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GIANCARLO PESAMOSCA

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ON THE SOLUTION OF A CLASS OF LOCATION PROBLEMS. A SAMPLE PROBLEM (*)

by Giancarlo PESAMOSCA ([1])

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Abstract. – *In Section 2 of this paper the optimality conditions for the EMFL (Euclidean Multi-Facility Location) minisum problem are shown to be a useful tool for finding the analytical solution of many simple problems. In Section 3 the problem of connecting by means of two new facilities the vertices of an isoceles triangle is completely solved.* © Elsevier, Paris

Keywords: Non-differentiable optimization, Euclidean Multifacility Location Problem.

Résumé. – *Dans cet article on fait voir que les conditions d'optimalité pour le problème EMFL (Euclidean Multi-Facility Location) sont un moyen efficace pour trouver analytiquement la solution de plusieurs simples problèmes. Ensuite le problème de connecter les vertices d'un triangle isocèle au moyen de deux nouvelles facilités est résolu.* © Elsevier, Paris

Mots clés : Optimisation non différentiable, Euclidean Multifacility Location problem.

1. INTRODUCTION

After the optimality conditions for the general EMFL minisum problem have been stated (*see* [1], [2], [3]), the analytical solution of many simple problems has become possible. The form of the optimality conditions presented in [2] gives an explicit expression of the subdifferential of the objective function and represents a useful tool for this purpose, suggesting the method proposed in Section 2. The problem solved in Section 3, as well as the case considered in [4], show that the analytical solution of some symmetric problems may present simple and aesthetic features.

2. THE OPTIMALITY CONDITIONS AND THE ANALYTICAL SOLUTION OF THE EMFL MINISUM PROBLEM

The optimality conditions in the form presented in [2] can be summarized as follows.

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([1]) Dip. Metodi e Mod. Matematici Univ. "La Sapienza" Via A. Scarpa 16, 00162 Roma.

Let $X_1, \ldots, X_m$ $(X_j = (x_j, y_j))$ be the New Facilities, $A_1, \ldots, A_n$ the Existing Facilities, and H the graph having X_j, A_r as nodes, and the links interconnecting them as edges. The edges are assumed to be ordered, for example in lexicographic order X_1X_2, $X_1X_3 \cdots X_1X_m$, $X_2X_3, \ldots, X_{m-1}X_m$, $X_1A_1, \ldots, X_mA_n$ (of course, only the edges present in H appear in the sequence).

The EMFL minisum problem can be stated as

$$\begin{aligned} F(X) = & F(X_1, \ldots, X_m) = \sum_{(j,r)\in\Omega} w_{jr}\|X_j - X_r\|_2 \\ & + \sum_{(j,r)\in\overline{\Omega}} \overline{w}_{jr}\|X_j - A_r\|_2 = \min \end{aligned} \tag{1}$$

where Ω, $\overline{\Omega}$ are respectively the sets of the pairs (j, r) such that the edge X_jX_r (or X_jA_r) exists in H, and w_{jr} (or $\overline{w}_{jr}$) are the corresponding positive weights. Let $\tilde{X} = [\tilde{X}_1, \ldots, \tilde{X}_m]$ be a given point of R^{2m}. Two facilities $\tilde{X}_j$, $\tilde{X}_r$ (or $\tilde{X}_j$, A_d) adjacent in H are "Interacting" if they overlap, *i.e.* $\tilde{X}_j = \tilde{X}_r$ (or $\tilde{X}_j = A_d$). The corresponding zero-length edges $\tilde{X}_j\tilde{X}_r$ (or $\tilde{X}_jA_d$) are "active edges". As has been proved in [2], the components of the subdifferential set $\partial F(\tilde{X}) = [x_1^*, y_1^*, \ldots, x_m^*, y_m^*]$ can be characterized as follows:

$$\begin{bmatrix} x_j^* \\ y_j^* \end{bmatrix} = \begin{bmatrix} g_{jx} \\ g_{jy} \end{bmatrix} - \sum_{r\in\Sigma_j^-} \begin{bmatrix} u_{rj} \\ v_{rj} \end{bmatrix} + \sum_{r\in\Sigma_j^+} \begin{bmatrix} u_{jr} \\ v_{jr} \end{bmatrix} + \begin{bmatrix} \overline{u}_{jd} \\ \overline{v}_{jd} \end{bmatrix}, \quad (j = 1, \ldots, m) \tag{2}$$

$$u_{jr}^2 + v_{jr}^2 \le w_{jr}^2, \quad \forall r \in \Sigma_j^+, \qquad \overline{u}_{jd}^2 + \overline{v}_{jd}^2 \le \overline{w}_{jd}^2 \tag{3}$$

where:

- $g_{jx} = \frac{\partial F}{\partial x_j}\big|_{\tilde{X}}$, $g_{jy} = \frac{\partial F}{\partial y_j}\big|_{\tilde{X}}$ are the partial derivatives with respect to x_j, y_j of the *differentiable part* of $F(X)$, *i.e.* the sums with respect to r of the derivatives of $w_{jr}\|X_j - X_r\|_2$ and $\overline{w}_{jr}\|X_j - A_r\|_2$, restricted to the non-interacting pairs X_jX_r and X_jA_r,
- Σ_j^-, Σ_j^+ are the sets of the indices r (respectively $r < j$ for Σ_j^- and $r > j$ for Σ_j^+) of the facilities X_r interacting with X_j. Hence, each pair of variables (u_{rj}, v_{rj}) [or (u_{jr}, v_{jr})] is associated with an active edge X_rX_j [or X_jX_r],
- the pair $(\overline{u}_{jd}, \overline{v}_{jd})$ is associated with the edge X_jA_d (if a facility A_d interacting with X_j exists).

A R^{2m} point $\tilde{X}$ is a minimizer for (1) iff $0 \in \partial F(\tilde{X})$, *i.e.* for any active edge a pair (u_{jr}, v_{jr}) or $(\overline{u}_{jd}, \overline{v}_{jd})$ exists which satisfies the inequalities (3) and the linear system of $2m$ equations obtained by setting (2) to zero. These conditions reduce to $\mathrm{Grad}\,[F(X)] = 0$ if no pair of interacting facilities exists. Let z be the number of active edges in $H(\tilde{X})$, and (j_1, r_1), $(j_2, r_2), \ldots, (j_z, r_z)$ be the ordered list of the pairs of indices of the corresponding interacting facilities. Since for any j the first components in (2) contains only g_{jx}, u_{rj}, u_{jr}, $\overline{u}_{jd}$ and the second component only g_{jy}, v_{rj}, v_{jr}, $\overline{v}_{jd}$, then (2) can be split into two parts $G_x + AU$, $G_y + AV$, and the related linear system $\partial F(\tilde{X}) = 0$ into two independent subsystems

$$G_x + AU = 0, \qquad G_y + AV = 0 \tag{4}$$

where:

• $U(z) = [u_{j_1,r_1}, u_{j_2,r_2}, \ldots, u_{j_z,r_z}]^T$; $V(z) = [v_{j_1,r_1}, v_{j_2,r_2}, \ldots, v_{j_z,r_z}]^T$ (some components of U, V can be of the type $(\overline{u}_{j_k,d_k}, \overline{v}_{j_k,d_k})$,

• $G_x(m) = [g_{1x}, g_{2x}, \ldots, g_{mx},]^T$, $G_y(m) = [g_{1y}, g_{2y}, \ldots, g_{my},]^T$,

• $A(mxz)$ is the coefficient matrix, whose entries are 1, -1, 0. In order to determine them, consider for $k = 1, 2, \ldots, z$ the pairs $u_{j_k r_k}$, $v_{j_k r_k}$. If X_{j_k}, X_{r_k} are new facilities (with $j_k < r_k$ since they are ordered), construct the k^{th} column of A by setting 1 in row j_k, -1 in row r_k, and 0 elsewhere. If the k^{th} pair is of the type $\overline{u}_{j_k d_k}$, $\overline{v}_{j_k d_k}$, then the kth column of A must have 1 in row j_k and 0 elsewhere.

In the current literature, the new facilities $\tilde{X}_j$ of a given R^{2m} point $\tilde{X}$ are classified as belonging to one of the following categories:

(i) *Isolated points.* $\tilde{X}_j$ is an isolated point if it does not interact with other facilities.

(ii) *Coinciding points.* $\tilde{X}_j$ is a coinciding point if it interacts with a facility A_d only.

(iii) *Isolated cluster.* A group of interacting new facilities is an isolated cluster if their common location is distinct from all other facility locations.

(iv) *Coinciding cluster.* A group of interacting new facilities is a coinciding cluster if their common location coincides with a facility A_d (interacting at least with a facility X_j of the cluster).

As has been proved in [2], the systems (4) can be split into as many independent subsystems as there are isolated points, coinciding points, and clusters in the point $\tilde{X}$ to be tested for optimality. Let K be a cluster, and let us suppose that K is formed by all the new facilities of H (if not so, a pair of subsystems must be considered instead of the systems (4)). If K is a tree, then the systems have a unique solution which can be computed by means of a recursive algorithm [2]. If K contains cycles, the systems have in general more unknowns than equations, so that the solutions depend on $p = z - r$ (with r rank of A) arbitrary parameters. As will be shown in the problem in Section 3 (region R_1), in the case of $p = 1$ the solutions of the two systems can be expressed by means of two parameters α, β, and the corresponding inequalities (3) represent circles of (α, β) plane. Therefore, (3) are satisfied if these circles have a common intersection.

If the solution of (1) is considered as function of the weights, it can be completely described by determining the sets of weights corresponding to any possible type of solution (*i.e.* any combination of isolated points, coinciding points, and clusters), and then partitioning the space of the weights into regions corresponding to these sets. The weights can be normalized so that one of them is 1.

3. THE CONSIDERED PROBLEM AND ITS SOLUTION

The symmetric EMFL problem shown in Figure 1 is considered. The Existing Facilities are $A_0 = (0, 0)$, $A_1 = (a, h)$, $A_2 = (-a, h)$, the New Facilities $X_1 = (x_1, y_1)$, $X_2 = (x_2, y_2)$, and the weights are μ for the edges A_0X_1, A_0X_2, 1 for X_1A_1, X_2A_2, and ν for X_1X_2.$L = \sqrt{a^2 + h^2}$ is the side of the triangle, and $\vartheta = 2 \operatorname{arctg}(a/h)$ is the angle $A_1\widehat{A_0}A_2$ [$\vartheta \in (0, 2\pi)$].

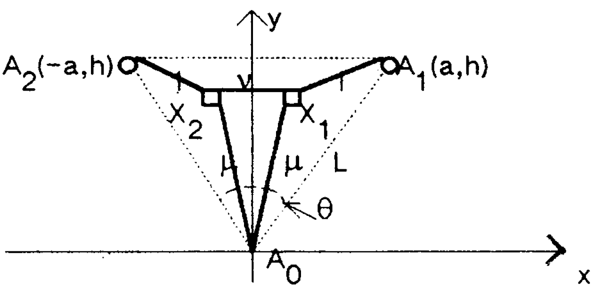

Figure 1.

It is required to locate X_1 and X_2 so as

$$F(X) = F(X_1, X_2)$$
$$= \mu[f_1(X) + f_2(X)] + \nu f_3(X) + f_4(X) + f_5(X) = \min \quad (5)$$

with

$$f_1(X) = \sqrt{x_1^2 + y_1^2}, \qquad f_2(X) = \sqrt{x_2^2 + y_2^2},$$

$$f_3(X) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

$$f_4(X) = \sqrt{(a - x_1)^2 + (h - y_1)^2},$$

$$f_5(X) = \sqrt{(-a - x_2)^2 + (h - y_2)^2}.$$

The solution of (5) is symmetric with respect the y-axis. A proof is given by the following

THEOREM 1: *Let* X_1, X_2 *be a solution of* (5) *with* $X_1 = (x, y)$, *and let* $\mu, \nu > 0$. *Then* $X_2 = (-x, y)$.

Proof: Let us first assume that X_1, X_2 are isolated points, and consider the two Weber problems having respectively X_1, X_2 as new facilities, and (A_0, A_1, X_2), (A_0, X_1, A_2) as existing facilities (*see* Fig. 1). Such problems must be also solved when X is optimal: in fact, if not so X_1 (or X_2) could be moved from its position, with a decrease of $F(X)$. For a property of the 3-point Weber problem, the angles

$$\alpha_1 = A_0\widehat{X}_1A_1, \qquad \alpha_2 = A_1\widehat{X}_1X_2, \qquad \alpha_3 = X_2\widehat{X}_1A_0,$$
$$\beta_1 = A_2\widehat{X}_2A_0, \qquad \beta_2 = A_2\widehat{X}_2X_1, \qquad \beta_3 = X_1\widehat{X}_2A_0$$

depend at optimality on the weights only, and their cosines (*see* [5]) are

$$\left.\begin{array}{ll} \cos\alpha_1 = (\nu^2 - 1 - \mu^2)/2\mu, & \cos\alpha_2 = (\mu^2 - 1 - \nu^2)/2\nu, \\ \cos\alpha_3 = (1 - \mu^2 - \nu^2)/2\mu\nu, & \cos\beta_j = \cos\alpha_j (j = 1, 2, 3). \end{array}\right\} \quad (6)$$

Therefore $\alpha_j = \beta_j$ $(j = 1, 2, 3)$. But these equalities can hold only if the solution is symmetric, *i.e.* if $X_1 = (x, y)$, $X_2 = (-x, y)$.

If X_1, X_2 are non isolated, two clearly symmetric cases are possible:

(i) $X_1 = X_2$ belonging to the y-axis;

(ii) $X_1 = A_1$, $X_2 = A_2$. Both can be thought as a limit of a case of isolated points. ◆

The problem (5) allows of four different types of solution, corresponding to the regions R_1, R_2, R_3, R_4 of the space of the weights as shown in Figure 2 for the special case of $\vartheta = \pi/4$. The coordinates of P, S, Q are

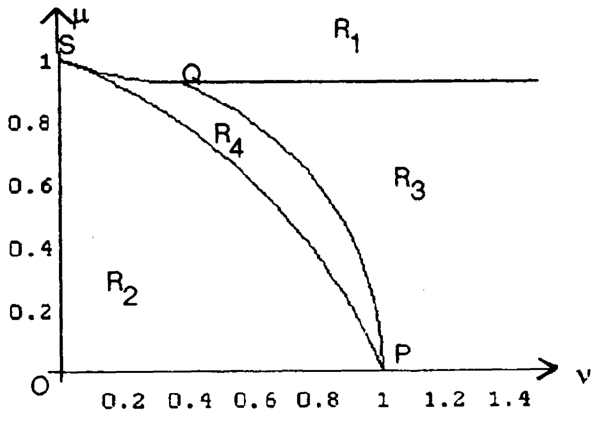

Figure 2.

$P(1, 0)$, $S(0, 1)$ and $Q(s, c)$, with $s = \sin(\vartheta/2)$, $c = \cos(\vartheta/2)$. The shape of the regions depends on ϑ, but it is independent of h. It can be observed that Q moves along the arc of circumference $\mu = \cos(\vartheta/2)$, $\nu = \sin(\vartheta/2)$ as ϑ varies in $(0, 2\pi)$.

The following statements define the regions and the corresponding type of solution.

1. R_1 is delimited by the ray $(S, +\infty)$ of the μ-axis, by the ray $(Q, +\infty)$ parallel to the ν-axis, and by the arc SQ of the hyperbole

$$\mu^2 - \nu^2 + 2\nu s = 1 \tag{7}$$

If $(\nu, \mu) \in R_1$, the coinciding cluster $X_1 = X_2 = A_0$ solves the problem (5).

2. R_2 is bounded by the segments OP, OS and by the arc PS of the ellipse

$$\mu^2 + \nu^2 + 2\mu\nu s = 1 \tag{8}$$

If $(\nu, \mu) \in R_2$ the solution is in the coinciding points $X_1 = A_1$, $X_2 = A_2$.

3. The region R_3 is bounded by the ray $(Q, +\infty)$ parallel to the ν axis and by the arc PQ of the curve

$$\frac{s^2}{s^2 + (c - y^*)^2} + \left[\frac{y^* - c}{\sqrt{s^2 + (c - y^*)^2}} + \mu\right]^2 = \nu^2 \tag{9}$$

with

$$y^* = c + s\frac{\mu}{\mu^2 - 1}\sqrt{1 - \mu^2}, \qquad (0 \le \mu \le c) \tag{10}$$

If $(\nu, \mu) \in R_3$ the solution is an "Isolated Cluster" $X_1 = X_2 = (0, y)$ located on the y-axis. The ordinate is $y = Ly^*$, with L side of the triangle.

4. The region R_4 is bounded by the arcs SQ, PS, PQ of (7), (8) and (9). If $(\nu, \mu) \in R_4$ the problem is solved by the isolated points $X_1(x, y)$, $X_2(-x, y)$, with

$$x = \frac{m_2 a - h}{m_2 - m_1}, \quad y = m_1 x \tag{11}$$

$$m_1 = \frac{\sqrt{4\nu^2\mu^2 - (1-\mu^2-\nu^2)^2}}{1-\mu^2-\nu^2}, \quad m_2 = -\frac{\sqrt{4\nu^2 - (\mu^2-1-\nu^2)^2}}{\mu^2-1-\nu^2} \tag{12}$$

These statements will now be proved.

1. As concerns the cluster $X_1 = X_2 = A_0$, tacking into account that

$$g_{1x} = \partial f_4/\partial x_1|_{A_0} = -s, \qquad g_{1y} = \partial f_4/\partial y_1|_{A_0} = -c$$

$$g_{2x} = \partial f_5/\partial x_2|_{A_0} = s, \qquad g_{2y} = \partial f_5/\partial y_2|_{A_0} = -c$$

the optimality conditions (4), (3) can be written as

$$\begin{cases} u_{12} + \overline{u}_{10} = s \\ -u_{12} + \overline{u}_{20} = -s \end{cases}, \qquad \begin{cases} v_{12} + \overline{v}_{10} = c \\ -v_{12} + \overline{v}_{20} = c \end{cases} \tag{13}$$

$$u_{12}^2 + v_{12}^2 \leq \nu^2, \qquad \overline{u}_{10}^2 + \overline{v}_{10}^2 \leq \mu^2, \qquad \overline{u}_{20}^2 + \overline{v}_{20}^2 \leq \mu^2 \tag{13'}$$

Let α, β be two real parameters. Then the general solutions of (13) are

$$U = \begin{bmatrix} u_{12} \\ \overline{u}_{10} \\ \overline{u}_{20} \end{bmatrix} = U_1 + U_2 = \begin{bmatrix} \alpha + s \\ -\alpha \\ \alpha \end{bmatrix},$$

$$V = \begin{bmatrix} v_{12} \\ \overline{v}_{10} \\ \overline{v}_{20} \end{bmatrix} = V_1 + V_2 = \begin{bmatrix} \beta \\ -\beta + c \\ \beta + c \end{bmatrix}$$

where $U_1 = [\alpha,\ -\alpha,\ \alpha]$, $V_1 = [\beta,\ -\beta,\ \beta]$ are the general solutions of the homogeneous parts, and $U_2 = [s,\ 0,\ 0]$, $V_2 = [0,\ c,\ c]$ are particular solutions of the complete systems. Hence the inequalities (13′) can be written as

$$\begin{cases} (\alpha + s)^2 + \beta^2 \leq \nu^2 \\ \alpha^2 + (-\beta + c)^2 \leq \mu^2 \\ \alpha^2 + (\beta + c)^2 \leq \mu^2 \end{cases}$$

which in an auxiliary $(\alpha,\ \beta)$ plane define three circles C_1, C_2, C_3 of centres $C_1(-s,\ 0)$, $C_2(0,\ c)$, $C_3(0,\ -c)$, and radii ν, μ, μ (*see* Fig. 3).

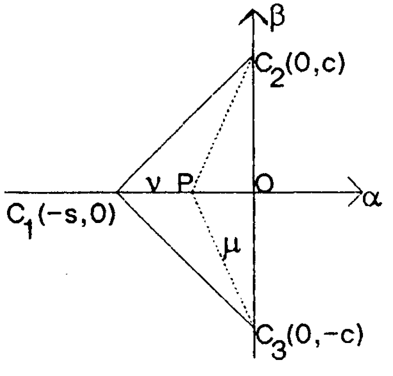

Figure 3.

It follows by simple inspection of the isosceles triangle $C_1C_2C_3$, that $\mu \geq c$ is a necessary condition for C_1, C_2, C_3 to have a common intersection. Let us assume that $\mu \geq c$ is satisfied, and $\nu = C_1P$. Then the circles intersect only if $\mu^2 \geq \overline{OC}_3^2 + \overline{OP}^2$, *i.e.* $\mu^2 \geq c^2 + (s - \nu)^2$ or $\mu^2 \geq 1 + \nu^2 - 2\nu s$, which is the hyperbole (7). If both these conditions are satisfied then $(\mu,\ \nu) \in R_1$ and (13), (13′) are also satisfied, so that the cluster $X_1 = X_2 = A_0$ solves the problem.

2. The conditions (4), (3) of optimality for $X_1 = A_1$, $X_2 = A_2$ can be written as follows:

$$\overline{u}_{11} = -g_{1x}, \qquad \overline{v}_{11} = -g_{1y} \Rightarrow g_{1x}^2 + g_{1y}^2 \leq 1 \tag{14}$$

$$\overline{u}_{22} = -g_{2x}, \qquad \overline{v}_{22} = -g_{2y} \Rightarrow g_{2x}^2 + g_{2y}^2 \leq 1 \tag{15}$$

Let us first consider (14) (condition for $X_1 = A_1$). After computing the derivatives

$$g_{1x} = \mu \frac{\partial f_1}{\partial x_1}\bigg|_{A_1} + \nu \frac{\partial f_3}{\partial x_1}\bigg|_{A_1, A_2}, \qquad g_{1y} = \mu \frac{\partial f_1}{\partial y_1}\bigg|_{A_1} + \nu \frac{\partial f_3}{\partial y_1}\bigg|_{A_1, A_2}$$

in $A_1 = (a, h)$ and $A_2 = (-a, h)$ and by using some well known trigonometric identities, the inequality in (14) becomes $\mu^2 + \nu^2 + 2\mu\nu s \leq 1$, which is the ellipsis delimiting the region R_2. The same result can be obtained by considering (15) instead of (14).

3. If the solution is not in the vertices, two cases can occur: an isolated cluster located on the y-axis or two isolated points $X_1 \neq X_2$. Let us consider the first case. The optimality conditions (4), (3) are:

$$\begin{cases} u_{12} = -g_{1x} \\ -u_{12} = -g_{2x} \end{cases}, \qquad \begin{cases} v_{12} = -g_{1y} \\ -v_{12} = -g_{2y} \end{cases} \tag{16}$$

$$u_{12}^2 + v_{12}^2 \leq \nu^2 \quad \Rightarrow \quad g_{1x}^2 + g_{1y}^2 \leq \nu^2$$

and the consistency of the systems requires

$$g_{1x} + g_{2x} = 0, \qquad g_{1y} + g_{2y} = 0 \tag{17}$$

Since a solution on the y-axis is required, we can set $X_1 = (0, y)$, $X_2 = (0, y)$ in the derivatives, thus obtaining (after introducing the notation $X^* = (X_1, X_2)$):

$$g_{1x} = \frac{\partial f_4}{\partial x_1}\Big|_{X^*} + \mu \frac{\partial f_1}{\partial x_1}\Big|_{X^*} = \frac{-a}{\sqrt{a^2 + (h-y)^2}},$$

$$g_{1y} = \frac{\partial f_4}{\partial y_1}\Big|_{X^*} + \mu \frac{\partial f_1}{\partial y_1}\Big|_{X^*} = \frac{y-h}{\sqrt{a^2 + (h-y)^2}} + \mu,$$

$$g_{2x} = \frac{\partial f_5}{\partial x_2}\Big|_{X^*} + \mu \frac{\partial f_2}{\partial x_2}\Big|_{X^*} = \frac{a}{\sqrt{a^2 + (h-y)^2}},$$

$$g_{2y} = \frac{\partial f_5}{\partial y_2}\Big|_{X^*} + \mu \frac{\partial f_2}{\partial y_2}\Big|_{X^*} = \frac{y-h}{\sqrt{a^2 + (h-y)^2}} + \mu.$$

Therefore, the first of (17) is always satisfied, and the second reduces to $g_{1y} = 0$ (or to $g_{2y} = 0$). Moreover the inequality in (16) becomes

$$\frac{a^2}{a^2 + (h-y)^2} + \left[\frac{y-h}{\sqrt{a^2 + (h-y)^2}} + \mu\right]^2 \leq \nu^2 \tag{18}$$

Since $h = Lc$ and $a = Ls$, the solution $y \in (0, h)$ of the equation $g_{1y} = 0$ can be written as

$$y = L\left(c + s\frac{\mu}{\mu^2 - 1}\sqrt{1 - \mu^2}\right) = Ly^* \tag{19}$$

By setting $y = Ly^*$ and taking the equality sign (18) can be rewritten in the form (9), (10) which is the left bound of the region R_3. The upper and lower bounds are given by the inequalities $0 \le \mu \le c$, which follow from (19) since $\lim_{\mu \to c} y = 0$, $\lim_{\mu \to 0} y = Lc = h$. Hence the isolated cluster $X_1 = X_2$ is optimal if $(\mu, \nu) \in R_3$.

4. The remaning case of isolated points $X_1 \neq X_2$ corresponds to the remaining region R_4. The systems (4) reduce to the gradient system $G_x = 0$, $G_y = 0$, which can be numerically solved. However a more simple way to obtain the solution is to intersect the two straight lines passing through A_0X_1, and A_1X_1:

$$y = m_1 x, \qquad y - h = m_2(x - a)$$

whose coefficients m_1 and m_2 can be easily computed, since the three angles in X_1 are known at optimality (their cosines are given by (6)). The solution is expressed by (11), (12).

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