RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

D. T. FINKBEINER
O. M. NIKODÝM

On convex sets in abstract linear spaces where no topology is assumed (Hamel bodies and linear boundedness)

Rendiconti del Seminario Matematico della Università di Padova, tome 23 (1954), p. 357-365
<http://www.numdam.org/item?id=RSMUP_1954__23__357_0>

ON CONVEX SETS IN ABSTRACT LINEAR SPACES WHERE NO TOPOLOGY IS ASSUMED
(HAMEL BODIES AND LINEAR BOUNDEDNESS)

Nota (*) *di* D. T. FINKBEINER *e di* O. M. NIKODÝM
(*Kenyon College Gambier, Ohio*)

1. Introduction.

This note continues the study, initiated by W. D. BERG and O. M. NIKODÝM, of basic notions concerning convex sets in general linear spaces. In [1] these authors exhibited many paradoxical properties of such sets in infinite dimensional spaces. The present investigation is concerned primarily with the concept of linear boundedness in infinite dimensional spaces, and attention is given to a particularly simple and useful type of linearly bounded convex sets, called Hamel bodies. It is shown that two Hamel bodies can be chosen in such a way that their vector sum is the entire space. Also for any convex body B there exists a symmetric image B' of B such that the vector sum of B and B' includes the entire line of symmetry and therefore is not linearly bounded.

2. Hamel Bodies.

Throughout this paper L will denote a real linear vector space. We first recall several basic definitions most of which were stated by Berg and Nikodým, in [1].

(1) A set $S \subseteq L$ is said to be *convex* if and only if the conditions $x, y \in S$ and $0 \leq \lambda \leq 1$ imply $\lambda x + (1 - \lambda) y \in S$.

(*) Pervenuta in Redazione il 19 Giugno 1954.

(2) A set S is said to be *linearly bounded* if and only if for each line l in L set $l \cap S$ is contained in a finite segment of l.

(3) A set S is said to be *linearly closed* if and only if for each line l in L the set $l \cap S$ is closed in the natural topology of l.

(4) A point x is said to be a *linearly inner* point of a set S if and only if on each line through x there exists an open interval containing x and belonging to S.

(5) A convex set which possesses at least one linearly inner point is called a *convex body*.

(6) A *Hamel basis* for L is a set H of linearly independent vectors with the property that if $x \in L$ there exist a finite sequence of vectors $h_1, h_2, ..., h_n \in H$ (where n depends on x) and a sequence of real numbers $\lambda_1, \lambda_2, ..., \lambda_n$ such that $x = \lambda_1 h_1 + \lambda_2 h_2 + ... + \lambda_n h_n$.

Let H be a Hamel basis for L and let $-H = \{ x \mid -x \in H \}$. The *Hamel body* $B(H)$ *determined by* H is defined to be the intersection of all convex sets containing both H and $-H$; that is,

$$B(H) = \text{hull}\,(H \cup (-H))\,.$$

We shall ses that $B(H)$ is a linearly closed, linearly bounded convex body, symmetric about the zero vector O which is a linearly inner point of $B(H)$. The usefulness of Hamel bodies is indicated by their relation to general convex bodies as given by the following result.

THEOREM 1. If B is a convex body and if $x \in B$, there exist a Hamel body $B(H)$ and a vector y such that $x \in B(H) + y \subseteq B$.

Proof. Let B be a convex body, $x \in B$, and z a linearly inner point of B. Then $y = (x + z)/2$ is also a linearly inner point of B. Let $B' = B - y = \{ b - y$, for all $b \in B \}$, the translation of B by the vector $-y$. Then $x - y \in B'$ and $z - y = -(x - y) \in B'$. Since B' is a convex body with 0 as a linearly inner point, a Hamel basis H for L may be formed by choosing $h_1 = x - y$ and then extending the set $\{ h_1 \}$ to a Hamel basis by choosing each h_α in such a way that h_α and $-h_\alpha$ are in B'. Then $x - y \in B(H) \subseteq B'$, and $x \in B(H) + y \subseteq B$.

An immediate consequence of Theorem 1 is that any convex body B is the union of all translated Hamel bodies contained in B. Since every convex body contains many translated Hamel bodies, it is natural to wonder whether each linearly bounded convex body is contained in a suitably chosen translated Hamel body. That this is not the case follows from a result of Klee [3] who showed that in any infinite dimensional space L there exists a linearly bounded, convex, ubiquitous subset U. This implies that $U \subset \text{cl}\, U = L$. On the other hand Theorem 3 below establishes that any Hamel body is linearly closed; hence U is contained in no translated Hamel body.

Our next theorem and its corollaries provide simple geometric descriptions of Hamel bodies.

THEOREM 2. If H is a Hamel basis, then the following are equivalent:

(1) $x \in B(H)$.

(2) there exist a finite sequence of real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ and a sequence of n distinct vectors $h_i \in H$ such that $\sum_1^n |\lambda_i| \leq 1$, and $x = \sum_1^n \lambda_i h_i$.

Proof. Let $x \in B(H) = \text{hull}\,(H \cup (-H))$. From [1], there exist two positive integers, m and k, vectors $f_1, \ldots, f_m \in H$ and $g_1, \ldots, g_k \in -H$, and non-negative real numbers $\alpha_1, \ldots, \alpha_m$, $\beta_1, \ldots, \beta_k$ such that $\sum_1^m \alpha_i + \sum_1^k \beta_i = 1$ and $x = \sum_1^m \alpha_i f_i + \sum_1^k \beta_i g_i$. It is quite possible that $f_i = -g_j$ for some i and some j. We may suppose that $f_1, \ldots, f_r$ are distinct from each other and from all g_i, and that $-g_1, \ldots, -g_s$ are distinct from each other and from all f_i (where $s = k - (m - r) \geq 0$), and that $f_{r+1} = -g_{s+1}, \ldots, f_{r+(m-r)} = -g_{s+(m-r)}$. Let $n = r + k = m + s$, let

$$h_i = \begin{cases} f_i & \text{if } l \leq i \leq r \\ f_i (= -g_{s-r+i}) & \text{if } r+1 \leq i \leq m \\ -g_{i-m} & \text{if } m+1 \leq i \leq n, \end{cases}$$

and let

$$\lambda_i = \begin{cases} \alpha_i & \text{if } 1 \leq i \leq r \\ \alpha_i - \beta_{s-r+i} & \text{if } r+1 \leq i \leq m \\ -\beta_{i-m} & \text{if } m+1 \leq i \leq n. \end{cases}$$

Then we have

$$x = \sum_1^r \alpha_i f_i + \sum_{r+1}^m (\alpha_i - \beta_{s-r+i}) f_i + \sum_{m+1}^n (-\beta_{i-m})(-g_i)$$
$$= \sum_1^n \lambda_i h_i .$$

Furthermore the vectors h_i are all distinct vectors of H, and

$$\sum_1^n |\lambda_i| = \sum_1^r |\alpha_i| + \sum_{r+1}^m |\alpha_i - \beta_{s-r+i}| + \sum_{m+1}^n |-\beta_{i-m}|$$
$$\leq \sum_1^r |\alpha_i| + \sum_{r+1}^m |\alpha_i| + \sum_1^s |-\beta_i| + \sum_{s+1}^k |-\beta_i| = 1.$$

This completes the first half of the proof. To prove the converse, let $x = \sum_1^n \lambda_i h_i$, where $h_i \in H$, the h_i are distinct, and $\sum_1^n |\lambda_i| \leq 1$. We may assume $\lambda_i > 0$ for $i \leq m$ and $\lambda_i < 0$ for $m + 1 \leq i \leq n$ Let $h_{n+1} \in H$ be distinct from $h_1, \ldots, h_n$, and let $\lambda_{n+1} = (1 - \sum_1^n |\lambda_i|)/2$. Then we have

$$x = (\sum_1^m \lambda_i h_i + \lambda_{n+1} h_{n+1}) + (\sum_{m+1}^n (-\lambda_i)(-h_i) + \lambda_{n+1}(-h_{n+1})).$$

Since this is a linear combination of vectors in $H \cup (-H)$ in which the coefficients are non-negative with sum equal to one, $x \in$ hull $(H \cup (-H) = B(H)$.

THEOREM 3.

(1) $B(H) = -B(H)$,

(2) $B(H)$ is a linearly bounded and linearly closed convex body.

Proof. From Theorem 2 it follows immediately that if $x \in B(H)$, then $-x \in B(H)$. The convexity of $B(H)$ is explicit in the definition of a Hamel body. To prove that $B(H)$ is a body, we show that 0 is a linearly inner point. Let l be an arbitrary line through 0. For $x \neq 0$ and $x \in l$, we have $x = \sum_1^n \alpha_i h_i$; let $y = (\sum_1^n |\alpha_i|)^{-1} x$. Then $y \neq 0$ and $y \in B(H) \cap l$. Thus the closed segment $[y, -y]$ of l is in $B(H)$, and 0 is a linearly inner point of $B(H)$.

To verify that $B(H)$ is linearly bounded, let $x = \sum_1^m \alpha_i h_i$ and $y = \sum_{m+1}^n \beta_i h_i$ be two arbitrary but distinct points. For each point z on the line determined by x and y there exists a real number λ such that $z = \lambda x + (1-\lambda)y$. Define n real numbers μ_i by letting $\mu_i = \lambda\alpha_i$ for $1 \leq i \leq m$ and $\mu_i = (l-\lambda)\beta_i$ for $m+1 \leq i \leq n$. Consider the real function ρ defined by $\rho(\lambda) = \Sigma\,|\,\mu_i\,| = |\,\lambda\,|\sum_1^m|\,\alpha_i\,| + |\,1-\lambda\,|\sum_{m+1}^n|\,\beta_i\,|$. Since $\sum_1^m|\,\alpha_i\,| + \sum_{m+1}^n|\,\beta_i\,| > 0$, $\rho(\lambda)$ increases without bound as $\lambda \to \infty$. Then for sufficiently large λ, $\rho(\lambda) > 1$, and $z \notin B(H)$. Thus $B(H)$ is linearly bounded. Furthermore, ρ is a non-negative, continuous function which assumes a minimum value $M = \min(\sum_1^m|\alpha_i|, \sum_{m+1}^n|\beta_i)$ at either $\lambda = 0$ or $\lambda = 1$. If $M > 1$, then the line joining x and y does not intersect $B(H)$. If $M \leq 1$, the equation $\rho(\lambda) = 1$ determines two and only two values λ_1 and λ_2 (possibly equal) such that $\rho(\lambda_1) = \rho(\lambda_2) = 1$, $\rho(\lambda) \leq 1$ for $\lambda_1 \leq \lambda \leq \lambda_2$, and $\rho(\lambda) > 1$ otherwise. Hence in all cases the intersection of $B(H)$ with the line joining x and y is a closed set in the natural topology of the line. Since x and y are arbitrary, $B(H)$ is linearly closed.

3. The Vector Sum of Hamel Bodies.

We first recall the definition of the vector sum of two sets, S_1 and S_2:

$$S_1 + S_2 = \{\, x = s_1 + s_2 \mid s_1 \in S_1 \text{ and } s_2 \in S_2 \,\}.$$

It has been shown by Berg and Nikodym [2] that the convex hull of two linearly bounded convex bodies is not necessarily linearly bounded. Here we obtain a corresponding result for the vector sum of two linearly bounded convex bodies by showing (in Theorem 4) the even more surprising result that the vector sum of two Hamel bodies may include the entire space.

LEMMA. If L is a linear space with a finite or denumerable Hamel basis H, then there exists a denumerable sequence of vectors $x_i \in L$ such that $L = \bigcup_1^\infty (B(H) + x_i)$.

Proof. We first recall the well known result of Löwig [4] that any two Hamel bases for a given space have the same cardinal number. This number is often called the dimension of the space. If a Hamel basis for L consists of a single vector h, then $B(H)$ is the segment $[h, -h]$, Clearly the entire real line is contained in the denumerable union of the translations of $B(H)$ by all integral multiples of h. We proceed by induction, assuming the lemma valid for any space with a finite Hamel basis of fewer than k vectors. Let $H = \{h_1, ..., h_k\}$ be a Hamel basis for L, and let L' be the space spanned by $H' = \{h_1, ..., h_{k-1}\}$. For $x \in L$ we have $x = x' + \lambda_k h_k$, where $x' \in L'$. By the induction hypothesis L' is contained in a denumerable union of translations of $B(H')$, and hence L' is contained in a denumerable union translations T_i, $i = 1, 2, ...,$ of $B(H)$. Likewise, the line generated by h_k is contained in a denumerable union of translations S_j, $j = 1, 2, ...,$ of $B(H)$. Let R_{ij} be the translation obtained by following T_i by S_j. Renumbering the R_{ij} by a diagonal process, we see that the R_{ij} are denumerable. Furthermore L is contained in the union of all the translations R_{ij} of $B(H)$. Hence the lemma is valid in any space with a finite Hamel basis. But any space with a denumerably infinite Hamel basis is the union of denumerably many spaces with finite Hamel basis, and a denumerable union of denumerable sets of translations is denumerable. Hence the lemma is valid in any space with a denumerable Hamel basis.

With the aid of this lemma we now show the existence of two Hamel bodies whose vector sum includes the entire space.

THEOREM 4. If L has a denumerably infinite Hamel basis H, there exists a Hamel basis G such that

$$L = B(H) + B(G).$$

Proof. From the lemma we obtain the representation $L = \bigcup_1^\infty B_i$, where each B_i is a translation of the Hamel body $B(H)$, and therefore each B_i is a convex body in L. For each $n = 1, 2, ...,$ choose $g_i \in B_i$ for $i \leq n$ such that the set $\{g_1, g_2, ..., g_n\}$ is linearly independent. This is clearly possible for $n=1$, For fixed m suppose that $g_i \in B_i$ for $i \leq m$ such that $\{g_1, g_2, \ldots, g_m\}$ is linearly independent. If for each $g_{m+1} \in B_{m+1}$ the set $\{g_1, \ldots, g_m, g_{m+1}\}$ is linearly dependent, then B_{m+1} is contained in the space spanned by the vectors $g_1, \ldots, g_m$, which contradicts the fact that B_{m+1} is a convex body in L. By induction, a Hamel basis $G=\{g_i\}$ may be chosen for L with $g_i \in B_i$. We next verify that $L=B(H)+$ $+B(G)$. Let $x \in L$, and consider the vector $2x$. For some integer k, $2x \in B(H)+y_k$ for some vector $y_k \in B_k$. But $g_k \in B_k$ so

$$g_k = b_1 + y_k \text{ for some } b_1 \in B(H), \text{ and likewise}$$
$$2x = b_2 + y_k \text{ for some } b_2 \in B(H).$$

Thus, $2x = g_k + (b_2 - b_1)$.

But $g_k \in B(G)$, the Hamel body determined by G, so $g_k/2 \in B(G)$.

Likewise $-b_1$, $b_2 \in B(H)$, so $(b_2 - b_1)/2 \in B(H)$. Therefore $x \in B(G)+B(H)$, which completes the proof.

COROLLARY. For every convex body B in a space L with a denumerable Hamel basis, there exists a Hamel body $B(G)$ such that

$$L = B + B(G).$$

Proof. By theorem 1, B contains a translated Hamel body, so for some $y \in L$ and some Hamel body $B(H)$, $B(H) \subseteq B - y$. From theorem 4 a Hamel body $B(G)$ exists such that $L = B(H) + B(G) \subseteq B - y + B(G)$. Since $L + y = L$, we have $B + B(G) = L$.

4. Symmetric Images of Linearly Bounded Convex Bodies.

In this final section we study the linear boundedness of the vector sum of two very simply related convex bodies, namely any convex body B and a symmetric image of B. To define the term *symmetric image*, we let M and N be complementary subspaces of L; that is, $L = M + N$ and $M \cap N = (0)$. It follows that every vector of L is the sum of two uniquely determined vectors, $x = m + n$, where $m \in M$ and $n \in N$. The *symmetric* *image* x' of $x = m + n$ *relative to the ordered pair* (M, N) is defined by $x' = m - n$.

The symmetric image S' of a set S relative to the ordered pair (M, N) is defined by $S' = \{ x' \mid x \in S \}$. From these definitions it is clear that $(S')' = S$ and $(S + T)' = S' + T'$. Also if S is a convex body, so are S' and $S + S'$, and if S is linearly bounded, so is S'.

We shall be concerned here with the particular case in which M is a one dimensional space l (a line through 0) and N is a hyperplane P. A set will be said to be *symmetric relative to* (l, P) if and only if $S = S'$. The line l is called the *line of symmetry*.

We shall show that for any convex body B in an infinite dimensional vector space there exists a hyperplane P such that if $l \subsetneqq P$ then the vector sum of B and it symmetric image B' relative to (l, P) contains the entire line of symmetry l. In this connection we shall make use of the following theorem [1].

THEOREM (Berg.Nikodým). If B is a convex body in an infinite dimensional vector space L, there exists a hyperplane P such that every translation of P intersects B.

This result may be restated in the following equivalent form.

THEOREM 5. If B is a convex body in an infinite dimensional vector space L, there exists a hyperplane P such that $L = P + B$.

Proof. Let B, L and P be as described in the Berg-Nikodym theorem. For each $x \in L$ there exists $b \in B$ such that $b \in P + x$. Then there exists $p \in P$ such that $b = p + x$; since P is a hyperplane, $-p \in P$ and hence $x = -p + b \in P + B$. A reversal of this argument shows that the Berg-Nikodym statement follows directly from theorem 5.

THEOREM 6. If B is a convex body in an infinite dimensional vector space L, if P is a hyperplane such that $L = P + B$, if l is a line such that $l \cap P = (0)$, and if B' is the symmetric image of B relative to (l, P), then $l \subseteq B + B'$.

Proof. Theorem 5 guarantees the existence of a hyperplane P such that $L = P + B$. Let l be any line for which

$l \cap P = (0)$. For any $x \in l$ we have

$$x = b + p, \text{ for some } b \in B \text{ and some } p \in P,$$
$$x' = b' + p' = b' - p.$$

Thus $x + x' = b + b' \in B + B'$. For any $y \in l$, $y/2 \in l$, and $y = y'$, so $y = y/2 + y'/2 \in B + B'$. Hence $B + B'$ contains the entire line of symmetry.

In an n-dimensional space the n-dimensional sphere with center at 0 has the following property: for every hyperplane P there exists a line l such that the sphere is symmetric relative to the pair (l, P). Hence the symmetry, of the sphere can be considered as being « perfect », and of course many bodies other than spheres have the same property. However, in an infinite dimensional space, no convex, linearly bounded body with « perfect » symmetry exists. This paradoxical absence of perfectly symmetric, convex, linearly bounded bodies is established as our final result.

THEOREM 7. If B is a linearly bounded convex body in an infinite dimensional space L, if P is a hyperplane such that $L = P + B$, and if l is any line for which $l \cap P = (0)$, then B is not symmetric relative to the pair (l, P).

Proof. Let B, L, P and l be as described in the hypothesis. If B is symmetric relative to (l, P), then by theorem 6

$$l \subseteq B + B' = B + B = 2B$$

since B is convex. Hence $2B$ is not linearly bounded, and neither is B, contradicting the hypothesis.

REFERENCES

1. BERG, W. D. and NIKODÝM, O. M., *Sur les espaces linéaires réels abstraits ou aucune topologie n'est admise.* Comptes Rendus de l'Académie de Science, Paris, T. 235 (1952), pp. 1005-1007.
2. BERG, W. D. and NIKODYM, O. M., *Sur les ensembles convexes dans l'espace linéaire ou aucune topologie n'est admise.* Comptes Rendus, Paris, T. 235 (1952), pp. 1096-1097.
3. KLEE, V. L., *Convex sets in linear spaces,* III. Duke Math J., vol. 20 (1953), pp. 105-111.
4. LÖWIG, H., *Über die Dimension linearer Räume.* Studia Math. 5 (1934), pp. 18-23.