RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

VÁCLAV HLAVATÝ

Report on the recent Einstein unified field theory

Rendiconti del Seminario Matematico della Università di Padova, tome 23 (1954), p. 316-332

<http://www.numdam.org/item?id=RSMUP_1954__23__316_0>

REPORT ON THE RECENT EINSTEIN UNIFIED FIELD THEORY [1])

Memoria (*) *di* VÁCLAV HLAVATÝ (*a Bloomington, Indiana*)

This short paper is a summary of some main results (without proofs) which I obtained in trying to find a physical interpretation of the new Einstein unified field theory. The proofs are given in the papers [2] - [14] mentioned at the end. The paper consists of four sections: I. Mathematics, II. Conjectures, III. Restricted case, IV. Applications.

* * *

I. Mathematics.

The gravitational field is represented in the four-space of the general relativity [2]) by ten components of a quadratic symmetric tensor, while the electromagnetic field is represented by six components of a skew symmetric quadratic tensor. Therefore, it is understandable why Einstein, in his recent attempt for unified field theory [1], started with sixteen components of a general quadratic real tensor $g_{\lambda\mu}$.

The existence of such a tensor imposes an algebraic structure on the four-space. In order to deal with some consequences of this algebraic structure, we introduce first the symmetric part $h_{\lambda\mu}$ of $g_{\lambda\mu}$

$$h_{\lambda\mu} \overset{\text{def}}{=} g_{(\lambda\mu)}.$$

(*) Pervenuta in Redazione il 29 marzo 1954.

1) Prepared under joint contract with the Office of Naval Research and the Army Office of Ordinance Research.

2) The general relativity will always be referred to as pure gravitational theory, even if the electromagnetic field is present.

Assume that its signature is $- - - +$, so that its determinant $\mathfrak{b}$ is negative, and use it throughout this paper as metric tensor. Moreover, denote by $k_{\lambda\mu}$ the skew symmetric part of $g_{\lambda\mu}$

$$k_{\lambda\mu} \stackrel{\text{def}}{=} g_{[\lambda\mu]} \neq 0$$

and introduce the determinant $\mathfrak{k}$ of $k_{\lambda\mu}$, which is obviously non-negative. The determinant $\mathfrak{g}$ of $g_{\lambda\mu}$ is assumed to be different from zero and by reason of continuity we have to assume $\mathfrak{g} < 0$. Later on we shall need also the following set of scalars

$$g \stackrel{\text{def}}{=} \mathfrak{g}/\mathfrak{b} > 0\,, \qquad k \stackrel{\text{def}}{=} \mathfrak{k}/\mathfrak{b} \leqq 0$$

$$D \stackrel{\text{def}}{=} \left(\frac{1}{2}\, k_{\alpha\beta}\, k^{\alpha\beta}\right)^2 - 4k$$

(1)a $$((g_{ab})) \stackrel{\text{def}}{=} \left(\begin{pmatrix} -1 & \alpha & 0 & 0 \\ -\alpha & -1 & 0 & 0 \\ 0 & 0 & -1 & \beta \\ 0 & 0 & -\beta & +1 \end{pmatrix}\right) \quad \text{whenever } kD \neq 0$$

(1)b $$((g_{ab})) \stackrel{\text{def}}{=} \left(\begin{pmatrix} -1 & \alpha & 0 & 0 \\ -\alpha & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}\right), \quad ((g_{ab})) \stackrel{\text{def}}{=} \left(\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & \beta \\ 0 & 0 & -\beta & +1 \end{pmatrix}\right)$$

whenever $k = 0,\ D \neq 0$

(1)c [3] $$((g_{ab})) \stackrel{\text{def}}{=} \left(\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & \gamma & \varepsilon\gamma \\ 0 & -\gamma & -1 & 0 \\ 0 & -\varepsilon\gamma & 0 & +1 \end{pmatrix}\right) \quad \text{whenever } k = 0,\ D = 0$$

$$(\alpha\beta\gamma \neq 0,\ \varepsilon = \pm 1,\ a, b = \mathrm{I, II, III, IV}).$$

---

[3]) There are three algebrically different classes of $g_{\lambda\mu}$:
$kD \neq 0$, the first class
$k = 0$, $D \neq 0$, the second class
$k = 0$, $D = 0$, the third class.
The second class splits into two different subclasses. The case $k \neq 0$, $D = 0$ cannot occur for a real tensor $g_{\lambda\mu}$.

The tensor $g_{\lambda\mu}$ may be written

$$g_{\lambda\mu} = g_{ab} U^a_\lambda U^b_\mu \tag{2}$$

where $\overset{\mathrm{I}}{U}_\lambda, \ldots, \overset{\mathrm{IV}}{U}_\lambda$ are four mutually perpendicular unit vectors. In the case of the first and second class, they are uniquely determined by $g_{\lambda\mu}$ in the following way: the vectors $\overset{\mathrm{I}}{U}_\lambda, \overset{\mathrm{II}}{U}_\lambda$ are uniquely determined up to ordinary rotations in the bivector $\overset{\mathrm{I}}{U}_{[\lambda} \overset{\mathrm{II}}{U}_{\mu]}$, the vectors $\overset{\mathrm{III}}{U}_\lambda, \overset{\mathrm{IV}}{U}_\lambda$ are uniquely determined up to Lorentz rotations in the bivector $\overset{\mathrm{III}}{U}_{[\lambda} \overset{\mathrm{IV}}{U}_{\mu]}$. In the case of the third class, only one nullvector, say $\overset{\mathrm{III}}{U}_\lambda + \varepsilon \overset{\mathrm{IV}}{U}_\lambda$, is uniquely determined (up to a factor) by $g_{\lambda\mu}$. This classification will be very handy later on. It has also a physical meaning with which we will deal later on.

We mention here, parenthetically, also another physical significance of the existence of $g_{\lambda\mu}$. It leads in the most natural way to spinor algebra and spinor analysis [2], [3], [9].

* * *

The connection $\Gamma^\nu_{\lambda\mu}$ of the Einstein unified theory (which replaces the connection of Christoffel symbols $\left\{ {\nu \atop \lambda\mu} \right\}$ of the pure gravitational theory based on $h_{\lambda\mu}$) is no longer symmetric

$$S^\nu_{\lambda\mu} \overset{\mathrm{def}}{=} \Gamma^\nu_{[\lambda\mu]} \neq 0 \tag{3}$$

and is defined by the set of 64 equations

$$\partial_\omega g_{\lambda\mu} = \Gamma^\alpha_{\lambda\omega} g_{\alpha\mu} + \Gamma^\alpha_{\omega\mu} g_{\lambda\alpha} \tag{4a}$$

equivalent to

$$D_\omega g_{\lambda\mu} = 2 S^\alpha_{\omega\mu} g_{\lambda\alpha} \,. \tag{4b}$$

Here D_ω is the symbol of covariant derivative with respect to the connection $\Gamma^\nu_{\lambda\mu}$. Before we try to solve (4) for $\Gamma^\nu_{\lambda\mu}$, let us make the following remarks: there are tensors $g_{\lambda\mu}$ with $g \neq 0$ for which (4) does not admit any solution $\Gamma^\nu_{\lambda\mu}$. There are tensors $g_{\lambda\mu}$ with $g \neq 0$ for which (4) admits more than one solution. Unlike in the pure gravitational theory where the Christoffel symbols are (uniquely) defined by the covariant constancy of the metric tensor, in the unified theory the connection $\Gamma^{\nu}_{\lambda\mu}$ is *not* defined by the covariant constancy of the basic tensor (cfr. (4)b). It may be proved that the requirement of the covariant constancy of $g_{\lambda\mu}$ leads to a solution $\Gamma^{\nu}_{\lambda\mu}$ only if $g = \text{const}$ (which condition will turn out to be a physical restriction).

Theoretically it is not difficult to solve 64 linear nonhomogeneous equations (4)a for 64 unknowns $\Gamma^{\nu}_{\lambda\mu}$. Practically, however, it is almost beyond control. Moreover, keeping in mind our main goal, i.e. the physical interpretation, we could need only a solution *in tensorial form* [4]). In order to obtain it, we start with the first 40 equations (4)b

$$D_{\omega} h_{\lambda\mu} = 2S^{\alpha}_{\omega(\mu} g_{\lambda)\alpha} \tag{5a}$$

which yield

$$\Gamma^{\nu}_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} + S^{\nu}_{\lambda\mu} + U^{\nu}_{\lambda\mu}. \tag{6}$$

Here $\left\{ {\nu \atop \lambda\mu} \right\}$ are Christoffel symbols of $h_{\lambda\mu}$, the tensor $S^{\nu}_{\lambda\mu}$ is defined by (3) and

$$U^{\nu}_{\lambda\mu} \overset{\text{def}}{=} 2h^{\nu\alpha} S^{\beta}_{\alpha(\lambda} k_{\mu)\beta}. \tag{7}$$

The equation (6) shows the tensorial form of the solution $\Gamma^{\nu}_{\lambda\mu} - \left\{ {\nu \atop \lambda\mu} \right\}$. In order to find $S^{\nu}_{\lambda\mu}$, we take in account the last 24 equations (4)b

$$D_{\omega} k_{\lambda\mu} = 2S^{\alpha}_{\omega[\mu} g_{\lambda]\alpha}. \tag{7b}$$

---

[4]) Schrodinger, who deals in [16] with a system of the form (4)a, writes (l. c. p. 166), «.... it is next to impossible to produce it (the solution) in a surveyable tensorial form». Cf. also [1]; for an (incomplete) bibliography of non-tensorial solution in some very special cases, see [11].

Substituting into (7)b from (6), one obtains finally

(8)a $$2S_{\alpha\beta\gamma}X^{\alpha\beta\gamma}_{\omega\mu\nu} = K_{\omega\mu\nu}\,.$$

Here

$$X^{\alpha\beta\gamma}_{\omega\mu\nu} \overset{\text{def}}{=} \delta^{[\alpha}_{[\omega}\delta^{\beta]}_{\mu]}\delta^{\gamma}_{\nu} - 2\delta^{[\alpha}_{\nu}k_{[\omega}{}^{\beta]}k_{\mu]}{}^{\gamma} - 2\delta^{[\alpha}_{[\mu}k_{\omega]}{}^{\beta]}k_{\nu}{}^{\gamma}$$

$$K_{\omega\mu\nu} \overset{\text{def}}{=} \nabla_{\omega}g_{\nu\mu} + \nabla_{\mu}g_{\omega\nu} + \nabla_{\nu}g_{\omega\mu}$$

and ∇_{μ} is the symbol of covariant derivative with respect to the connection $\left\{ {\nu \atop \mu\lambda} \right\}$. In order to solve the system of 24 linear non-homogeneous equations (8)a for 24 unknows $S_{\alpha\beta\gamma}$ we transcribe it first in the non-holonomic frame given by U^{a}_{λ}

(8)b $$2S_{efg}X^{efg}_{abc} = K_{abc} \qquad a, \ldots, g = \text{I, II, III, IV}.$$

With the aid of (1) and (2), the system (8)b splits in 6 systems of four equations for four unknows each. Hence, it is not difficult to solve each of these six systems (and therefore also (8)) as well as to find *a necessary and sufficient condition for the uniqueness of the solution* $S_{\alpha\beta\gamma}$, namely

$$g \neq 0 \quad \text{whenever} \quad k \neq 0$$
$$g(g-2) \neq 0 \quad \text{whenever} \quad k = 0\,.$$

If this condition is satisfied, then there exists a unique inverse tensor $Y^{\omega\mu\nu}_{\xi\eta\zeta}$ to $X^{\alpha\beta\gamma}_{\omega\mu\nu}$ such that

(9) $$2S_{\xi\eta\zeta} = K_{\omega\mu\nu}Y^{\omega\mu\nu}_{\xi\eta\zeta}$$

and (6) yields the unique solution $\Gamma^{\nu}_{\mu\lambda}$ of (4) in the form

(10) $$\Gamma^{\nu}_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} + \frac{1}{2}[\nabla_{(\alpha}g_{\gamma)\beta} + \nabla_{\beta}g_{\alpha\gamma}] \times$$
$$\times\, Y^{\alpha\beta\gamma}_{\xi\eta\zeta}[h^{\zeta\nu}\delta^{\xi}_{\lambda}\delta^{\eta}_{\mu} + 2h^{\xi\nu}\delta^{\eta}_{(\lambda}k_{\mu)}{}^{\zeta}]\,.$$

Denoting by $R^{\nu}_{\omega\mu\lambda}$ $(H^{\nu}_{\omega\mu\lambda})$ the curvature tensors of $\Gamma^{\nu}_{\lambda\mu}$ (of $\left\{ {\nu \atop \lambda\mu} \right\}$) we introduce the tensors

$$R_{\mu\lambda} \overset{\text{def}}{=} R^{\alpha}_{\alpha\mu\lambda}, \quad H_{\mu\lambda} \overset{\text{def}}{=} H^{\beta}_{\alpha\mu\lambda}, \quad S_{\lambda} \overset{\text{def}}{=} S^{\alpha}_{\lambda\alpha},\; U_{\lambda} \overset{\text{del}}{=} U^{\alpha}_{\alpha\lambda}$$

which are related by

(11)a $$R_{(\mu\lambda)} = H_{\mu\lambda} + S^{\beta}_{\alpha\mu} S^{\alpha}_{\lambda\beta} - (S_\beta + U_\beta) U^{\beta}_{\lambda\mu} - \nabla_\alpha U^{\alpha}_{\lambda\mu} + U^{\beta}_{\alpha\mu} U^{\alpha}_{\beta\lambda} + \nabla_{(\mu}(U_{\lambda)} + S_{\lambda)}) .$$

(11)b $$R_{[\omega\mu]} = \nabla_\alpha S^{\alpha}_{\omega\mu} + 2U^{\beta}_{\alpha[\omega} S^{\alpha}_{\mu]\beta} + (U_\beta + S_\beta) S^{\beta}_{\omega\mu} + \partial_{[\omega} S_{\mu]}$$

where one has to substitute for $S^{\nu}_{\lambda\mu}$ from (9) and for $U^{\beta}_{\lambda\mu}$ from (7) and (9). Put

$$P_{\mu\lambda} \overset{\text{def}}{=} \partial_\alpha \Gamma^{\alpha}_{\mu\lambda} - \Gamma^{\alpha}_{\mu\beta} \Gamma^{\beta}_{\alpha\lambda} - \frac{1}{2} [\partial_\lambda \Gamma^{\alpha}_{(\mu\alpha)} + \partial_\mu \Gamma^{\alpha}_{(\lambda\alpha)}] + \Gamma^{\beta}_{\mu\lambda} \Gamma^{\alpha}_{(\alpha\beta)} .$$

Then it turns out that

(12) $$P_{\lambda\mu} + R_{\mu\lambda} = D_\mu S_\lambda .$$

In order to obtain the field $g_{\lambda\mu}$ from $\Gamma^{\nu}_{\lambda\mu}$, one has to impose some conditions on the connection. Einstein proposes (among other conditions) to use either

(13)a $$P_{\lambda\mu} = 0$$

or

(13)b $$P_{(\lambda\mu)} = 0 \qquad \partial_{[\omega} P_{\mu\lambda]} = 0$$

in both cases coupled with

(14)a $$S_\lambda = 0 .$$

The conditions (13) may be condensed by virtue of (12) and (14)a to

(14)b $$R_{\mu\lambda} = \partial_{[\mu} X_{\lambda]}$$

where X_λ is either an arbitrary gradient (in the case of (13)a) or a suitably chosen vector (in the case of (13)b). The equations (14) are to be looked upon as gravitational and electromagnetic field equations.

Remark. The 84 equations (4) and (14) for 84 unknowns $\Gamma^{\nu}_{\lambda\mu}$, $g_{\lambda\mu}$ and X_{λ} constitute a compatible system [11]. Cfr. also [15].

II. Conjectures.

So far we played mathematics only and the results of the previous section might be looked upon as describing the elementary geometric properties of the four-space of Einstein's unified theory, *regardless of its physical interpretation.* If we want to find a *physical* interpretation of this purely *geometrical* description of the four-space, we have to make several conjectures about the *physical* significance of these *geometrical* conditions. There are possibly many systems of such conjectures. We will follow here one such system without claiming that it is the only one which leads to physical interpretation.

The symmetric part of I(14)b together with I(11)a leads to

$$H_{\lambda\mu} - \frac{1}{2} H h_{\lambda\mu} = M_{\lambda\mu} \qquad (H \overset{\text{def}}{=} H_{\lambda\mu} h^{\lambda\mu}) \tag{1}$$

where $M_{\lambda\mu}$ is a suitable function of $S^{\nu}_{\lambda\mu}$ and $U^{\nu}_{\mu\lambda}$. The form of (1) is familiar from the pure gravitational theory and *suggests the identification of $h_{\lambda\mu}$ with gravitational potentials.* In the pure gravitational theory, the gravitational field equation (1) is a generalization of the Poisson equation, where on the left hand side we have the second derivative of the potential function. This is also true for the left hand term in (1) if we agree to look upon $h_{\lambda\mu}$ (and not on its derivatives) as gravitational potential. This gives us a hint where to look for the electromagnetic field, which has to satisfy Maxwell's equations (at least in a certain approximation). The Maxwell equations contain only first derivatives of the field. Led by analogy with the identification of the basic tensor $h_{\lambda\mu}$ in the pure gravitational theory, we require that in the unified theory the Maxwell equations be expressible in terms of the components of the basic tensor $g_{\lambda\mu}$ and its first derivatives alone.

In other words *we require that the electromagnetic field be a function of $g_{\lambda\mu}$ alone.*

As far as the relativistic Maxwell equations for an electromagnetic field $m_{\lambda\mu} = m_{[\lambda\mu]}$ are concerned, we may assume them in the « idealized » form

$$(2) \qquad \text{a) } \partial_{[\omega} m_{\mu\lambda]} = 0, \qquad \text{b) } \mathfrak{v}^{\nu} = \partial_{\lambda} \mathfrak{m}^{\lambda\nu}$$
$$(\mathfrak{m}^{\lambda\nu} \overset{\text{def}}{=} m^{\lambda\nu} \sqrt{|\mathfrak{b}|}).$$

Let us concentrate first on (2)a alone. According to our assumption about the electromagnetic field (expressible by means of $g_{\lambda\mu}$ alone) this equation cannot be a consequence of I(14)b. Hence, it must be a consequence of I(14)a. Now the following problem has to be solved next. Is there any skew symmetric tensor $m_{\lambda\mu}$, expressible by means of $g_{\lambda\mu}$ alone, such that it satisfies by virtue of I(14)a the equation (2)a

a) either in a certain approximation (for instance for a weak field only) [5]);

b) or exactly?

It turns out [14] that infinitely many tensors $m_{\lambda\mu}$ satisfy the requirement a) and that there is exactly one tensor (up to a multiplicative constant) which satisfies the requirement b), namely

$$(3) \qquad m_{\lambda\nu} \overset{\text{def}}{=} \frac{1}{\sqrt{g}} \left[\varkappa \sqrt{|k|}\, k_{\lambda\nu} - \frac{1}{2} \mathfrak{e}_{\lambda\nu\alpha\beta} \sqrt{|\mathfrak{b}|}\, k^{\alpha\beta} \right] {}^{6)}.$$

Therefore we identify the electromagnetic field with the tensor $m_{\lambda\mu}$ given by (3). If we define the class of $m_{\lambda\mu}$ in the same way as we did for $k_{\lambda\mu}$, then *the tensors* $k_{\lambda\mu}$ *and* $m_{\lambda\mu}$ *are of the same class*. Therefore, there are three classes of electromagnetic fields.

---

[5]) The exact meaning of the word « approximation » is given in [14].

[6]) $\mathfrak{e}_{\lambda\nu\alpha\beta}$ is the covariant tensor density, skew symmetric in all indices whose components in every coordinate system are $+1$, -1, 0. Moreover, $\varkappa = -\operatorname{sgn} \mathfrak{e}_{\lambda\nu\alpha\beta} k^{\lambda\nu} k^{\alpha\beta}$.

The general algorithm may now be described as follows: the skew symmetric part of I(14)b leads to (2)b, while the symmetric part of I(14)b leads to « gravitational » field equations (1).

The resulting equations expressed in terms of $h_{\lambda\mu}$ and $m_{\lambda\mu}$ are somehow complicated. Therefore, we shall apply in the next section the general algorithm to a restricted case *which leads more or less easily to test devices.*

The pure gravitational theory applied to celestial mechanics leads to prediction of three phenomena, namely the advance of perihelion of Mercury, the deflection of light, and the red shift. The « confirmation » of these predictions by observation tests have been looked upon as satisfactory. Therefore we shall try in the last section to apply the unified theory to the celestial mechanics for the special purpose of obtaining the corresponding predictions of the mentioned phenomena. We shall carry out the computation under the same assumption as used in the pure gravitational theory in the presence of the electromagnetic field of the sun: *We assume that the electromagnetic field $m_{\lambda\mu}$ of the sun is a weak field so that the quadratic (and higher) products of its components may be neglected.* The equation (3) shows that this is possible only if the field $k_{\lambda\nu}$ is so weak that the quadratic (and higher) products of its components may be neglected. Therefore the condition

$$k^{\beta}_{(\mu} K_{\lambda)\nu\beta} = 0 \tag{4}$$

is automatically satisfied. On the other hand, the equation (4) is a necessary and sufficient condition for

$$U^{\nu}_{\lambda\mu} = 0 \tag{5}$$

so that

$$S^{\nu}_{\lambda\mu} = \frac{1}{2} K^{\nu}_{\lambda\mu}\,. \tag{6}$$

Therefore, we shall apply in the next section the general algorithm to the restricted case characterized by (4) or (5), without making first any assumption about the weakness of the field $m_{\lambda\mu}$. Then, of course, (4) (or (5)) represent a *structural* condition imposed on the field $g_{\lambda\mu}$. Then in the last section we shall apply the obtained results to celestial mechanics under the above mentioned assumption about the weakness of the field $m_{\mu\lambda}$ so that (4) (or (5)) becomes a *quantitative* condition rather than a *structural* one.

III. The restricted case.

One of the first consequences of (4) and (5) is

(1) $$g = \text{const}, \quad k_{\alpha\beta}k^{\alpha\beta} = \text{const}, \quad k = 0.$$

Moreover, we assume

(2) $$S_{[\omega\mu\lambda]}S^{\omega\mu\lambda} \leqq 0, \quad S_{[\omega\mu\lambda]} \neq 0$$ [7]

and define

(3) $$\sqrt{g}\,\mathfrak{v}^{\nu} \overset{\text{def}}{=} \mathfrak{E}^{\nu\omega\mu\lambda} S_{\omega\mu\lambda}$$ [8]

so that we obtain an *identity*

(4) $$\mathfrak{v}^{\nu} \equiv \partial_{\lambda}\mathfrak{m}^{\lambda\nu}$$

and consequently

(5) $$\partial_{\lambda}\mathfrak{v}^{\lambda} = 0.$$

On the other hand II(2)a is equivalent to

(6) $$m_{\lambda\mu} = \partial_{[\lambda}m_{\mu]}$$

where m_{λ} is a suitably chosen non-gradient. Using now II(3), II(5), II(6) as well as the identity (4), one sees that

---

[7]) These conditions are necessary if dealing with charged particles. In the case of radiation they have to be replaced by

(2)* $$S_{[\omega\mu\lambda]} = 0.$$

[8]) $\mathfrak{E}^{\nu\omega\mu\lambda}$ is the contravariant tensor density, skew symmetric in all indices whose components in all coordinate systems are $+1$, -1, 0.

the skew symmetric part of I(14)b is equivalent to six equations

$$\Delta \mathfrak{m}^{\omega\nu} - \nabla^{[\omega} \mathfrak{v}^{\nu]} = \frac{1}{2\sqrt{g}} \mathfrak{E}^{\mu\lambda\omega\nu} \partial_\mu X_\lambda \tag{7}$$

$$(\Delta \overset{\text{def}}{=} h^{\lambda\mu} \nabla_\lambda \nabla_\mu)$$

for twelve unknowns m_μ, $\mathfrak{v}^\nu$, X_λ (while $h_{\lambda\mu}$ are looked upon as auxiliary variables). Assuming that $\mathfrak{v}^\nu$ is not a gradient, we substitute the solution $\mathfrak{v}^\nu$ of (7) into the identity (4) and obtain in this way *the equation* II(2)b. *Therefore, we identify* $\mathfrak{v}^\nu$ *with the current vector density.* Whenever $\mathfrak{w}^2 \overset{\text{def}}{=} \mathfrak{v}^\nu \mathfrak{v}_\nu \neq 0$, we identify $\mathfrak{w}$ with the electric charge density. The condition (2) yields $\mathfrak{v}^\nu \neq 0$ and $\mathfrak{v}^\nu \mathfrak{v}_\nu \geq 0$. Later on we shall use also the vector and the scalar

$$w^\nu \overset{\text{def}}{=} \mathfrak{v}^\nu / \sqrt{|\mathfrak{b}|} \qquad w^2 \overset{\text{def}}{=} \mathfrak{w}^2 / |\mathfrak{b}|$$

and in the case $w^2 \neq 0$ also the vector

$$u^\nu \overset{\text{def}}{=} \mathfrak{v}^\nu / \mathfrak{w} = w^\nu / w$$

(the velocity « four-vector »).

A similar method applied to the symmetric part of I(14)b yields

$$H_{\lambda\mu} - \frac{1}{2} H h_{\lambda\mu} = \frac{g}{2} T_{\lambda\mu} \tag{8}$$

where

$$T_{\lambda\mu} \overset{\text{def}}{=} w_\lambda w_\mu - Q_{\lambda\mu} \tag{9)a}$$

$$Q_{\lambda\mu} \overset{\text{def}}{=} \frac{3}{2} w^2 h_{\lambda\mu} - 2w^\alpha \nabla_{(\lambda} m_{\mu)\alpha} \tag{9)b}$$

$$+ \frac{1}{2} h_{\gamma\lambda} \mathfrak{e}_{\mu\alpha\xi\eta} \mathfrak{E}^{\gamma\beta\rho\nu} (\nabla_\beta m^{\xi\eta})(\nabla^\alpha m_{\rho\nu}).$$

Hence, we have 20 equations II(2)b (7) (8) for twenty-two unknowns m_λ, $\mathfrak{v}^\nu$, X_λ, $h_{\lambda\mu}$. Therefore, we may as well prescribe at least one condition more

$$(10) \qquad 2\sqrt{g}\, m_{\omega\nu} \nabla^{\omega} \mathfrak{v}^{\nu} + \mathfrak{E}^{\mu\lambda\omega\nu} m_{\omega\nu} \partial_{\mu} X_{\lambda} = 0$$ [9])

which enables us to define a (rest) mass density $\mathfrak{M}$ by the following requirements similar to the corresponding requirements in the pure gravitational theory: $\mathfrak{M}$ is a non-negative function of H such that $\mathfrak{M} = 0$ only if $H = 0$ and moreover, it satisfies the (relativistic) continuity equations. These requirements yield

$$(11) \qquad \mathfrak{M} = |\mathfrak{w}/\mu|$$

where $\mu \neq 0$ is a function of position (to be found experimentally) which is constant along the trajectory of the particle

$$u^{\lambda} \partial_{\lambda} \mu = 0 .$$

Adopting this definition we see that $\mathfrak{M} = 0$ if and only if the particle moves with the velocity of light (photons). Moreover, we see from (11) that $T_{\lambda\mu}$ has (besides μ) the same arguments as the total momentum energy tensor in the pure gravitational theory whenever the electromagnetic field is present. *Hence, we identify $T_{\lambda\mu}$ with the total momentum energy tensor of the unified theory and moreover,* (still on the basis of (8)) *we identify $g \neq 0$ with a constant multiple of the gravitational constant* [10]).

The ten equations II(2)b and (7) may be looked upon as defining (besides $\mathfrak{v}^{\nu}$) *the electromagnetic field $m_{\lambda\mu}$ in terms of the gravitational field $h_{\lambda\mu}$* (and X_{λ}). On the other hand, the equations (8) may be looked upon as defining *the gravitational field $h_{\lambda\mu}$ in terms of the electromagnetic field $m_{\lambda\mu}$*. The absence of $m_{\lambda\mu}$ is here excluded by the assumption $g_{[\lambda\mu]} \neq 0$ while there are cases (as we shall see later) without

---

[9]) As a matter of fact, we already used this condition for deriving (8) and (9).

[10]) g is constant if and only if $U^{\alpha}_{\alpha\mu} = 0$. This conditions is satisfied here by virtue of II(5).

gravitational field. Therefore we look at the electromagnetic field as a primary field, so that the previous statements may be worded as follows: *The gravitational field as well as* $\mathfrak{M}$ *are built up from the electromagnetic field.* However, there are electromagnetic fields, which do not create any gravitation. Thus, for instance, if we are dealing with radiation only so that (2) has to be replaced by (2)* then the system of equations II(2), (7), (8), (10) (as well as (2)*, II(5)) admits a solution

$$((m_{\lambda\mu})) = \left(\left(\begin{matrix} 0 & 0 & -a & a \\ 0 & 0 & -b & b \\ a & b & 0 & 0 \\ -a & -b & 0 & 0 \end{matrix}\right)\right), \quad ((h_{\lambda\nu})) = \left(\left(\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right)\right),$$

$$\mathfrak{v}^{\nu} = 0, \quad X_{\lambda} = \partial_{\lambda} X$$

where $a = a(Z)$, $b = b(Z)$ are periodic functions of the argument $Z = x^3 - x^4$ so that $m_{\lambda\nu}$ *represents the plane wave of the electromagnetic theory of light.* Hence, *this electromagnetic field does not create any gravitation* as it was to be expected. Incidently, this case might be looked upon as *a first confirmation of the unified theory,* at least in this very special case.

The equations of motion follow from

$$(12) \qquad \nabla_{\lambda} T^{\lambda\mu} = 0$$

which is a consequence of (8) and of the well known Bianchi identity for $H_{\omega\mu\lambda}^{\nu}$. They are equivalent to

$$(13) \qquad w^{\lambda} \nabla_{\lambda} w^{\nu} = \nabla_{\lambda} Q^{\lambda\nu}.$$

* * *

The last result suggests a slight change of the theory which would result in the law of inertia in the usual form: The skew symmetric part of I(14)b is equivalent to

$$\partial_{[\omega} R_{\lambda\mu]} = 0\,.$$

Let us replace this system of four equations by another one of the same number of equations

(14) $$D_\lambda \mathfrak{E}^{\alpha\beta\gamma\lambda} S_{\alpha\beta\gamma} S_{[\xi\eta\zeta]} = - \sigma \sqrt{|\mathfrak{g}|}\, S_{[\xi\eta\zeta]} .$$

Here σ is a scalar to be chosen in such a way that

(15) $$m_{\lambda\mu} \triangle m^{\lambda\mu} = 0$$

which is equivalent to (10). Then we obtain again the Maxwell equations II(2) and the gravitational field equations (8), but the equations of motion derived from (12) are now

(16)a $$w^\lambda D_\lambda w^\nu = \sigma w^\nu$$

or in equivalent form

(16)b $$w^\lambda \nabla_\lambda w^\nu = \sigma w^\nu .$$

Hence the law of inertia based on (16) runs as follows: *If a particle moves in the field* $g_{\lambda\mu}$ *without external forces, then it describes a path which is an autoparallel line of the connection* $\Gamma^\nu_{\lambda\mu}$ *given by* I(4) (*cf. also* I(10)). *Despite the presence of the electromagnetic field, this path is the same as in the pure gravitational theory in the absence of the electromagnetic field, namely a geodesic of* $h_{\mu\lambda}$.

IV. Applications.

Let us now apply the previous results to celestial mechanics under the assumption that the electromagnetic field $m_{\mu\lambda}$ of the sun is weak so that the quadratic products of its components may be neglected (and consequently II(5) is automatically satisfied).

Under this assumption, the equation III(8) reduces to

(1) $$H_{\lambda\mu} = 0 ,$$

From now on we denote by $h_{\lambda\mu}$ the Schwarzschild solution of (1). Moreover applying (as in the pure gravitational theory) the method of successive approximation (i.e. neglecting in

III(12) only cubic and higher products of $m_{\lambda\mu}$), we obtain the equations of motion in the form III(13) where now ∇_μ refers to the Schwarzschild solution $h_{\lambda\mu}$. If we assume for the time being that we know the electromagnetic field $m_{\lambda\mu}$ of the sun and substitute it into III(13), then the solution w^ν of III(13) gives rise in the usual way to the path of a particle. This path has to be identified either with an orbit of a planet (if $w \neq 0$) or with the path of a photon (for $w = 0$). Hence, we are able (at least theoretically) to predict the three phenomena dealt with in the pure gravitational theory: The advance of perihelion of Mercury, the deflection of light as well as the red shift. *However this prediction is based on the knowledge of the electromagnetic field of the sun, which is a solution of* III(7) *and* II(2).

* * *

The situation is much simpler in the changed theory based on III(14). We obtain again (1) from III(8), and denote by $h_{\lambda\mu}$ its Schwarzschild solution. Then the equations III(16)b define the geodesics of $h_{\lambda\mu}$ so that we do not need the knowledge of the electromagnetic field of the sun to define them. Identifying these geodesics either with orbits of planets (for $w \neq 0$) or with paths of photons ($w = 0$) *we obtain exactly the same predictions for the three above mentioned phenomena as in the pure gravitational theory.*

However the equations III(16)b are a consequence of

$$\nabla_\lambda Q^{\lambda\nu} = \sigma w^\nu \quad [11] \tag{2}$$

equivalent to III(14). *Consequently, the crucial test for this theory* (at least as far as the three above mentioned phenomena are concerned) *will consist in checking whether the electromagnetic field of the sun satisfies* (2). Here the last word belongs to the astrophysicist.

---

[11]) Here w^ν is a known solution of III(16)b and $\mathfrak{v}^\nu = w^\nu \sqrt{|\mathfrak{b}|}$ has to satisfy II(2)b.

This statement might be looked upon from another angle (which leads to the same results): The confirmation of the (pure gravitational) relativistic predictions of the three phenomena has been looked upon as a sufficient proof for the pure gravitational relativistic theory (in these three cases). We have obtained the same predictions in the unified theory. Hence, adopting the same judgement we may claim that the unified theory has been proved (in these three cases). *Then* (2) *has to be looked upon as a prediction for te electromagnetic field of the sun.* Here again the last word belongs to the astrophysicists.

BIBLIOGRAPHY

[1] EINSTEIN ALBERT, *The Meaning of Relativity*, 3rd ed., Princeton, 1950; 4th ed., Princeton, 1953, Appendix II.

[2] HLAVATY V., « Spinor Space and Line Geometry », *Canadian Journal of Mathematics*, Vol. III, No. 4, 1951, pp. 442-459.

[3] HLAVATY V., « Spinor Space and Line Geometry II », *Journal of Rational Mechanics and Analysis*, Vol. 1, No. 2, April 1952, pp. 321-339.

[4] HLAVATY V., « The Elementary Basic Principles of the Unified Theory of Relativity », *Proceedings of the National Academy of Sciences*, Vol. 38, No. 3, pp. 343-347, March 1952.

[5] HLAVATY V., « The Einstein Connection of the Unified Theory of Relativity », *Proceedings of the National Academy of Sciences*, Vol. 38, No. 5, pp. 415-419, May 1952.

[6] HLAVATY V., « The Elementary Basic Principles of the Unified Theory of Relativity *A* », *Journal of Rational Mechanics and Analysis*, Vol. 1, No. 4, October 1952, pp. 539-562.

[7] HLAVATY V., « The Elementary Basic Principles of the Unified Theory of Relativity *B* », *Journal of Rational Mechanics and Analysis*, Vol. 2, No. 1, January 1953, pp. 1-32.

[8] HLAVATY V., « The Schrödinger Final Affine Field Laws », *Proceedings of the National Academy of Sciences*, Vol. 38, No. 12, pp. 1052-1058, December 1952.

[9] HLAVATY V., « The Spinor Connection in the Unified Einstein Theory of Relativity », *Proceedings of the National Academy of Sciences*, Vol. 39, No. 6, pp. 501-506, June 1953.

[10] HLAVATY V., « Connections Between Einstein's Two Unified Theories of Relativity », *Proceedings of the National Academy of Sciences*, Vol. 39, No. 6, pp. 507-510, June 1953.

[11] HLAVATY V. and SÁENZ A. W., « Uniqueness Theorems in the Unified Theory of Relativity », *Journal of Rational Mechanics and Analysis*, Vol. 2, No. 3, July 1953, pp. 523-536.

[12] HLAVATY V., « The Elementary Basic Principles of the Unified Theory of Relativity C_1: Introduction », *Journal of Rational Mechanics and Analysis*, Vol. 3, No. 1, January 1954, pp. 103-1446.

[13] HLAVATY V., « The Elementary Basic Principles of the Unified Theory of Relativity C_2: Application », *Journal of Rational Mechanics and Analysis*, Vol. 3, No. 2, March 1954, pp. 147-179.

[14] HLAVATY V., « Maxwell's Field in the Einstein Unified Field Theory », to appear soon in *Niew Archief for Wiskunde.*

[15] LICHNEROWICZ A., « Compatibilité des Équations de la Théorie Unitaire d'Einstein-Schrödinger », *Comptes Rendus des Séances de l'Academie des Sciences*, t. 237, pp. 1383-1386, séance du 30 Novembre 1953.

[16] SCHRÖDINGER E., « The Final Affine Field Laws I », *Proc. Roy. Irish Acad.* 51, 168-171 (1947).