $[\varphi]$ $[(\varphi)]$ or $([\varphi])$ respectively. In accordance with [§ 6; 13], and [Preface 8], $(\varphi) \subseteq (\psi)$ will mean that (φ) is an order-genuine strict subordering of (ψ). If $[\varphi]$, $[\psi]$ are fields, then $[\varphi] \subseteq [\psi]$ will mean that $[\varphi]$ is an all-operations-genuine strict subfield of $[\psi]$, (partial field of $[\psi]$). $[(\varphi)] \subseteq [(\psi)]$ will mean that the linearly ordered field $[(\varphi)]$ is an order - and operation - genuine strict subfield of the linearly ordered field $[(\psi)]$.

3. - Let

$$Q_1, Q_2, \ldots, Q_\lambda, \ldots \tag{1}$$

be a distinguished well ordering of different elements of any kind, with power $\aleph$, where $\aleph > \max(\operatorname{card} \varphi_0, \operatorname{card} B')$.

Its domain will be denoted by $\} Q_\lambda \}$. Let

(2) $$s_1, s_2, \dots, s_\beta, \dots$$

be a well ordering of different somata with domain B'.

4. - Let v_0 be the linear space composed of all

(3) $B\varphi_0$–aggregates (see [§ 3]).

We define the functional $g_0(x)$ of the variable $B\varphi_0$–aggregate x, as, (see [§ 5; 14])

(4) $$g_0(x) \underset{df}{=} \sum_{i=1}^{n} \mu(a_i) \cdot q_i,$$

whenever

(5) $$x = \sum_{i=1}^{n} a_i q_i \quad \text{with} \quad a_1, \dots, a_n \in B,\ q_i \in \varphi,$$

$n \geq 1$, where $a_1 + \dots + a_n = 1$ and where $a_1, \dots, a_n$ are disjoint. We know that this definition is independent of the choice of the representation (5) of x. We also know that $g_0(x)$ is a linear functional with multipliers taken from φ_0. It is φ_0 — valued and its domain is composed of all $B\varphi_0$ — aggregates.

5. - If $X = \sum_{k=1}^{m} A_k \lambda_k$, $(m \geq 1)$, where $A_k \in B'$ and where λ_k are quantities of a given linearly ordered field ψ, denote by $N^\psi(X)$ (see [§ 5]) the $B\mu$–norm of X in ψ. This is a left ending in (ψ).

We know that

(6) $$g_0(x) \leq N^{\varphi_0} x \quad \text{for all } x \in v_0, \quad [§ 5; 14].$$

6. - We shall construct, as in [19], a linearly ordered field ([Φ]) whose quantities belong to $\} Q_\lambda \}$ and which is order — and operations — isomorphic to ([φ_0]).

Let Φ_0 be the smallest ideal (segment) in the well ordering (1), such that card $\Phi_0 =$ card φ_0. Choose a $1 \to 1$ mapping α_0 of φ_0 onto Φ_0. This correspondence induces on Φ_0 a chain ordering (Φ_0), defined by

$$Q' \overset{(\Phi_0)}{\leq} Q'' \cdot \rightleftarrows \cdot a_0^{-1} Q' \overset{(\varphi_0)}{\leq} a_0^{-1} Q''.$$

It also organizes Φ_0 into a field $[\Phi_0]$ by means of the definitions

$$Q' +^{[\Phi_0]} Q'' \underset{df}{=} a_0(a_0^{-1}Q' +^{[\varphi_0]} a_0^{-1}Q''),$$

$$Q' \cdot^{[\Phi_0]} Q'' \underset{df}{=} a_0(a_0^{-1}Q' \cdot^{[\varphi_0]} a_0^{-1}Q''),$$

thus a_0 gives a linearly ordered field $([\Phi_0])$ isomorphic to $([\varphi])$ through a_0^{-1}. Its quantities belong to $\{Q_\lambda\}$.

6.1. - The correspondence a_0 induces another one, also denoted by a_0, which transforms the aggregates $x = \sum_i a_i q_i$, where $a_i \in B$, $q_i \in \varphi_0$ into $X = \sum_i a_i a_0(q_i) = a_0(X)$, which are $B\Phi_0$-aggregates. Their collection makes up a linear space, denoted by V_0 whith multipliers taken from Φ_0. It also transforms the functional $g_0(x)$ into $G_0(X)$ defined by

$$G_0(X) = G_0(a_0(x)) = \sum_i a_0(\mu(a_i)) \cdot a_0(q_i) \quad \text{whenever} \quad x = \sum_i a_i q_i.$$

The quantity $a_0(\mu(a))$ for $a \in B$ is a finitely additive Φ_0-valued measure on B. $G_0(X)$ is a Φ_0-valued linear functional defined on V_0 with multipliers taken from Φ_0.

6.2. - We have for all $X \in V$:

$$G_0(X) \leq N^{\Phi_0}(X).$$

We have here the $B - a_0(\mu) -$ norm of X in Φ_0. It is a left ending in (Φ_0).

7. - Let us denote by α the logical type of the elements Q_λ. In [§ 6; 36] we have shown that an elementary proper or apparent extension of $([\Phi_0])$ can be made so as to have the type $\sigma(\alpha)$ for the quantities of the amplified field, this type being the same whatever be the extension, transcendental, proper algebraic or apparent. Now, consider all chains (C) whose domains are composed of elements with type α, and also consider all chains (C') whose domains are composed of elements with type $\sigma(\alpha)$. Having two chains (C), (C') and supposing that card $C =$ card C', there may exists an order-isomorphism m from (C) onto (C'). Such an isomorphism is a correspondence between elements of type α and elements of type $\sigma(\alpha)$. Consider the class of all order-isomorphism having the above logical type. Let us order it well, getting a well ordering

$$(7) \qquad m_1, m_2, \ldots, m_\gamma, \ldots$$

where all elements of (7) are different.

8. - LEMMA. If 1. C_1', C_2' are fields with elements of type $\sigma(\alpha)$, 2. $(C_1') \subseteq (C_2')$, 3. card $C_2' < \aleph$, 4. $([C_1])$ is a field whith $C_1 \subseteq \{Q_\lambda\}$, 5. t_1 is an order - and operation-isomorphism from (C_1') onto (C_1), then there exists C_2 with $C_2 \subseteq \{Q_\lambda\}$, and an order and operation-isomorphism t_2 from C_2' onto C_2 such that

$$(C_1) \subseteq (C_2), \quad t_1 = C_1' \mathbin{1} t_2 .$$

In addition to that t_1^{-1} and t_2^{-1} belong to (7).

Proof. It suffices to suppose that $(C_1) \neq (C_2)$. As card $\{Q_\lambda\} = \aleph$ and card $C_1 =$ card $C_1' \leq$ card $C_2' < \aleph$, it follows that $\{Q_\lambda\} - C_1$ has the power $\aleph$. Since card $(C_2' - C_1') \leq$ card $C_2' < \aleph$, there exists in $\{Q_\lambda\} - C_1$ a set D such that card $D =$ card $(C_2' - C_1')$. Putting $D' \underset{df}{=} C_2' - C_1'$, there exists a $1 \rightarrow 1$ correspondence s from D' onto D. Put $C_2 \underset{df}{=} C_1 \cup D$. We have $C_1 \cap D = 0$.

Let t_2 be the correspondence, defined by $C_1' \mathbin{1} t_2 = t_1$, $D' \mathbin{1} t_2 = s$.

t_2 is a one-to-one - mapping carrying C_2' into C_2, and C_1' into C_1; t_2 induces on C_2 an ordering (C_2) and it reproduces on C_1 the given ordering (C_1). We have $(C_1) \subseteq (C_2)$. We also have $([C_1]) \subseteq ([C_2])$. Since t_0^{-1} and t_0^{-1} are order isomorphisms between chains with elements of type α and chains with elements of type $\sigma(\alpha)$, therefore t_1^{-1} and t_2^{-1} are elements elements of the well ordering (m_γ).

9. - Let

$$(7a) \qquad W_1, W_2, \ldots, W_\delta, \ldots$$

be well ordering of different elements, each of which being a linear superfield of $([\Phi_0])$, with elements belonging to $\{Q_\alpha\}$.

Starting with the entities $([\Phi_0])$, V_0, $G_0(X)$, defined in [6] and [6.1], we shall construct a well ordering

$$([\Phi_0]),\ ([\Phi_1]),\ \dots,\ ([\Phi]),\ \dots$$

of linearly ordered, more and more ample fields with domains $\subseteq \{Q_\lambda\}$. The construction will be based on the correspondence a_0 from [6], on the well ordering $\{Q_\lambda\}$, on (s_β) from [3], on (m_γ) from [7], and on [9; (7a)] which were chosen arbitrarily. The construction will yield well determined entities.

Let $\alpha \geq 1$ be an ordinal where card $\alpha < \aleph$.

Suppose that we have already constructed the fields $([\Phi_\beta])$, the spaces V_β and the linear functionals $G_\beta(X)$ for every $\beta < \alpha$, $\beta \geq 0$, and suppose that the following hypotheses are satisfied for every $\beta < \alpha$, $\beta \geq 0$: 1) $\Phi_\beta \subseteq \{Q_\lambda\}$; 2) if $\beta' \leq \beta'' \leq \beta$, then $([\Phi_{\beta'}]) \subseteq ([\Phi_{\beta''}])$; 3) V_β is a linear space with multipliers taken from Φ_0 and composed of some $B'\Phi_0$–aggregates; 4) if $\beta' \leq \beta'' \leq \beta$, then $V_{\beta'} \subseteq V_{\beta''}$; 5) $G_\beta(X)$ is a linear Φ_β–valued functional, defined on V_β with multipliers taken from Φ_0; 6) if $\beta' \leq \beta'' \leq \beta$, and $X \in V_{\beta'}$, then $G_{\beta'}(X) = G_{\beta''}(X)$; 7) for every $X \in V_\beta$ we have $G_\beta(X) \leq N^{\Phi_\beta}(X)$, the B, $a_0(\mu)$–norm of X in Φ_β). 8) card $\Phi_\beta < \aleph$.

We shall construct $([\Phi_\alpha])$, V_α and $G_\alpha(X)$ and prove that for $\beta = \alpha$ the conditions 1) - 8) are satisfied.

10. - Notice that from 1) - 8) the following properties are resulting:

2°) if $\beta' < \beta'' < \alpha$, then $([\Phi_{\beta'}]) \subseteq ([\Phi_{\beta''}])$;

4°) if $\beta' < \beta'' < \alpha$, then $V_{\beta'} \subseteq V_{\beta''}$;

6°) if $\beta' < \beta'' < \alpha$, then for $X \in V_{\beta'}$ we have $G_{\beta'}(X) = G_{\beta''}(X)$.

11. - First we shall define the auxiliary entities

$$([\Phi_\alpha^\bullet]),\ V_\alpha^\bullet,\ G_\alpha^\bullet(X)$$

as follows: The type of all sets Φ_β being the same, [Hyp. 1] and since, by $\alpha \geq 1$, there exists β with $0 \leq \beta \leq \alpha$, the union

$$(8) \qquad \Phi_\alpha^\bullet \overline{\overline{df}} \bigcup_{\beta<\alpha} \Phi_\beta$$

is meaningful. We shall organize $\Phi_\alpha^\bullet$ into a chain by the definition: If Q', $Q'' \in \Phi_\alpha^\bullet$, then there exist β', β'' with $\beta' < \alpha$, $\beta'' < \alpha$ such that

$$(9) \qquad Q' \in \Phi_{\beta'}, \quad Q'' \in \Phi_{\beta''}.$$

Supposing $\beta' < \beta''$, we have $\Phi_{\beta'} \subseteq \Phi_{\beta''}$, [§ 7; 10, 2°)]. Now, if $Q' \gtreqless Q''$ in $(\Phi_{\beta''})$, we put $Q' \gtreqless Q''$ in $(\Phi_\alpha^\bullet)$.

This definition does not depend on the choice of β', β'' satisfying (9), and it organizes $\Phi_\alpha^\bullet$ into a chain $(\Phi_\alpha^\bullet)$. We have for all $\beta < \alpha$

$$(10) \qquad (\Phi_\beta) \subseteq (\Phi_\alpha^\bullet).$$

We shall organize $\Phi_\alpha^\bullet$ into a field as follows: Admitting (9) and $\beta' \leq \beta''$, we define $Q' +^{\Phi_\alpha^\bullet} Q''$ and $Q' \cdot^{\Phi_\alpha^\bullet} Q''$ as $Q' + Q''$ and $Q' \cdot Q''$ in $[\Phi_{\beta''}]$.

This definition does not depend on the choice of β' and β'', [§ 7; 10, 2°)], and organizes $\Phi_\alpha^\bullet$ into a field $[\Phi_\alpha^\bullet]$. Thus we obtain the linearly ordered field $([\Phi_\alpha^\bullet])$ such that for every $\beta < \alpha$ we have

$$(10.1) \qquad ([\Phi_\beta]) \subseteq ([\Phi_\alpha^\bullet]).$$

Since card $\alpha < \aleph$, and as, by [Hyp 8)], card $\Phi_\beta < \aleph$, it follows that

$$(10.2) \qquad \text{card}\, \Phi_\alpha^\bullet < \aleph.$$

12. - By [Hyp. 3)], V_β is composed of some $B'\Phi_0$–aggregates, and there exists β with $\beta < \alpha$. Consequently the union

$$(11) \qquad V_\alpha^\bullet \overline{\overline{df}} \bigcup_{\beta<\alpha} V_\beta$$

is meaningful. $V_\alpha^\bullet$ is composed of some $B'\Phi$–aggregates. $V_\alpha^\bullet$ is a linear space with multipliers taken from Φ_0.

13. - By [Hyp. 5)] $G_\beta(X)$ is a linear Φ-valued functional, defined on V_β, $(\beta < \alpha)$, with multipliers taken from Φ_0. Hence, by (8),

$$\mathrm{D}\, G_\beta \subseteq \Phi^\bullet_\alpha .$$

We define $G^\bullet_\alpha(X)$ on $V^\bullet_\alpha$ as follows: Let $X \in V^\bullet_\alpha$. There exists, by (11), $\beta < \alpha$ such that $X \in V_\beta$. We put

$$G^\bullet_\alpha(X) \underset{df}{=} G_\beta(X). \tag{12}$$

This definition does not depend on the choice of β, this by [§ 7; 10, 6°)]. We have $\mathrm{C}\, G^\bullet_\alpha = V^\bullet_\alpha$. We see that $G^\bullet_\alpha(X)$ is a $\Phi^\bullet_\alpha$-valued linear functional on $V^\bullet_\alpha$ with multipliers taken from Φ_0.

14. - Let $X \in V^\bullet_\alpha$. There exists $\beta < \alpha$ such that $X \in V_\beta$. By [Hyp 7)] we have

$$G_\beta(X) < N^{\Phi_\beta}(X),$$

where the right hand side is the B, $\alpha_0(\mu)$ — norm of X in Φ_β. It is a left ending in (Φ_β). Now, by [§ 7; 11, (10)], $(\Phi_\beta) \subseteq (\Phi^\bullet_\alpha)$.

We get $G_\alpha(X) \leq N^{\Phi^\bullet_\alpha}(X)$, and then, by Def (12), $G^\bullet_\alpha(X) \leq$ $\leq N^{\Phi^\bullet_\alpha}(X)$ for all $X \in V^\bullet_\alpha$, (14) where the right hand side is a left ending in $\Phi^\bullet_\alpha$. It is the $B\alpha_0(\mu)$-norm of X in $\Phi^\bullet_\alpha$.

15. - Notice that if $\alpha - 1$ exists, then the entities $([\Phi^\bullet_\alpha])$, $V^\bullet_\alpha$ and $G^\bullet_\alpha(X)$ coincide with $([\Phi_{\alpha-1}])$, $V_{\alpha-1}$ and $G_{\alpha-1}(X)$ respectively.

16. - It may happen that the space $V^\bullet_\alpha$ contains the characteristic aggregats of all somata of (B'). If so, the process stops through the formation of $([\Phi^\bullet_\alpha])$.

17. - Let us suppose that $V^\bullet_\alpha$ does not contain the characteristic aggregates of all somata of (B). Under this hypothesis we are going to construct the entities $([\Phi_\alpha])$, V_α and $G_\alpha(X)$ as follows. For the sake of simplicity we shall drop the index α, if no ambiguity will be feared of.

18. - There exists a soma A_0 of (B') whose characteristic $B'\Phi_0$-aggregate (see [§ 5; 11]) $X_0 = \Omega_{A_0} = A_0 \cdot 1 + \mathrm{co}\, A_0 \cdot 0$ does not belong to $V^\bullet$. (The coefficients 1,0 can be understood as taken from Φ. We shall apply the known Banach device (1):

Let $x_1, x_2 \in V^\bullet$. We have, by (14),

$$G^\bullet(x_2 - x_1) \le N^{\Phi^\bullet}(x_2 - x_1) = N^{\Phi^\bullet}[(-x_1 - X_0) + (x_2 + X_0)].$$

Now, the $Ba_0(\mu)$-norm in $\Phi^\bullet$ has a meaning for all $B'\Phi_0$-aggregates.

Since $-x_1 - X_0$, $x_2 + X_0$ are so, [12], we get, by [§ 5; 6],

$$G^\bullet(x_2 - x_1) \le N^{\Phi^\bullet}(-x_1 - X_0) + N^{\Phi^\bullet}(x_2 + X_0).$$

Hence, by [13] and [§ 5; 8],

$$G^\bullet(x_2) - G^\bullet(x_1) \le N^{\Phi^\bullet}(x_1 + X_0) + N^{\Phi^\bullet}(x_2 + X_0).$$

Applying [§ 1G; 4], we get

$$(15) \qquad G^\bullet(x_1) - N(x_1 + X_0) < -G^\bullet(x_2) + N(x_2 + X_0)$$

which is valid for all $x_1, x_2 \in V^\bullet$.

The left hand side in (15) is a right ending in $(\Phi^\bullet)$, and the right hand side is a left ending in $(\Phi^\bullet)$.

By [§ 1G; 6] we obtain for iterated endings:

$$C^* \underset{df}{=} \mathfrak{RE}\{-G^\bullet(x) - N(x + X_0) \mid x \in V^\bullet\} \le$$

$$\le \mathfrak{LE}\{-G^\bullet(x) + N(x + X_0) \mid x \in V^\bullet\} \underset{df}{=} {}^*D,$$

in the chain of all $\Phi^\bullet$-endings. By [§ 1K; 2, 3] there exists a unique smallest right $\Phi^\bullet$-ending γ^* such that $C^* \le \gamma^*$, and there exists a unique greatest $(\Phi^\bullet)$-ending ${}^*\delta$, such that ${}^*\delta \le {}^*D$. Besides we have $\gamma^* \le {}^*\delta$. Thus we get for $\Phi^\bullet$-endings the relation

$$(16) \quad -G^\bullet_\alpha(x) - N^{\Phi^\bullet_\alpha}(x + X_0) \le \gamma^* \le {}^*\delta \le -G^\bullet_\alpha(x) + N^{\Phi^\bullet_\alpha}(x + X_0)$$

valid for all $x \in V^\bullet_\alpha$.

19. - Concerning γ^*, $^*\delta$ two cases may occur. It may happen that there exists a quantity $\rho' \in \Phi^\bullet$ such that

$$\gamma^* \leq \rho' \leq {}^*\delta. \tag{17}$$

Suppose this be so. We take the element Q_β from [§ 7; 3, (1)] with the smallest index for which $\gamma^* \leq Q_\beta \leq {}^*\delta$. We denote this Q_β bx ρ'', getting

$$\gamma^* \leq \rho'' \leq {}^*\delta. \tag{18}$$

20. - It may happen that the quantity δ', mentioned above, does not exist. In this case the couple $(\gamma^*, {}^*\delta)$ of endings represents a gap in $(\Phi^\bullet)$ which is either algebraic or transcendental, (see [§ 6; 31, 34, 37]). To proceed with necessary correctness, take the case [§ 7; 19.1] and denote by φ_α the linearly ordered field, with elements of type $\sigma(\alpha)$ which constitutes the corresponding elementary linear extension of $([\Phi^\bullet])$ through a well determined isomorphism denoted by $A_\alpha^{\bullet -1}$. Let $\varphi_\alpha^\bullet$ be the $A_\alpha^{\bullet -1}$ image of $\Phi_\alpha^\bullet$, and let ρ'' (of type $\sigma(\alpha)$) be a quantity in φ_α which fills up the gap, according to [§ 6; 37]. We have $([\varphi_\alpha^\bullet]) \subseteq ([\varphi_\alpha])$ and $A_\alpha^\bullet$ is the isomorphism from $\varphi_\alpha^\bullet$ onto $\Phi_\alpha^\bullet$, where

$$\Phi_\alpha^\bullet \subseteq \{ Q_\lambda \}. \tag{19}$$

In the case (19) denote by $([\varphi_\alpha])$ the linearly ordered field with elements of type $\sigma(\alpha)$ which constitutes the isomorphic image of $\Phi_\alpha^\bullet$ through isomorphism denoted by $A_\alpha^{\bullet -1}$. Put $\varphi_\alpha^\bullet = \varphi_\alpha$. We have, as in (19), $([\varphi_\alpha^\bullet]) \subseteq ([\varphi_\alpha])$ and $A_\alpha^\bullet$ is the isomorphism from $([\varphi_\alpha^\bullet])$ onto $([\Phi_\alpha^\bullet])$, where

$$\Phi_\alpha^\bullet \subseteq \{ Q_\lambda \}. \tag{20}$$

We are in the conditions of Lemma [8]. Indeed, we have (19) and (20), and, by [11; (11.2)] card $\Phi_\alpha^\bullet < \aleph$, hence card $\varphi_\alpha^\bullet < \aleph$. In the case, considered in [19], we have card $\varphi_\alpha < \aleph$. Now consider the case involving a true extension.

The extension is defined by polynomials in $\Phi_\alpha^\bullet$ or by couples of polynomials. The cardinal number of the class of all polynomial of the n^{th} order is $(\text{card}\, \Phi_\alpha^\bullet)^{n+1} = \text{card}\, \Phi_\alpha^\bullet$, hence the cardinal of the class of all polynomials is $\aleph_0 \cdot$ $\cdot \operatorname{card} \Phi_\alpha^\bullet$. Since the fields are linearly ordered, their characteristic is 0 and they contain a partial field isomorphic with the field of all ordinary rational numbers. Hence $\aleph_0 \leq \operatorname{card} \Phi_\alpha^\bullet$. Consequently the power of the class of all ordered couples of polynomials is $(\operatorname{card} \Phi_\alpha^\bullet)^2 = \operatorname{card} \Phi_\alpha^\bullet$. Consequently we have $\operatorname{card} \varphi_\alpha < \aleph$, whenever the case may be. By Lemma [8] there exists an isomorphism m_Υ, taken from [7, (7)] such that $a_\alpha^\bullet = \varphi_\alpha^\bullet \, 1 \, m_\Upsilon^{-1}$ and where $([\Phi_\alpha^\bullet]) \subseteq ([m_\Upsilon^{-1}(\varphi_\alpha)])$. Take m_Υ with the smallest index and denote it by a_α^{-1}. Put

$$(20.1) \qquad ([\Phi_\alpha]) \underset{df}{=} a_\alpha([\varphi_\alpha]).$$

We have

$$(21) \qquad ([\Phi_\alpha^\bullet]) \subseteq ([\Phi_\alpha]), \qquad \Phi_\alpha \subseteq \{ Q_\Upsilon \}$$

and

$$(22) \qquad \begin{cases} ([\varphi_\alpha]) \xrightarrow{a_\alpha} ([\Phi_\alpha]), \\ ([\varphi_\alpha^\bullet]) \xrightarrow{a_\alpha} ([\Phi_\alpha^\bullet]). \end{cases}$$

Define

$$(23) \qquad \rho_\alpha \underset{df}{=} a_\alpha \rho''.$$

We have $\rho_\alpha \in \Phi_\alpha$.

If there is no true extension of $\Phi_\alpha^\bullet$, we get back $[\Phi_\alpha^\bullet]) =$ $= ([\Phi_\alpha])$, and $\rho_\alpha = \rho''$.

If the gap yields an apparent extension, it is well determined, and so is also the extension if the gap is transcendental. However an algebraic gap may admit many extensions with type $\sigma(\alpha)$. For each such extension we obtain by te above construction a well determined element

$$\rho_\alpha = a_\alpha \rho''.$$

To have a well defined construction, take this extension of $\Phi_\alpha^\bullet$ in $\{ Q_\Upsilon \}$ which in [9, (7a)] has the smallest index, and if, nevertheless, there will be more than one element ρ_α which fills up the gap, take among them one which has in $\{ Q_\Upsilon \}$ the smallest index. In this way the extension and the element ρ_α will be well determined.

21. - Consider the relation (16) in [18]. Since $(\Phi_\alpha^\bullet) \subseteq (\Phi_\alpha)$, the ordering of endings in $(\Phi_\alpha^\bullet)$ constitutes an order (and operation)-genuine subordering of the ordering of endings in Φ_α.

We get for the isomorphic images of endings

$$-G_\alpha^\bullet(x) - N^{\Phi_\alpha}(x + X_0) \leq \gamma^* \leq {}^*\delta \leq -G_\alpha^\bullet(x) + N^{\Phi_\alpha}(x + X_0)$$

in the ordering of Φ_α-endings. Since in Φ_α we have

$$\gamma^* \leq \rho_\alpha \leq {}^*\delta, \quad \rho_\alpha \in \Phi_\alpha, \tag{23.1}$$

we obtain

$$-G_\alpha^\bullet(x) - N^{\Phi_\alpha}(x + X_0) \leq \rho_\alpha \leq -G_\alpha^\bullet(x) + N^{\Phi_\alpha}(x + X_0) \tag{24}$$

in Φ_α, valid for all $x \in V_\alpha^\bullet$.

22. - Define the space V_α as the set of all aggregates $x + tX_0$, where $x \in V_\alpha$, $t \in \Phi_0$. First we notice that if $Y \in V_\alpha$, then Y admits a unique representation

$$Y = x + tX_0 \tag{24.1}$$

where

$$x \in V_\alpha^\bullet, \quad t \in \Phi_0.$$

Indeed let

$$Y = x_1 + t_1X_0 = x_2 + t_2X_0,$$

where

$$x_1,\ x_2 \in V_\alpha^\bullet;\ t_1,\ t_2 \in \Phi_0.$$

We get

$$x_1 - x_2 = (t_2 - t_1)X_0.$$

If we had $t_2 \neq t_1$, we would get

$$X_0 = \frac{x_1 - x_2}{t_2 - t_1} \in V_\alpha^\bullet$$

which we excluded in [17], Hence $t_2 = t_1$, and then $x_1 = x_2$.

I say that V_α is a linear space with multipliers taken from Φ_0. It is composed of some $B'\Phi_0$-aggregates. Indeed

$$(x_1 + t_1X_0) + (x_2 + t_2X_0) = (x_1 + x_2) + (t_1 + t_2)X_0,$$

where

$$x_1 + x_2 \in V_\alpha^\bullet, \quad t_1 + t_2 \in \Phi_0 .$$

If $\lambda \in \Phi_0$ we have

$$(x + tX_0)\lambda = \lambda x + (t\lambda)X_0 ,$$

where

$$\lambda x \in V_\alpha^\bullet \quad \text{and} \quad t\lambda \in \Phi_0 .$$

Let $x + tX_0 \in V_\alpha$. Since $x \in V_\alpha^\bullet$, it is a $B'\Phi_0$–aggregate and since $t \in \Phi_0$, tX_0 is also a $B'\Phi_0$–aggregate, therefore $x + tX_0$ is a $B'\Phi_0$–aggregate.

We have

$$V_\alpha^\bullet \subseteq V_\alpha . \tag{25}$$

Indeed if $x \in V_\alpha^\bullet$, we can write $x = x + 0 \cdot X_0$.

23. - Define for all aggregates $Y = x + tX_0 \in V_\alpha$ the functional $G_\alpha(Y)$, by putting

$$G_\alpha(x + tX_0) \underset{df}{=} G_\alpha^\bullet(x) + t\rho_\alpha . \tag{26}$$

The definition is meaningful on account of the uniqueness of representation (24.1) of Y. We see that G_α is a linear functional in V_α with multipliers taken from Φ_0. Indeed, let

$$Y_1 = x_1 + t_1X_0 , \quad Y_2 = x_2 + t_2X_0$$

be elements of V_α.

We have

$$G_\alpha(Y_1) = G_\alpha^\bullet(x_1) + t_1\rho_\alpha , \quad G_\alpha(Y_2) = G_\alpha^\bullet(x_2) + t_2\rho_\alpha ,$$

hence

$$\begin{aligned} G_\alpha(Y_1) + G_\alpha(Y_2) &= [G_\alpha^\bullet(x_1) + G_\alpha^\bullet(x_2)] + (t_1 + t_2)\rho_\alpha = \\ &= G_\alpha^\bullet(x_1 + x_2) + (t_1 + t_2)\rho_\alpha = G_\alpha[x_1 + x_2 + (t_1 + t_2)X_0] = \\ &= G_\alpha[(x_1 + t_1X_0) + (x_2 + t_2X_0)] = G_\alpha(Y_1 + Y_2). \end{aligned}$$

Let $Y = x + tX_0 \in V_\alpha$ and $\lambda \in \Phi_0$. We have

$$\begin{aligned} G_\alpha(\lambda Y) &= G_\alpha[\lambda x + (t\lambda)X_0] = G_\alpha^\bullet(\lambda x) + (t\lambda)\rho_\alpha = \\ &= \lambda G_\alpha^\bullet(x) + \lambda(t\rho_\alpha) = \lambda[G_\alpha(x + tX_0)] = \lambda G_\alpha(Y). \end{aligned}$$

If $Y \in V^{\bullet}_{\alpha}$, then $G_{\alpha}(Y) = G^{\bullet}_{\alpha}(Y)$. Indeed $Y = Y + 0 \cdot X_0$. Hence $G_{\alpha}(Y) = G^{\bullet}_{\alpha}(Y) + 0 \cdot \rho_{\alpha}$.

24. - We shall prove that if $Y \in V_{\alpha}$, then

$$G_{\alpha}(Y) \leq N^{\Phi_{\alpha}}(Y) \tag{27}$$

i. e. the $Ba_0(\mu)$-norm of Y in (Φ_{α}).

Proof. Let $Y = x + tX_0$. Case 1) $t = 0$. We have, by [23], $G_{\alpha}(Y) = G^{\bullet}_{\alpha}(Y)$, and then, by [14, (14)] $G_{\alpha}(Y) \leq N^{\Phi^{\bullet}_{\alpha}}(Y)$, hence, since [(21), 10],

$$G_{\alpha}(Y) \leq N^{\Phi_{\alpha}}(Y).$$

Case 2), $t > 0$. We have, by [21, (24)]

$$\rho_{\alpha} \leq -G^{\bullet}_{\alpha}(x) + N^{\Phi_{\alpha}}(x + X_0)$$

for every $x \in V^{\bullet}_{\alpha}$. Since $\frac{x}{t} \in V^{\bullet}_{\alpha}$, we get $\left(\text{for } \frac{1}{t} \in \Phi_0\right)$,

$$\rho_{\alpha} \leq -G^{\bullet}_{\alpha}\left(\frac{x}{t}\right) + N^{\Phi_{\alpha}}\left(\frac{x}{t} + X_0\right).$$

Hence, by [§ 5; 7],

$$\rho_{\alpha} \leq -\frac{1}{t} G^{\bullet}_{\alpha}(x) + \frac{1}{t} N(x + tX_0).$$

Hence, by [§ 1F; 7], since $t > 0$,

$$\rho_{\alpha} \leq \frac{1}{t} [-G^{\bullet}_{\alpha}(x) + N(x + tX_0)].$$

Hence, by [§ 1F; 10, 7],

$$t\rho_{\alpha} \leq -G^{\bullet}_{\alpha}(x) + N(x + tX_0).$$

Hence, by [§ 1G; 3],

$$G^{\bullet}_{\alpha}(x) + t\rho_{\alpha} \leq N(x + tX_0),$$

and then

$$G_{\alpha}(Y) \leq N(Y).$$

Case 3), $t < 0$. We have, by [§ 7; 21, (24)],

$$-G_\alpha^\bullet(x) - N(x + X_0) \leq \rho_\alpha$$

for all $x \in V_\alpha^\bullet$. Since $\frac{x}{t} \in V_\alpha^\bullet$, we get

$$-G_\alpha^\bullet\left(\frac{x}{t}\right) - N\left(\frac{x}{t} + X_0\right) \leq \rho_\alpha.$$

Applying [§ 1F; 27], we get $-G_x^\bullet - N \leq -0 + \rho_\alpha$ hence, by a theorem similar to [§ 1G; 4], $0 - G_\alpha^\bullet \leq N + \rho_\alpha$ and, by [§ 1F; 27]

$$-G_\alpha^\bullet \leq N + \rho_\alpha, \quad -G_x^\bullet - \rho_\alpha \leq N, \text{ i. e. } -G_\alpha^\bullet\left(\frac{x}{t}\right) - \rho_\alpha \leq N\left(\frac{x}{t} + X_0\right).$$

Hence, by [5; 8].

$$\left(-\frac{1}{t}\right) G_\alpha^\bullet(x) - \rho_\alpha \leq N\left(\frac{x + tX_0}{-t}\right).$$

Hence, by [§ 5; 7],

$$-\left(\frac{1}{t}\right)[G_\alpha^\bullet(x) = t\rho_\alpha] \leq \frac{1}{-t} N(x + tX_\delta), \text{ i. e. } \left(-\frac{1}{t}\right) G_\alpha(Y) \leq \frac{1}{-t} \cdot N(Y).$$

Using [§ 1F; 10], we get $G_\alpha(Y) \leq N(Y)$, so the assertion (27) is proved in all cases.

25. - Having defined $([\Phi_\alpha]) \cdot V_\alpha$ and $G_\alpha(x)$, we shall prove that all properties 1) - 8) in [9] hold true for $\beta = \alpha$. We have $\Phi_x \subseteq \{ Q \}$. This follows from (21).

If $\beta' \leq \beta'' \leq \alpha$, then $([\Phi_{\beta'}]) \subseteq ([\Phi_{\beta''}])$. This follows from [9, 3)], [11, (11.2)] and [20, (21)].

V_α is a linear space with multipliers from Φ_0, and its elements are $B'\Phi_0$-aggregates. This follows from [22].

If $\beta' \leq \beta'' \leq \alpha$, then $V_{\beta'} \subseteq V_{\beta''}$. This follows from [9, 3], [22, (25)] and [12, (11)].

$G_x(X)$ is a linear Φ_α-valued functional, defined on V_x with multipliers, taken from Φ_0. This follows frem [23].

If $\beta' \leq \beta'' \leq \alpha$, and $X \in V_{\beta'}$, then $G_{\beta'}(X) = G_{\beta''}(X)$. This follows from [9, 6], [23] and [13].

For every $X \in V_\alpha$ we have $G_\alpha(X) \leq N^{\Phi_\alpha}(X)$. This was proved in [24].

We have card $\Phi_\alpha < \aleph$. This follows from [20], because card $\varphi_\alpha < \aleph$ and Φ_α is an isomorphic image of φ_α.

26. - This concludes the inductive process. We have got a well ordering

$$[(\Phi_0)] \subseteq [(\Phi_1)] \subseteq \dots \subseteq [(\Phi_\alpha)] \subseteq \dots, \tag{28}$$

of linearly ordered fields, a well ordering of linear spaces

$$V_0 \subseteq V_1 \subseteq \dots \subseteq V_\alpha \subseteq \dots, \tag{29}$$

and a well ordering of linear functionals

$$G_0(X),\ G_1(X), \dots,\ G_\alpha(X), \dots, \tag{30}$$

defined on (29) respectively.

The elements of all spaces (29) are $B'\Phi_0$-aggregates; $G_\alpha(X)$ is Φ_α-valued with multipliers taken from Φ_0.

All fields (28) have the same logical type.

§ 8. - The extended measure.

1. - The construction process exhibited in § 7 must stop at $\Phi_\gamma^\bullet$ for a certain index γ, where card $\gamma < \aleph$. Indeed, each step of the process is conditioned by the existence of a soma $\in B'$, whose characteristic aggregate does not belong to $V_\alpha^\bullet$.

Now, since card $B < \aleph$, the well ordering must have its power $< \aleph$, Take this index, consider the field $([\Phi_\gamma^\bullet])$, the space $V_\gamma^\bullet$ and the linear functional $G_\gamma^\bullet(X)$. The space $V_\gamma^\bullet$ contains the characteristic aggregates of all somata belonging to B'. We have

$$\text{card}\ \gamma \leq \text{card}\ B'. \tag{1}$$

To simplify notions, put

$$\Phi \mathrel{\overline{\overline{df}}} \Phi_\gamma^\bullet,\quad V \mathrel{\overline{\overline{df}}} V_\gamma^\bullet,\quad G(X) \mathrel{\overline{\overline{df}}} G_\gamma^\bullet(X).$$

2. - We define for $A \in B'$:

$$\mu^{\bullet}(A) \underset{df}{=} G(\Omega_A), \quad \text{where} \quad \Omega_A = A \cdot 1 + \operatorname{co} A \cdot 0.$$

We have

$$\mu^{\bullet}(A) \in \Phi. \tag{2}$$

If A_1, $A_2 \in B$, $A_1 \cdot A_2 = 0$, we have $\Omega_{A_1+A_2} = \Omega_{A_1} + \Omega_{A_2}$, and then

$$G(\Omega_{A_1+A_2}) = G(\Omega_{A_1}) + G(\Omega_{A_2}), \text{ i. e. } \mu^{\bullet}(A_1 + A_2) = \mu^{\bullet}(A_1) + \mu^{\bullet}(A_2). \tag{3}$$

If $A \in B$, then

$$\mu^{\bullet}(A) = G(\Omega_A) = a_0(\mu(A)).$$

Consequently $\mu^{\bullet}(A)$ represents the extended measure μ from B over all B'; it is finitely additive.

3. We have supposed that the measure $\mu(a)$, given on B is non negative. We shall prove that $\mu^{\bullet}$ is also non negative. To do this, consider the formula (27)

$$G(Y) \leq N^{\Phi\dot{\alpha}}(Y),$$

proved in [24, (27)].

If $Y = \Omega_A$ where $A \in B'$, then by [§ 5; 11], we have for $G(Y)$, $N(Y) = {}^*\mu(A)$. Hence $G(Y) \leq {}^*\mu(A)$, and then $G(Y) \leq$ $\leq {}^*\mu(A)$ which gives

$$\mu^{\bullet}(A) \leq {}^*\mu(A). \tag{4}$$

This is true for any $A \in B'$. Applying (4) to $\operatorname{co} A$, we get

$$\mu^{\bullet}(\operatorname{co} A) \leq {}^*\mu(\operatorname{co} A),$$

hence

$$\mu(1) - \mu^{\bullet}(A) \leq \mu(1) - \mu^*(A),$$

and then

$$\mu^*(A) \leq \mu^{\bullet}(A). \tag{5}$$

Since $\mu^*(A) \geq 0$ for every $A \in B'$, it follows that

$$\mu^{\bullet}(A) \geq 0.$$

We have proved the following

Theorem. If 1. B and B' are finitely additive Boolean tribes,

2. B is a finitely genuine strict subtribe of B', i. e. with equality, finite operations, zero and unit taken from B',

3. φ_0 is a non trivial linearly ordered field,

4. $\mu(a)$ is an φ_0-valued, finitely additive, non negative measure on B,

then there exists a linearly ordered field Φ and a Φ-valued, finitely additive, non negative measure $M(a)$ on B' such that

1) φ_0 is an operation and order genuine subfield of Φ through isomorphism,

2) if we denote this isomorphism by t, then $M(a) = t\mu(a)$ for all $a \in B$,

3) for every $A \in B'$ we have

$$t(\mu^*_{B\mu}(A)) \leq M(A) \leq t({}^*\mu_{A\mu}(A)),$$

where μ^*, ${}^*\mu$ are Jordan $B\varphi_0$-exterior and exterior, ending valued measures on B',

4) card $\Phi \leq$ max (card φ_0, card B').

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