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OPTIMAL REPLACEMENT POLICY FOR A DETERIORATING PRODUCTION SYSTEM

by M. BRAGLIA

Communicated by Naoto KAÏO

Abstract. – *An optimal replacement policy for a deteriorating production system is considered. A minimal repair model is defined which assumes the system to be replaced after a specified time, during which a number of failures may occur. The probability of failure can be given any arbitrary (increasing) form. The model is based on a technique which permits to deal with large repair times (non-negligible with respect to the replacement time and the interval time) and to calculate the probability of k breakdowns over the replacement interval rigorously. The average downtime relevant to k minimal repairs is obtained, on the contrary, from a semiempirical formula that appears in good agreement with the results of Monte Carlo simulations. The theory permits an accurate analysis of the error that is introduced when more conventional and approximate models are used.*

Keywords: Maintenance, breakdowns, replacement, integral equations.

Résumé. – *On considère une politique optimale de substitution pour un système de production qui va se détériorer par l'usage. Un modèle de renouvellement périodique avec réparation minimale en cas de panne est défini où le système est replacé depuis un intervalle de temps bien spécifié pendant lequel un certain nombre de pannes peuvent être arrivées. La probabilité de panne peut avoir n'importe quelle forme arbitraire (croissante). Le modèle est fondé sur une technique intégrale qui permet de considérer de grands temps de réparation (c'est-à-dire qui ne sont pas trascurables par rapport au temps de remplacement et à l'intervalle de temps) et de calculer rigoureusement la probabilité de k pannes sur l'intervalle de renouvellement. Le temps moyen dépensé pour k réparations est fondé au contraire sur une formule semi-empirique qui présente un bon accord avec les résultats de simulations Monte Carlo. La théorie permet une analyse détaillée de l'erreur qu'on introduit quand des modèles plus conventionnels et approximés sont utilisés.*

Mots clés : Entretien, pannes, remplacement, équations intégrales.

1. INTRODUCTION

Production systems undergo deterioration with usage and age. In case of breakdown, production is temporarely interrupted until machines are repaired or replaced. This implies higher costs and lower productivity and

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quality. Therefore, it becomes important to define an optimal scheduling of preventive maintenance in order to reduce the time-intervals in which the system is out-of-service. For this reason, various papers on the development of such policies have appeared in the literature. The differences between contributions of different authors may concern the mathematical approach (queueing theory, renewal theory, dynamic programming,...), the degree of knowledge of the system state (preventive maintenance model, preparedness model, inspection model) or different aspects of the problem (minimal repair model, shock model, miscellaneous replacement model). References can be found in three major surveys which cover the last forty years of scientific production: McCall [1] (up to 1964), Pierskalla and Voelker [2] (up to 1975), Valdez-Flores and Feldman [3] (up to 1989).

In this paper we consider a *minimal repair model* [4] relevant to a single-unit system. In our model: (1) the system failure rate (*i.e.*, the probability of failure *per unit time*) $\gamma(t)$ is an arbitrarily increasing function of t, (2) minimal repairs (or unscheduled maintenances) do not affect $\gamma(t)$ while a replacement (or preventive maintenance) makes the probability to vanish, (3) system failures are immediately detected and, finally, (4) the wasted time τ per single minimal repair is less than the time θ required for replacing the entire system (*i.e.*, per single ordinary maintenance). The model permits to calculate the (optimal) interval between successive preventive maintenances which minimizes the waste of time. In fact, our treatment is relevant to *times* rather than *costs* of the two different types of maintenances. Generally, it has been preferred to associate a cost c_f to each minimal repair and a cost $c_r < c_f$ to the replacement and also to neglect the waste of time caused by minimal repairs, as the assumption drastically simplifies the calculations of the relevant (renewal) theory [5]. Our model, on the contrary, permits a *rigorous* calculation of the probabiity of k breakdowns over the replacement interval in situations in which the latter approximation becomes invalid. In fact, the time required by each minimal (unscheduled) repair may be large (even comparable with the same replacement interval). As mentioned, our only requirement is that $\tau < \theta$. The calculation of the average downtime relevant to k minimal repairs is based on a convenient *semiempirical* formula which is found in very good agreement with corresponding Monte Carlo simulations. Thus, the theory permits an accurate analysis of the error that is introduced when more conventional and approximate models are used. As far as possible, we try to find an analytical solution to the problem of calculating the *optimal* interval of *real (or total)* time between successive ordinary maintenances, even in the general case in which $\gamma(t)$ is arbitrary. Numerical applications are considered in the special (linear) case $\gamma(t) = \alpha + 2\beta t$ for various values of the parameters α and β.

2. OPTIMAL TIME OF EFFECTIVE SERVICE BETWEEN ORDINARY MAINTENANCES

Consider a production system which, at any instant of time, has a certain probability to undergo breakdowns. Let

τ = (average) downtime per minimal repair (or extraordinary maintenance);

θ = (average) downtime per ordinary maintenance;

T_s = time of *effective service* between successive ordinary maintenances;

T_w = waste of time caused by (ordinary and extraordinary) maintenances in the interval between completions of consecutive ordinary maintenances;

$T_r = T_s + T_w - \theta$ = time interval between completion of an ordinary maintenance and beginning of the successive one.

To take account of production-system deterioration, we will abandon the common assumption of constant failure rate. We will assume this probability to increase with time according to the law $\gamma = \gamma(t)$. When $\gamma(t)$ is known, the problem becomes that of calculating the probability $p(k, T)$ of k breakdowns in $(0, T)$. Let T be the effective-service time. We are then able to determine immediately the optimal interval T_0 *which minimizes the ratio between time T_w in which the system is out-of-service and total time $T_w + T_s$ between successive ordinary maintenances.* In fact,

$$R(T) = \frac{T_w(T)}{T_w(T) + T} \qquad (T \equiv T_s) \tag{1}$$

with

$$T_w = \theta + \tau \sum_{k=1}^{\infty} kp(k, T) = \vartheta + \tau \langle k \rangle_T \tag{2}$$

Note that, as $\gamma(t)$ is an increasing function of the service time, substantial difference exists between the two kinds of maintenances. While the ordinary maintenance implies a complete revision of the production system with a consequent reduction of $\gamma(t)$ to the (initial) value $\gamma(0)$, a minimal repair does not change the value of $\gamma(t)$.

2.1. Probability $p(k, T)$ and analytical solution of the problem

As the waste of time due to minimal repairs is not included in T, the breakdowns may be treated as *instantaneous events*. As a consequence $p(k, T)$ is a *generalized Poisson* distribution with mean and variance

$$\Gamma(T) \equiv \int_0^T \gamma(t)\, dt \tag{3}$$

Of course, we do not intend to dwell on this well known result. However, in view of the theoretical developments of the successive sections, it may be useful to mention how the calculation of $p(k, T)$, usually based on the solution of differential equations, can also be based on less familiar solutions of *integral* equations.

A derivation of $p(k, T)$ which in our context assumes special interest may be that of starting from the following evident integral form

$$p(k, T) = \underbrace{\int_0^T d\xi \int_\xi^T d\eta \ldots \int_\chi^T}_{(k \text{ times})} d\zeta\, Q_0(0, \xi)\, \gamma(\xi)\, Q_0(\xi, \eta)\, \gamma(\eta) \ldots Q_0(\chi, \zeta)\, \gamma(\zeta)\, Q_0(\zeta, T) \tag{4}$$

where

$$Q_0(t', t) = e^{-\{\Gamma(t) - \Gamma(t')\}} \xrightarrow[t' \to 0]{} Q_0(0, t) = p(0, t) = e^{-\Gamma(t)} \tag{5}$$

is the probability of no breakdown in (t', t), given that the system has not failed until t'. In fact,

$$p(k, T) = e^{-\Gamma(T)} \int_0^T \gamma(\xi)\, d\xi \int_\xi^T \gamma(\eta)\, d\eta \ldots \int_\chi^T \gamma(\zeta)\, d\zeta = \frac{\Gamma^k(T)}{k!}\, e^{-\Gamma(T)} \tag{6}$$

To return to our problem, as the average number of breakdowns in $(0, T)$ is $\langle k \rangle_T = \Gamma(T)$, eq. (1) can be given the form

$$R = \frac{\theta + \tau\Gamma(T)}{\theta + \tau\Gamma(T) + T} \tag{7}$$

Therefore, the optimal interval T_0 between successive ordinary maintenances is easily seen to satisfy the equation

$$T_0 = \frac{\theta + \tau \Gamma (T_0)}{\tau \gamma (T_0)} \tag{8}$$

For $\gamma (t) = \alpha + 2\beta t$, it follows that $T_0 = \sqrt{\theta/(\beta\tau)}$ in correspondence of which ratio (7) assumes the (minimum) value

$$R = R_m = \frac{\theta + \tau (\alpha T_0 + \beta T_0^2)}{\theta + \tau (\alpha T_0 + \beta T_0^2) + T_0} \tag{9}$$

2.2. Generalization

We can also assume that the minimal-repair time, say ϑ, is distributed according to a given law $q(\vartheta)$. For instance, if $q(\vartheta) = \mu e^{-\mu\vartheta}$, then $\langle\vartheta\rangle = \mu^{-1} = \tau$, and the time spent for k minimal repairs will be distributed according to the k-th convolution of $q(\vartheta)$, that is the k-Erlang(τ) distribution. In any case the probability of k breakdowns in (0, T) is independent of the successive values of ϑ as the clock is stopped when a breakdown occurs. Therefore, $p(k, T)$ maintains form (6) and the mean waste of time caused by minimal repairs in (0, T) is still given by $\langle k\rangle_T \tau = \Gamma(T)\tau$.

2.3. A numerical case

Most results obtained so far are well known (e.g., [6] and [7]). They have been mentioned mainly for comparison with corresponding ones of successive sections. With the same aim, we now consider a numerical application. To this end, in figure 1 we represent the behavior of the ratio $R(T)$ given by (7) in the particular case $\alpha = 0.3$, $\theta = 2$ and $\tau = 0.2$ for $\beta = 0.0$, 0.1, 0.2 and 0.3.

The curves of figure 1 report $R(T)$ as a function of time of *effective* service $T \equiv T_s$. From this point of view, it is not even necessary to specify the distribution of the repair times. However, if this law is known, in principle it is possible to obtain $R(T)$ as a function of the total time T_r *really* elapsed

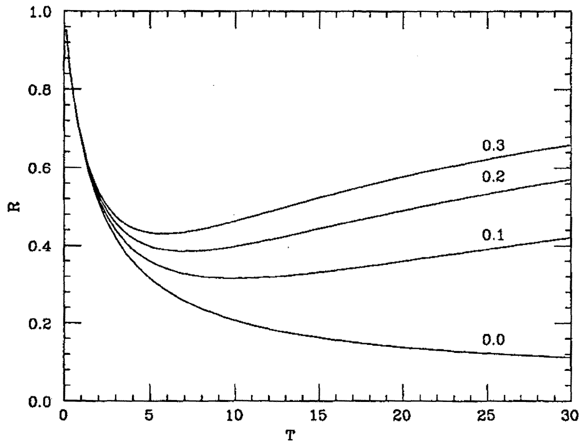


Figure 1. – Behavior of ratio (7) as a function of the time of effective service $T = T_s$ for $\alpha = 0.3$, $\theta = 2$ and $\tau = 0.2$. The curves are characterized by the corresponding values of β.

from the beginning of service. In figure 2 we report the same curves of figure 1 in this alternative time scale under the assumption that the minimal repair time τ is constant. The time-scale expansion is performed according to the equation $T = T_s + \langle k \rangle_T \tau = T_s + (\alpha T_s + \beta T_s^2) \tau$. As one can see the curves are now flatter than the corresponding ones of figure 1. For sufficiently elevated values of T the ratio $R(T)$ becomes almost independent of T. We will return to this question, for comparisons, in section 3.3.

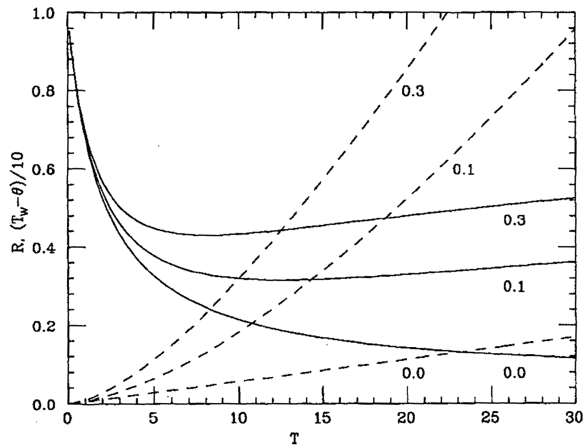


Figure 2. – Behavior of ratio (7) as a function of the real (*i.e.* total) time $T = T_r$ for $\alpha = 0.3$, $\theta = 2$ and $\tau = 0.2$ (continuous curves). The curves are marked with the corresponding values of β. The dashed curves report the corresponding waste of time caused by breakdowns.

3. OPTIMAL INTERVAL OF REAL TIME BETWEEN SUCCESSIVE ORDINARY MAINTENANCES

Now we consider the case in which the optimal interval between successive ordinary maintenances is calculated on the *real* (*i.e.*, *total*) time T_r. This case is of special value when considering that, in the majority of practical situations, the scheduling of maintenances must be planned on a temporal horizon (weekly, monthly...). In fact, one must define exactly *when* ordinary maintenances have to be made. But the approach of section 2.1 permits only approximate evaluations, that is for $\tau \ll T_r$, and the intervals between ordinary maintenances cannot be defined exactly.

As done in section 2.1, here also we are interested to determine the time T which minimizes the ratio $R(T)$ defined by (1). But now $T = T_w + T_s$ is the total time elapsed since the completion of the last ordinary maintenance.

Therefore, if $\tau_k(T) \equiv (k-1)\tau + \tau_k^*(T)$ with $\tau_0(T) \equiv 0$ is the waste of time caused by minimal repairs in $(0, T)$, eq. (1) assumes the new form

$$R(T) = \frac{T_w(T)}{T} \qquad (T \equiv T_s + T_w) \tag{10}$$

where

$$\begin{aligned} T_w = \theta + \sum_{k=1}^{\infty} \tau_k(T)\, p(k, T) = \theta + \{\langle k\rangle_T - (1 - p(0, T))\}\,\tau \\ + \sum_{k=1}^{\infty} \tau_k^*(T)\, p(k, T) \end{aligned} \tag{11}$$

Note that we permit the k-th breakdown to terminate in ordinary maintenance. In fact, $\tau_k^*(T)$ represents the waste of time caused by the last breakdown. Of course, the problem requires again to give explicit form to the probability $p(k, T)$ of k breakdowns in $(0, T)$. But with the new meaning of T the problem becomes a bit more complicated than that considered in section 2.1, even if still amenable to analytical solution.

3.1. Probability $p(k, T)$ of k breakdowns in the interval of time $T = T_r$

We pass now to the calculation of the probability of k interruptions, each one lasting τ, in the interval of time T. In this case we must take into account that: 1) if a breakdown occurs at t a successive breakdown can only occur in $(t+\tau,\, T)$, 2) there is no deterioration in $(t,\, t+\tau)$ and 3) the last breakdown may terminate in ordinary maintenance.

As regards the probability $p(0,\, T)$ of absence of interruptions in $(0,\, T)$, again it is given by eq. (5) of the preceding case. On the contrary, for $k = 1, 2, \ldots$, it becomes necessary to distinguish different situations which can occur dependently on the values of T and τ. First it must be considered that k interruptions cannot occur in a time-interval lower than $(k-1)\,\tau$. Thus, one must distinguish the case $T > k\,\tau$ (in which k interruptions may be entirely contained within T) from the case $(k-1)\,\tau \le T \le k\,\tau$ (in which this cannot occur). The two situations are different from the analytical point of view. In fact, for $(k-1)\,\tau \le T \le k\,\tau$, we will write that

$$\begin{aligned} p(k,\, T) = & \int_0^{T-k\,\tau} d\xi \int_{\xi+\tau}^{T-(k-1)\,\tau} d\eta \ldots \int_{\ldots}^{T-\tau} d\chi \\ & \times \int_{\chi+\tau}^{T} d\zeta\, Q_0(0,\, \xi)\,\gamma(\xi)\, Q_0(\xi,\, \eta-\tau)\,\gamma(\eta-\tau) \\ & \ldots Q_0(\chi-(k-1)\,\tau,\, \zeta-k\,\tau)\,\gamma(\zeta-k\,\tau) \end{aligned}$$

$$\begin{aligned} = & \int_0^{T-k\,\tau} \gamma(\xi)\, d\xi \int_{\xi+\tau}^{t-(k-1)\,\tau} \gamma(\eta-\tau)\, d\eta \\ & \ldots \int_{\chi+\tau}^{T} Q_0(0,\, \zeta-k\,\tau)\,\gamma(\zeta-k\,\tau)\, d\zeta \end{aligned} \qquad (12)_1$$

whereas for $T > k\tau$ the equation for $p(k,\, T)$ becomes

$$\begin{aligned} p(k,\, T) = & \int_0^{T-k\,\tau} \gamma(\xi)\, d\xi \int_{\xi+\tau}^{T-(k-1)\,\tau} \gamma(\eta-\tau)\, d\eta \\ & \ldots \int_{\ldots}^{T-2\,\tau} \gamma(\chi-(k-2)\,\tau)\, d\chi \\ & \times \left\{ \int_{\chi+\tau}^{T-\tau} Q_0(0,\, T-k\,\tau)\,\gamma(\zeta-(k-1)\,\tau)\, d\zeta \right. \\ & \left. + \int_{T-\tau}^{T} Q_0(0,\, \zeta-(k-1)\,\tau)\,\gamma(\zeta-(k-1)\,\tau)\, d\zeta \right\} \end{aligned}$$

$$+\int_{T-k\tau}^{T-(k-1)\tau}\gamma(\xi)\,d\xi\int_{\xi+\tau}^{T-(k-2)\tau}\gamma(\eta-\tau)\,d\eta$$
$$\ldots\int_{\ldots}^{T-2\tau}\gamma(\chi-(k-2)\tau)\,d\chi$$
$$\times\int_{\chi+\tau}^{T}Q_0(0,\zeta-(k-1)\tau)\,\gamma(\zeta-(k-1)\tau)\,d\zeta \qquad (12)_2$$

Despite the apparent complexity of these equations, it is easy to give $p(k, T)$ explicit forms. The integrations are performed without difficulty. The calculations are reported in Appendix A. The result is the following:

1) <u>For $T \leq (k-1)\tau$</u>, $\forall k = 1, 2, \ldots, T > 0, \tau > 0$:

$$p(k, T) = 0 \qquad (13)_1$$

2) <u>For $(k-1)\tau \leq T \leq k\tau$</u>, $\forall\ k = 1, 2, \ldots, \tau > 0$:

$$p(k, T) = 1 - \sum_{0}^{k-1}{}_j \frac{\Gamma^j(T-(k-1)\tau)}{j!}\,e^{-\Gamma(T-(k-1)\tau)} \qquad (13)_2$$

3) <u>For $T > k\tau$</u>, $\forall\ k = 1, 2, \ldots, \tau > 0$:

$$p(k, T) = \sum_{0}^{k}{}_j \frac{\Gamma^j(T-k\tau)}{j!}\,e^{-\Gamma(T-k\tau)} - \sum_{0}^{k-1}{}_j \frac{\Gamma^j(T-(k-1)\tau)}{j!}\,e^{-\Gamma(T-(k-1)\tau)} \qquad (13)_3$$

Note that *in the limit* $\tau \to 0$ *we reobtain distribution* (6).

One of the referees has suggested a rearrangement of these equations and an alternative proof that deserves to be reported for its simplicity. The rearrangement is the following. Let $k\tau \leq T \leq (k+1)\tau$. Then

$$p(0, T) = e^{-\Gamma(T)} \qquad (14)_1$$

$$p(j,T)=\sum_{0}^{j}{}_{i}\ \frac{\Gamma^{i}(T-j\,\tau)}{i!}\,e^{-\Gamma(T-j\,\tau)}$$
$$-\sum_{0}^{j-1}{}_{i}\ \frac{\Gamma^{i}(T-(j-1)\,\tau)}{i!}\,e^{-\Gamma(T-(j-1)\,\tau)}$$
$$\text{for}\quad j=1,2,\ldots,k \qquad (14)_2$$

$$p(k+1)=1-\sum_{0}^{k}{}_{i}\ \frac{\Gamma^{i}(T-k\,\tau)}{i!}\,e^{-\Gamma(T-k\,\tau)} \qquad (14)_3$$

$$p(j,T)=0 \qquad \text{for}\quad j\geq k+2 \qquad (14)_4$$

To prove these equations, let X_i be the failure time from the completion of the $(i-1)$-th minimal repair, where $X_0 \equiv 0$. Let $Y=\sum_{i=1}^{n} X_i$. Then

$$\Pr\{Y_n>t\}=\sum_{0}^{n-1}{}_{i}\ \frac{\Gamma^{i}(t)}{i!}\,e^{-\Gamma(t)} \qquad (15)$$

Equations $(14)_1$ and $(14)_4$ are evident. Concerning $(14)_2$ we have

$$\begin{aligned}p(j,T)&=\Pr\{(X_1+\tau)+\ldots+(X_j+\tau)\\&\qquad<T\leq(X_1+\tau)+\ldots+(X_j+\tau)+X_{j+1}\}\\&\quad+\Pr\{(X_1+\tau)+\ldots+(X_{j-1}+\tau)+X_j\\&\qquad<T\leq(X_1+\tau)+\ldots+(X_{j-1}+\tau)+(X_j+\tau)\}\\&=\Pr\{Y_j<T-j\,\tau\leq Y_{j+1}\}\\&\quad+\Pr\{T-j\,\tau<Y_j\leq T-(j-1)\,\tau\}\end{aligned} \qquad (16)$$

which, in virtue of (15) agrees with $(14)_2$. Similarly, eq. $(14)_3$ is obtained when noting that

$$\begin{aligned}p(k+1,T)&=\Pr\{Y_{k+1}\leq T-(k+1)\tau<Y_{k+2}\}\\&\quad+\Pr\{T-(k+1)\,\tau<Y_{k+1}\leq T-k\,\tau\}\\&=\Pr\{Y_{k+1}\leq T-k\,\tau\}\end{aligned} \qquad (17)$$

To give a numerical example, in figure 3 we represent the $p(k, T)$'s for $k = 0, \ldots, 4$ in the special case $\alpha = 0$ and $\beta = 0.3$ for $\tau = 2$.

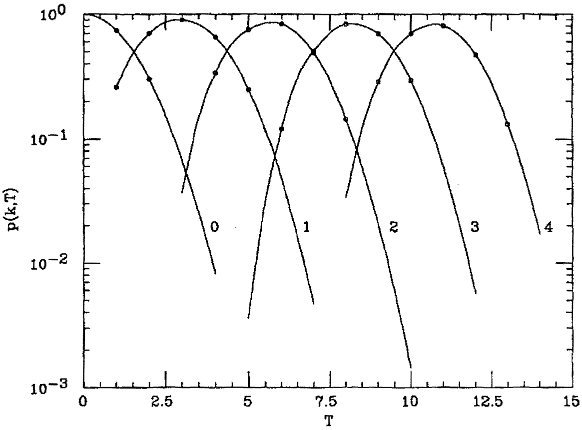


Figure 3. – Distributions $p(k, T)$ for $\alpha = 0.0, \beta = 0.3$ and $\tau = 2$. The curves are characterized by the value of k. The points are results of corresponding Monte Carlo simulations.

For comparison, the corresponding distributions obtained by Monte Carlo simulations are also reported. As one can see, analytical and Monte Carlo results are in perfect agreement.

3.2. Mean downtime relevant to k breakdowns

At this point we must calculate the average waste of time caused by k (minimal) repairs. The problem is particularly difficult when considering that the last interruption may terminate beyond T (*i.e.* between T and $T + \theta$). This happens whenever the breakdown occurs in $(T - \tau, T)$. In that case, only a fraction of τ must be included in the calculation of the mean time. The remaining part is treated as ordinary maintenance. (We have assumed $\tau \leq \theta$ to avoid to terminate beyond $T + \theta$ in the successive service interval.)

As shown in Appendix B, the analytical calculation of the mean waste of time caused by breakdowns becomes heavy even for $k = 1$. The calculation becomes impracticable for $k > 1$. However, the behavior of the $\tau_k(T)$'s as a function of T can be represented with good approximation by simple expressions. Some approximate analytical representations of the rigorous equations are derived in Appendix B. The results can be briefly recapitulated as follows.

For $\gamma(t) = \alpha$ (constant), if we write that $\tau_k(T) \equiv (k-1)\tau + \tau_k^*(T)$, then

$$\tau_k^*(T) = \begin{cases} \dfrac{T-(k-1)\tau}{(k+1)} & \text{for } (k-1)\tau \leq T < \vartheta_k \\ \tau \exp\left(-\dfrac{k\tau}{2(T-(k-1)\tau)}\right) & \text{for } T \geq \vartheta_k \end{cases} \tag{18$_1$}$$

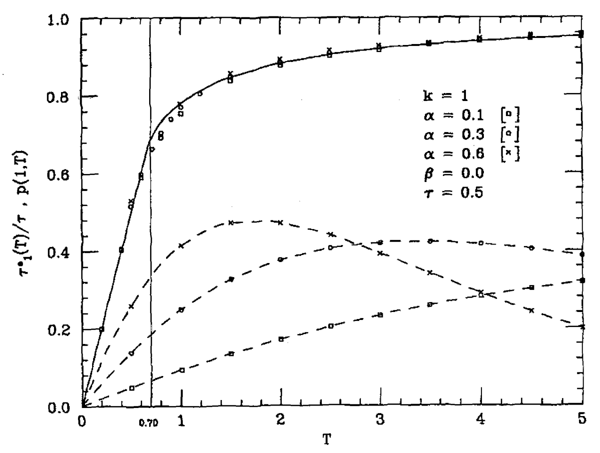
ϑ_k being the (iterative) solution of the equation

$$\frac{T-(k-1)\tau}{(k+1)\tau} = \exp\left(-\frac{k\tau}{2(T-(k-1)\tau)}\right)$$

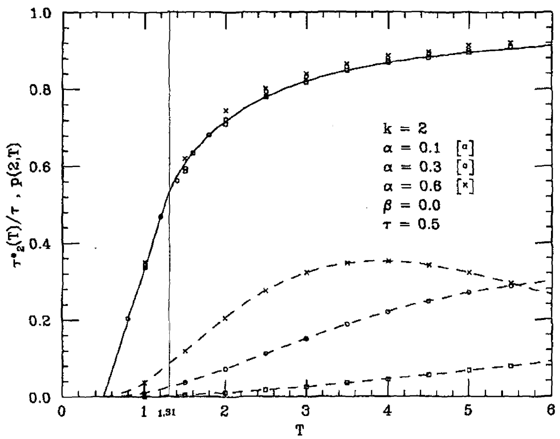
$$\text{that is } \begin{cases} X \equiv \dfrac{k\tau}{2(T-(k-1)\tau)} \\ X_{r+1} = \dfrac{k}{2(k+1)} \exp(X_r) \end{cases} \tag{18$_2$}$$

Note that the ϑ_k's appear to be independent of α. But $(18)_2$ has only solution for $k \leq 2$. For $k > 2$ various expedients can be used to joint the curve relevant to high values of T to the linear behavior obtained for $T \to (k-1)\tau$. However, a good choice is that of assuming the exponential form found for $T > \vartheta_k$ to be valid for $T \geq (k-1)\tau$.

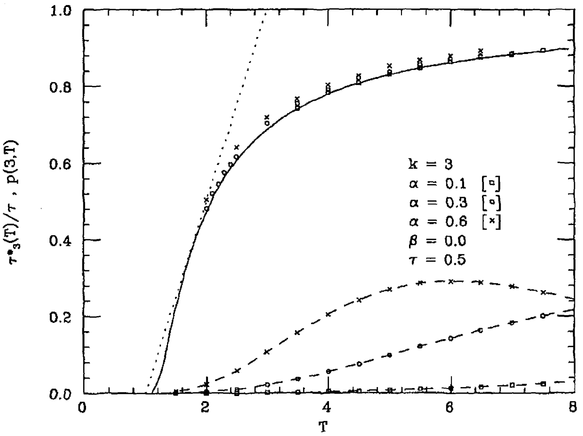
Monte Carlo simulations with different values of τ have confirmed that (18) represent very well the behavior of $\tau_k(T)$ for any $T \geq (k-1)\tau$. Examples of comparisons for $k = 1$, 2 and 3 are reported in figures 4a-c for three different values of α and $\tau = 0.5$ for which $\vartheta_1 = 0.70$ and $\vartheta_2 = 1.31$. (For



(a)



(b)



(c)

Figure 4. – (a) Behavior of $\tau_k^*(T)/\tau$ (full curve) and $p(k, T)$ (dashed curves) for $k = 1$, $\beta = 0$ and three different values of α. The points are results of Monte Carlo simulations; (b) same quantities of figure 4a but for $k = 2$; (c) same quantities of figure 4a and 4b but for $k = 3$. An identical result has been obtained also for $k = 4$.

convenience, in the same figures the corresponding distributions $p(k, T)$ are also reported.) As one can see, the agreement between Monte Carlo and theory is remarkable. Of course, as the approach is approximate, the analytical representation will require that α and τ are sufficiently small. However, even for the extreme condition $\alpha = 1$ and $\tau = 1$ the analytical representation remains good. This is shown in figure 5.

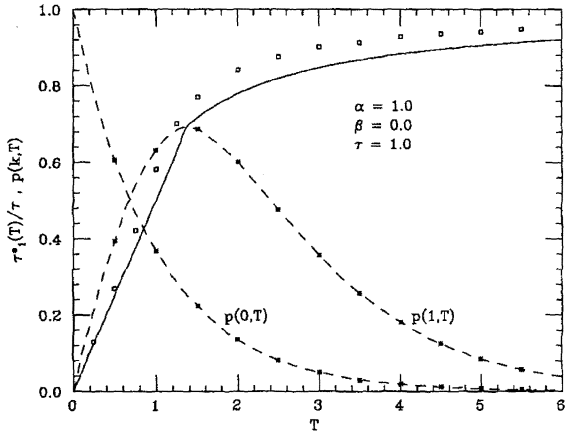


Figure 5. – **Behavior of $\tau_1^*(T)/\tau$ for $\beta = 0$, $\tau = 1$ and $\alpha = 1$. The points are results of Monte Carlo simulations. The dashed curves represent the corresponding distributions $p(0, T)$ and $p(1, T)$.**

For $\gamma(t) = 2\beta t$, on the contrary, it is found that a good representation of $\tau_k^*(T)$ is given by the (semi-empirical) formula

$$\tau_k^*(T) = C(\beta) \begin{cases} \dfrac{T - (k-1)\tau}{(2k+1)} & \text{for} \quad (k-1)\tau \le T < \vartheta_k \\ \dfrac{\tau \exp(2/3)}{2} \exp\left\{ -\dfrac{(2k+1)\tau}{3(T-(k-1)\tau)} \right. \\ \qquad \left. \times \exp\left(-\dfrac{2T - (4k-1)\tau}{10} \right) \right\} \\ \text{for} \quad T \ge \vartheta_k \end{cases} \tag{19$_1$}$$

where

$$\vartheta_k = (2k - 1/2)\tau$$

and (19)$_2$

$$C(\beta) = [1 + (2k/9)(\beta - 0.3)\exp(-T/15)]$$

Various simulations have been made to assess the accuracy of this approximation. Some typical results are given in figure 6 for $k = 1$ and various values of β. Identical results have been obtained also for $k = 2$, 3 and 4 [9].

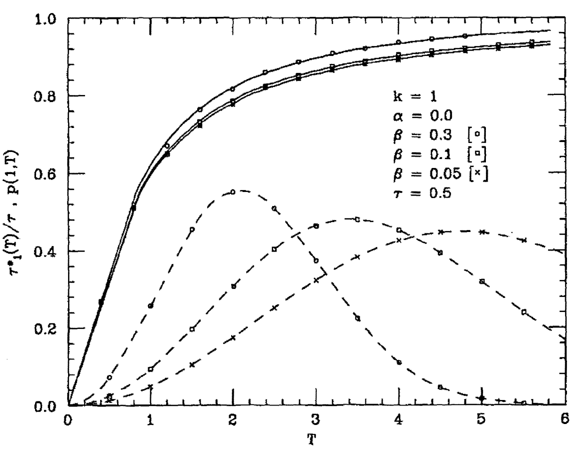


Figure 6. – Behavior of $\tau_k^*(T)/\tau$ (full curves) and $p(k, T)$ (dashed curves) for $k = 1$, $\alpha = 0$ and three different values of β. The points are results of Monte Carlo simulations. Identical results have been obtained also for $k = 2$, 3 and 4.

For $\gamma(t) = \alpha + 2\beta t$ the value of $\tau_k(T)$ can be obtained by weighting the two preceding values obtained for $\alpha = 0$ and $\beta = 0$. Weights are required to satisfy certain asymptotic behaviors, e.g. the behavior of $\tau_1(T)$ for $T \to 0$.

In this way it is found that [*cf.* Appendix B, *i.e.* (B.1) and (B.2)]

$$\tau_k^*(T) = \frac{\alpha\,(\tau_k^*(T))_{\beta=0} + \beta\,T\,(\tau_k^*(T))_{\alpha=0}}{\alpha + \beta\,T} \tag{20}$$

Figure 7 reports a comparison between theory and Monte Carlo simulations relevant to this general situation in which both $\alpha \neq 0$ and $\beta \neq 0$. Identical results have been obtained also for $k = 2$, 3 and 4 [9].

The results of figures 4-7 are interesting as they reveal the good agreement between analytical representations of $\tau_k(T)$ and Monte Carlo results. However, there are various considerations that can be drawn from the data reported in these figures. First of all it is evident that $\tau_k(T)$ differs from $k\,\tau$ mainly in the neighborhood of $T = k\,\tau$. This was an expected result as $\tau_k(T) = 0$ for $T \leq (k-1)\,\tau$. On the other hand, *the more the interval* $(0, T)$ *is restricted the more the event corresponding to the occurrence of k breakdowns becomes improbable.* (It becomes impossible for $T \leq (k-1)\,\tau$). Then, *the event becomes scarcely probable for values of T for which the difference between* $\tau_k(T)$ *and* $k\,\tau$ *becomes more pronounced,* that is for $T \to (k-1)\,\tau$.

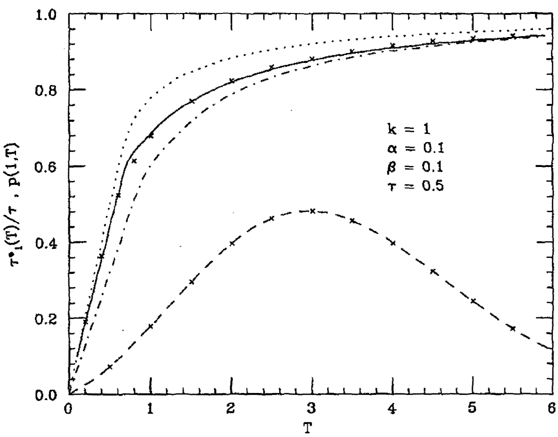
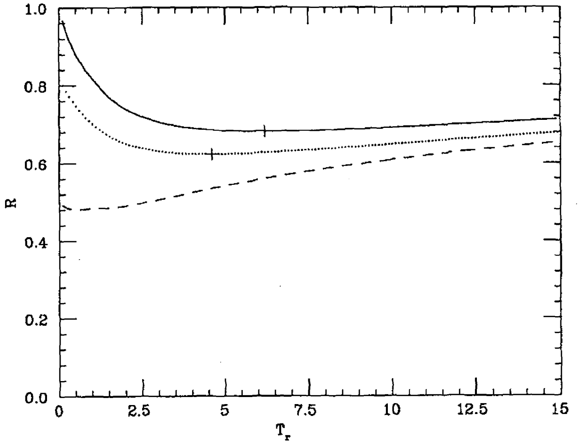


Figure 7. – Behavior of $\tau_k^*(T)/\tau$ (full curve) and $p(k, T)$ (dashed curve) for $k = 1$, $\alpha = 0.1$, $\beta = 0.1$ and $\tau = 0.5$. The points are results of Monte Carlo simulations. The dotted curve and the dot-dashed curve report the behaviours of $\tau_k^*(T)/\tau$ for $\alpha = 0.1$, $\beta = 0$ and $\alpha = 0$, $\beta = 0.1$, respectively. Identical results have been obtained also for $k = 2$, 3 and 4.

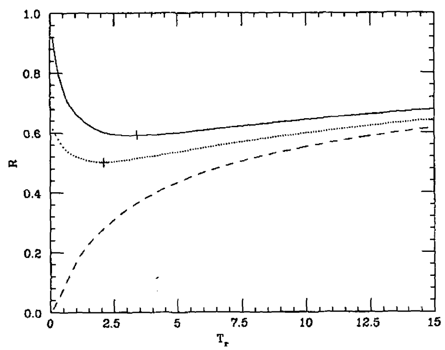
These considerations can lead to conclude that the assumption $\tau_k(T) = k\,\tau$ is quite reasonable. In fact, it is good in many cases. But this is not true when τ/T is not sufficiently small or, as the above figures reveal, when the failure rate $\gamma(t)$ (*i.e.* α and β) is not sufficiently small. In that case the assumption $\tau_k(T) = k\,\tau$ can lead to considerable error in the calculation of the optimal interval T_0 between ordinary maintenances.

3.3. Optimal interval between ordinary maintenances. Numerical examples

At this point we have the elements to determine the optimal time-interval between ordinary maintenances. Unfortunately, contrary to the case treated in section 2.1, it becomes difficult (if not impossible) to proceed analytically. But there is no difficulty to continue our analysis numerically. In fact this has been done under various conditions. Some typical results are reported in figure 8 where the ratio $R(T)$ is given as a function of $T = T_r$. We have considered different cases in which $\tau_k(T) = k\,\tau$ (full curve) and $\tau_k(T) = (k-1)\,\tau + \tau_k^*(T)$ (dotted curve). In the same figures we also report the behavior of $R(t)$ as obtained for $\tau_k(T) = (k-1)\,\tau$ (dashed curve). As one can see, the dotted curve is always in the area delimited by the two other curves which fix the upper and lower limits, respectively. As expected, if τ is sufficiently small there is no appreciable distinction between the two cases $\tau_k(T) = (k-1)\,\tau + \tau_k^*(T)$ and $\tau_k(T) = k\,\tau$. That is, full



(a)



(b)

Figure 8. – (a) Behaviour of the ratio R as a function of T_r for $\alpha = 0.3$, $\beta = 0.3$, $\tau = 1$ and $\theta = 2$ under the assumptions that the loss of time produced by k breakdowns is $k\tau$ (full curve), $(k-1)\tau$ (dashed curve) and $(k-1)\tau + \tau_k^*$ (dotted curve). The mark indicates the position of the minimum; (b) same quantities as in figure 8a but for $\alpha = 0.1$, $\beta = 0.3$, $\tau = 1$ and $\theta = 1$.

curve and dotted curve are coincident. We have verified this conclusion for $\tau = 0.2$ with the indicated values of the other parameters [9]. In this case the optimal time-interval between ordinary maintenances agrees with that evaluated with the technique of section 2. (*Cf.*, in particular, *fig.* 2). On the contrary, when τ is increased the differences between dotted and full curves become more and more pronounced and, as a consequence, the difference between the corresponding values of the optimal time between ordinary maintenances becomes pronounced. Figures 8 *a-b* report some examples which show the entity of the differences when varying the parameters of the problem. Note that, in all of the cases, the value of $T = T_0$ relevant to the minimum of the dotted curve is lower than that given by the full curve. Of course, the difference between the two values of T_0 depends on the failure rate. It tends to increase when α and β (i.e. $\gamma(t)$) are increased and vice versa. A 2^3 *Factorial-design* analysis shows that this difference depends mainly on τ and, in order of importance, on β and α.

3.4. Discussion and conclusions

We have calculated the optimal time-interval between successive ordinary maintenances. This interval can be relevant to the time of *effective service* or the *total* time really elapsed, including in this latter case the time spent for (minimal or extraordinary) repairs. In particular we have worked under conditions in which the conventional assumptions of: 1) very short interruptions (with respect to the effective service time between ordinary maintenances) and/or 2) sufficiently small breakdown probability, are not satisfied. Some analytical expressions that have been derived [e.g. eqs. (13), or (14), for $p(k, t)$] have a general validity. They can be applied to any function $\gamma(t)$ which can be treated as probability of beakdown per unit time. Other expressions (e.g., those of $\tau_k^*(T)$) are semi-empirical and necessarily approximate, even if in good agreement with results of Monte Carlo simulations. In fact, they permit a good evaluation of the optimal time-interval between successive maintenances even in situations in which the time spent per single extraordinary maintenance is not small with respect to the interval between ordinary maintenances. Our analysis shows that the error influencing the optimal time-interval can become large when evaluated under the conventional condition $\tau \to 0$, the correct value being generally lower than the approximate one.

APPENDIX A

Probability of k breakdowns in $(0, T)$

Since $(k-1)\tau$ may be considered the lower limit for T and k complete breakdowns can only occur in a time $T > k\tau$, we distinguish two cases. [8, 9].

1) $\underline{(k-1)\,\tau \leq T \leq k\,\tau}$. In this case we have

$$\begin{aligned}
p(1, T) &= \int_0^T Q_0(0, \xi)\,\gamma(\xi)\,d\xi = 1 - e^{-\Gamma(T)} \qquad \dots\ 0 \leq T \leq \tau \\
p(2, T) &= \int_0^{T-\tau} d\xi \int_{\xi+\tau}^{T} d\eta\, Q_0(0, \xi)\,\gamma(\xi)\,Q_0(\xi, \eta-\tau)\,\gamma(\eta-\tau) \\
&= \int_0^{T-\tau} \gamma(\xi)\,d\xi \int_{\xi+\tau}^{T} Q_0(0, \eta-\tau)\,\gamma(\eta-\tau)\,d\eta \\
&= 1 - e^{-\Gamma(T-\tau)} - \Gamma(T-\tau)\,e^{-\Gamma(T-\tau)} \qquad \dots\ \tau \leq T \leq 2\,\tau
\end{aligned}$$

where we have paid attention to write: 1) $Q_0(\xi, \eta-\tau)$ instead of $Q_0(\xi+\tau, \eta)$ for the probability of no breakdowns between $\xi+\tau$ and η (as the system does not work between ξ and $\xi+\tau$ and $\gamma(t)$ cannot change) and 2) $Q_0(\eta-\tau, T-2\,\tau) = 1$ as it is certain that further breakdowns are not possible if one of them occurs in $(T-\tau, T)$. Analogously,

$$\begin{aligned}
p(3, T) &= \int_0^{T-2\,\tau} d\xi \int_{\xi+\tau}^{T-\tau} d\eta \int_{\eta+\tau}^{T} d\zeta\, Q_0(0, \xi)\,\gamma(\xi)\,Q_0(\xi, \eta-\tau) \\
&\quad \times \gamma(\eta-\tau)\,Q_0(\eta-\tau, \zeta-2\,\tau)\,\gamma(\eta-2\,\tau) \\
&= \int_0^{T-2\,\tau} \gamma(\xi)\,d\xi \int_{\xi+\tau}^{T-\tau} \gamma(\eta-\tau)\,d\eta \\
&\quad \times \int_{\eta+\tau}^{T} Q_0(0, \zeta-2\,\tau)\,\gamma(\zeta-2\,\tau)\,d\zeta \\
&= 1 - e^{-\Gamma(T-2\,\tau)} - \Gamma(T-2\,\tau)\,e^{-\Gamma(T-2\,\tau)} \\
&\quad - \frac{\Gamma^2(T-2\,\tau)}{2!}\,e^{-\Gamma(T-2\,\tau)} \qquad \dots\ 2\,\tau \leq T \leq 3\,\tau
\end{aligned}$$

In the same way one finds that, in the general case,

$$p(k, T) = 1 - \sum_{0}^{k-1}{}_{j} \frac{\Gamma^j(T-(k-1)\,\tau)}{j!}\,e^{-\Gamma(T-(k-1)\,\tau)}$$
$$\dots\ (k-1)\,\tau \leq T \leq k\,\tau$$

2) $\underline{T > k\,\tau}$. In this case we have

$$
\begin{aligned}
p(1, T) &= \int_0^{T-\tau} d\xi\, Q_0(0, \xi)\, \gamma(\xi)\, Q_0(\xi, T-\tau) + \int_{T-\tau}^{T} Q_0(0, \xi)\, \gamma(\xi)\, d\xi \\
&= \int_0^{T-\tau} Q_0(0, T-\tau)\, \gamma(\xi)\, d\xi + \int_{T-\tau}^{T} Q_0(0, \xi)\, \gamma(\xi)\, d\xi \\
&= \Gamma(T-\tau)\, e^{-\Gamma(T-\tau)} + \{e^{-\Gamma(T-\tau)} - e^{-\Gamma(T)}\} \quad \dots T \geq \tau
\end{aligned}
$$

$$
\begin{aligned}
p(2, T) &= \int_0^{T-2\tau} \gamma(\xi)\, d\xi \\
&\times \left\{ \int_{\xi+\tau}^{T-\tau} Q_0(0, T-2\tau)\, \gamma(\eta-\tau)\, d\eta + \int_{T-\tau}^{T} Q_0(0, \eta-\tau)\, \gamma(\eta-\tau)\, d\eta \right\} \\
&+ \int_{T-2\tau}^{T-\tau} \gamma(\xi)\, d\xi \int_{\xi+\tau}^{T} Q_0(0, \eta-\tau)\, \gamma(\eta-\tau)\, d\eta \\
&= \frac{\Gamma^2(T-2\tau)}{2!}\, e^{-\Gamma(T-2\tau)} + \Gamma(T-2\tau)\, e^{-\Gamma(T-2\tau)} + e^{-\Gamma(T-2\tau)} \\
&- \{\Gamma(T-\tau)\, e^{-\Gamma(T-\tau)} + e^{-\Gamma(T-\tau)}\} \quad \dots T \geq 2\tau
\end{aligned}
$$

$$
\begin{aligned}
p(3, T) &= \int_0^{T-3\tau} \gamma(\xi)\, d\xi \int_{\xi+\tau}^{T-2\tau} \gamma(\eta-\tau)\, d\eta \\
&\times \left\{ \int_{\eta+\tau}^{T-\tau} Q_0(0, T-3\tau)\, \gamma(\zeta-2\tau)\, d\zeta + \int_{T-\tau}^{T} Q_0(0, \zeta-2\tau)\, \gamma(\zeta-2\tau)\, d\zeta \right\} \\
&+ \int_{T-3\tau}^{T-2\tau} \gamma(\xi)\, d\xi \int_{\xi+\tau}^{T-\tau} \gamma(\eta-\tau)\, d\eta \\
&\times \int_{\eta+\tau}^{T} Q_0(0, \zeta-2\tau)\, \gamma(\zeta-2\tau)\, d\zeta \\
&= \frac{\Gamma^3(T-3\tau)}{3!}\, e^{-\Gamma(T-3\tau)} + \frac{\Gamma^2(T-3\tau)}{2!}\, e^{-\Gamma(T-3\tau)} \\
&+ \frac{\Gamma(T-3\tau)}{1!}\, e^{-\Gamma(T-3\tau)} + e^{-\Gamma(T-3\tau)} \\
&- \left\{ \frac{\Gamma^2(T-2\tau)}{2!}\, e^{-\Gamma(T-2\tau)} + \frac{\Gamma(T-2\tau)}{1!}\, e^{-\Gamma(T-2\tau)} + e^{-\Gamma(T-2\tau)} \right\} \dots T \geq 3\tau
\end{aligned}
$$

In the general case,

$$p(k, t) = \sum_{0}^{k}{}_{j} \frac{\Gamma^{j}(T - k\tau)}{j!} e^{-\Gamma(T-k\tau)} - \sum_{0}^{k-1}{}_{j} \frac{\Gamma^{j}(T - (k-1)\tau)}{j!} e^{-\Gamma(T-(k-1)\tau)}$$

As regards the normalizaton of $p(k, T)$, it is not difficult to show that $\sum_{0}^{\infty}{}_{k} p(k, T) = 1$.

APPENDIX B

Mean waste of time caused by k breakdowns

The simplest situation is that of single breakdown, *i.e.* $k = 1$. If $\tau_1(T)$ is the average time that is lost in $(0, T)$, then

1) For $T \leq \tau$:

$$\tau_1(T) = \frac{\int_0^T (T - \xi) Q_0(0, \xi) \gamma(\xi) d\xi}{p(1, T)} = T - \frac{-T e^{-\Gamma(T)} + \int_0^T e^{-\Gamma(\xi)} d\xi}{1 - e^{-\Gamma(T)}}$$

For $\gamma(t) = \alpha$ (constant) [8, 9]

$$\tau_1(T) = T - \frac{-T e^{-\alpha T} + \left(1 - e^{-\alpha T}\right) \alpha^{-1}}{1 - e^{-\alpha T}} \xrightarrow[\alpha T \to 0]{} T/2$$

while for $\gamma(t) = \alpha + 2\beta t$ with $\alpha T \to 0$ and $\beta T^2 \to 0$, to the first order with respect to α and β,

$$\tau_1(T) \approx T - \frac{\alpha T/2 + 2\beta T^2/3}{\alpha + \beta T} = \frac{\alpha(T/2) + \beta T(T/3)}{\alpha + \beta T} = \begin{cases} T/2 & \text{for } \beta = 0 \\ T/3 & \text{for } \alpha = 0 \end{cases} \tag{B.1}$$

2) For $T \geq \tau$:

$$\tau_1(T) = \frac{\left\{\begin{array}{l} \displaystyle\int_0^{T-\tau} \tau\, Q_0(0, \xi)\, \gamma(\xi)\, Q_0(\xi, T-\tau)\, d\xi \\ \displaystyle\qquad + \int_{T-\tau}^{T} (T-\xi)\, Q_0(0, \xi)\, \gamma(\xi)\, d\xi \end{array}\right\}}{p(1, T)}$$

$$= \frac{\left\{\begin{array}{l} \tau\, \Gamma(T-\tau)\, e^{-\Gamma(T-\tau)} + T\{e^{-\Gamma(T-\tau)} - e^{-\Gamma(T)}\} \\ \qquad\qquad + \{\phi(T-\tau) - \phi(T)\} \end{array}\right\}}{\Gamma(T-\tau)\, e^{-\Gamma(T-\tau)} + \{e^{-\Gamma(T-\tau)} - e^{-\Gamma(T)}\}}$$

where

$$\phi(t) = \int_0^t t e^{-\Gamma(t)}\, \gamma(t)\, dt.$$

In strict analogy with the preceding case, it can be proved that for $\alpha t \to 0$ and $\beta T^2 \to 0$, to the first order with respect to α and β,

$$\tau_1(T) \approx \frac{\alpha\tau(1 - \tau/(2T)) + \beta T \tau(1 - \tau/T + \tau^2/(3T^2))}{\alpha + \beta T}$$

$$= \begin{cases} \tau\{1 - \tau/(2T)\} & \text{for } \beta = 0 \\ \tau\{1 - \tau/T + \tau^2/(3T^2)\} & \text{for } \alpha = 0 \end{cases} \tag{B.2}$$

As expected, for $T = \tau$ we find again (B.1). However, this result does not represent well $\tau_1(T)$ for $\tau \ll T$ and elevated T. For $\tau \ll T$, we can approximate $\tau_1(T)$ as follows

$$\tau_1(T) \approx \tau - \frac{1}{2}\frac{\gamma(T)\,\tau}{\Gamma(T)}\,\Delta$$

$$\approx \begin{cases} \tau\left(1 - \dfrac{1}{2}\dfrac{\tau}{T}\right) \approx \tau \exp\left(-\dfrac{1}{2}\dfrac{\tau}{T}\right) & \text{for } \gamma(t) = \alpha \\ \tau\left(1 - \dfrac{2}{3}\dfrac{\tau}{T}\right) \approx \tau \exp\left(-\dfrac{2}{3}\dfrac{\tau}{T}\right) & \text{for } \gamma(t) = 2\beta t \end{cases}$$

Δ being the average waste of time relevant to the breakdowns that occur in $(T - \tau, T)$. In fact, $\tau_1(T)$ has not the same asymptotic behavior of the preceding representation. A different more appropriate analysis of the problem is then necessary, possibly valid for any k. In practice, it is important to find an approximate handy form of $\tau_k(T)$ which permits a more precise calculation of R_m and T_0 than that provided by the simple choice $\tau_k(T) = k\,\tau$.

To this end, suppose first that $(k-1)\,\tau \leq T \leq k\,\tau$ and let

$$\tau_k(T) = (k-1)\,\tau + (T - \langle\xi\rangle_k)$$

$\langle\xi\rangle_k$ being the average time at which the k-th breakdown occurs. The rigorous explicit form of $\langle\xi\rangle_k$ is not simple to be obtained even for $k = 2$, but not even convenient. The rigorous expression of $\langle\xi\rangle_1$ is already too complicated for practical use. Appropriate forms can be easily obtained for small T. In particular, for $\gamma(t) = \alpha$ we have found [*see* (B.1)] that $\langle\xi\rangle_1 = T/2$ while for $\gamma(t) = 2\,\beta\,t$ we have found $\langle\xi\rangle_1 = 2\,T/3$. Analogously, for $\gamma(t) = \alpha$ we are then led to assume that $\langle\xi\rangle_2 = 2\,(T-\tau)/3$ and, for $\gamma(t) = 2\,\beta\,t$, that $\langle\xi\rangle_2 = 4\,(T-\tau)/5$. In fact, if a single breakdown occurs, on the average, at $T/2$ (or $2\,T/3$), in case of two breakdowns the second one occurs, on the average, at $2\,(T-\tau)/3$ [or $4\,(T-\tau)/5$)). In general, we are then led to assume that

$$\begin{aligned}
\tau_k(T) &= (k-1)\,\tau + T - \frac{k\,(T-(k-1)\,\tau)}{(k+1)} \\
&= (k-1)\,\tau + \frac{T-(k-1)\,\tau}{(k+1)} \qquad \text{for} \quad \gamma(t) = \alpha\,t \\
\tau_k(T) &= (k-1)\,\tau + T - \frac{2\,k\,(T-(k-1)\,\tau)}{(2\,k+1)} \\
&= (k-1)\,\tau + \frac{T-(k-1)\,\tau}{(2\,k+1)} \qquad \text{for} \quad \gamma(t) = 2\,\beta\,t
\end{aligned}$$

For $T > k\,\tau$ a convenient procedure can be the following. Suppose first that $\gamma(t) = \alpha$ and write that

$$\begin{aligned}
\tau_k(T) = (k-1)\,\tau + \left\{\frac{T^* - \tau}{T^*}\,\tau + \frac{\tau}{T^*}\,\frac{\tau}{2}\right\} &= (k-1)\,\tau + \tau\left(1 - \frac{\tau}{2\,T^*}\right) \\
&\approx (k-1)\,\tau + \tau\exp\left(-\frac{\tau}{2\,T^*}\right)
\end{aligned}$$

Such an equation takes account that: 1) only the last breakdown may be incomplete and 2) only a part T^* of T must be considered in the average reported in braces. Thus, the problem becomes that of giving an appropriate explicit form to T^*. For $k = 1$ there is no problem as $T^* = T$. That is, in accord with (B.2),

$$\tau_1 (T) = \frac{T - \tau}{T} \tau + \frac{\tau}{T} \frac{\tau}{2} = \tau \left(1 - \frac{\tau}{2T}\right) \approx \tau \exp\left(-\frac{\tau}{2T}\right)$$

For $k = 2$ we have, with good approximation, $T^* = (T - \tau)/2$, that is

$$\tau_2 (T) = \tau + \tau \left(1 - \frac{\tau}{T - \tau}\right) \approx \tau + \tau \exp\left(-\frac{\tau}{T - \tau}\right)$$

Analogously, it is expected that for arbitrary k, a good approximation is that of assuming $T^* = (T - (k-1)\tau)/k$ since, on the average, we assign the same fraction $1/k$ of service time $T - (k-1)\tau$ to each breakdown. Thus,

$$\begin{aligned} \tau_k (T) &= (k-1)\tau + \tau \left(1 - \frac{k\tau}{2(T - (k-1)\tau)}\right) \\ &\approx (k-1)\tau + \tau \exp\left(-\frac{k\tau}{2(T - (k-1)\tau)}\right) \end{aligned}$$

From this result we could be led to write that

$$\tau_k (T) = (k-1)\tau + \begin{cases} \dfrac{T - (k-1)\tau}{(k+1)} & \text{for} \quad (k-1)\tau \le T < k\tau \\ \tau \exp\left(-\dfrac{k\tau}{2(T - (k-1)\tau)}\right) & \text{for} \quad T \ge k\tau \end{cases}$$

However, the choice would be inappropriate since the connection of the two behaviours of $\tau_k (T)$ at $T = k\tau$ is not continuous. More correctly we impose that

$$\frac{T - (k-1)\tau}{(k+1)\tau} = \exp\left(-\frac{k\tau}{2(T - (k-1)\tau)}\right)$$

$$\text{that is} \begin{cases} X = X(k) \equiv \dfrac{k\tau}{2(T - (k-1)\tau)} \\ X_{r+1} = \dfrac{k}{2(k+1)} \exp(X_r) \end{cases}$$

and obtain (iteratively) the value of $T = \vartheta_k$ in correspondence of which the two behaviors are connected with continuity. Finally, we assume

$$\tau_k(T) = (k-1)\,\tau + \begin{cases} \dfrac{T-(k-1)\,\tau}{(k+1)} & \text{for } (\mathrm{k}-1)\,\tau \le \mathrm{T} < \vartheta_{\mathrm{k}} \\ \tau \exp\left(-\dfrac{k\,\tau}{2\,(T-(k-1)\,\tau)}\right) & \text{for } \mathrm{T} \ge \vartheta_{\mathrm{k}} \end{cases}$$

Note that: 1) the values of ϑ_k do not depend on α and 2) only two solutions exist in correspondence of $X(1) = 0.3574$ and $X(2) = 0.6191$. For $k > 2$ there are various ways to have a satisfactory connection at $T = k\,\tau$. Much more simply, we may assume the exponential behavior to be valid for any $T > (k-1)\,\tau$. Generally the approximation is quite good. Various Monte Carlo simulations for different values of τ have revealed that these results represent very well the rigorous behavior of $\tau_k(T)$ for any permitted value of T (*i.e.*, $T \ge (k-1)\,\tau$) and reasonable values of α and τ. Comparisons between Monte Carlo and theory are reported in figure 4 of the text.

To extend these results to more general forms of $\gamma(t)$, suppose first that $\gamma(t) = 2\,\beta\,t$. The main difference with the case treated above is that the distribution in time of the breakdowns is no longer uniform which complicates the evaluation of quantities such as T^*.

In consideration of these difficulties and with the purpose to give the best representation to $\tau_k^*(T)$ we will adopt the following *semiempirical* approach. First we observe that the linear behavior adopted in the interval $((k-1)\,\tau,\ k\,\tau)$ represents well $\tau_k^*(T)/\tau$ up to values greater than about 1/2. This is clearly shown by Monte Carlo simulations. Assume then that

$$\tau_k^*(T) = \frac{T-(k-1)\,\tau}{(2k+1)} \qquad \text{for} \quad (k-1)\,\tau \le T \le (2\,k-1/2)\,\tau$$

is the behavior of the mean time that is lost in the k-th breakdown in the indicated interval of time. Note that the upper limit $(2\,k-1/2)\,\tau$ corresponds to the value of T for which $\tau_k^*(T) = \tau/2$. At this point, we can use the above considerations to say that $\tau_k^*(T)$ is reasonably represented by the form

$$\tau_k^*(T) = C\,\tau \exp\left(-A\,\frac{\tau}{T-(k-1)\,\tau}\,\phi_k(T)\right)$$

Placed that $\phi_k((2k-1/2)\tau)=1$, we must fix the constants A and C in order to satisfy the condition of passage for the point $((2k-1/2)\tau, 1/2)$ of the ratio $\tau_k^*(T)/\tau$. This request can be satisfied with $A=(2k+1)/3$ and $C=(1/2)\exp(2/3)$, *i.e.* when writing that

$$\tau_k^*(T)=\frac{\tau\exp(2/3)}{2}\exp\left(-\frac{(2k+1)}{3}\frac{\tau\phi_k(T)}{T-(k-1)\tau}\right)$$
$$\approx\tau\exp\left(-\frac{(2k+1)}{3}\frac{\tau\phi_k(T)}{T-(k-1)\tau}\right)$$

For $k=1$ and $\phi_k(T)=1$ this form has the expected asymptotic behavior. Thus, it remains only to define the function $\phi_k(T)$. We have already assumed that $\phi_k(T)=1$ for $T=(2k-1/2)\tau$. Then, we may start by assuming that $\phi_k(T)=1$ for any T. But in this case it is found that $\tau_k^*(T)$ tends to τ when increasing T less rapidly than indicated by Monte Carlo simulations. In other words, contrary to the case $\gamma(t)=\alpha$, it is no longer possible to represent the correct behavior of $\tau_k^*(T)$ only with linear terms in $\tau/(T-(k-1)\tau)$ in the exponential. Having ascertained that $\phi_k(T)$ must be a decreasing function of T not so different from one in the time interval of interest, we may assume that

$$\phi_k(T)=\exp\left\{-B_k\left(T-\frac{(4k-1)\tau}{2}\right)\right\}$$

B_k being an appropriate constant. In order to agree with the Monte Carlo data we must write that $B_k\approx B=1/5$ independently of k. Thus, we can finally assume that

$$\phi_k(T)=\phi(T)=\exp\left(-\frac{2T-(4k-1)\tau}{10}\right)$$

This equation, permits to represent $\tau_k^*(T)$ very well. The accuracy of the approximation depends on the parameters β and τ. In fact the dependence of $\tau_k^*(T)$ on β becomes weak if β is sufficiently small. But it is not difficult to introduce an appropriate dependence on β of the factor C to further improve the analytical representation of $\tau_k^*(T)$. Taking into account that the constants reported above optimize the representation of $\tau_k^*(T)$ for $\beta=0.3$, from an examination of the Monte Carlo behaviors we deduce that the form

$$C(\beta)=\left\{1+\frac{2}{9}k(\beta-0.3)\exp\left(-\frac{T}{15}\right)\right\}$$

is appropriate. As required, $C(\beta)=1$ for $\beta=0.3$, $C(\beta)\to 1$ for $T\to\infty$ and, finally, $C(\beta)$ increases (linearly) with k as revealed by the simulations.

On the other hand, even the constants appearing in the expression of $C(\beta)$ have been deduced by Monte Carlo simulations. Comparisons between analytical (semi-empirical) behaviors and Monte Carlo results are reported in figure 6 of the text.

The extension of these results to the general case in which the form of $\gamma(t)$ is not specified is difficult but not necessary to our ends. For this reason, we will assume as final case that $\gamma(t) = \alpha + 2\beta t$. We must find a suitable procedure to *weight* $(\tau_k^*(T))_{\alpha=0}$ and $(\tau_k^*(T))_{\beta=0}$. An appropriate way seems that of imposing that (B.1) and (B.2) obtained for $k = 1$ are satisfied. Thus we write that

$$\tau_k^*(T) = \frac{\alpha(\tau_k^*(T))_{\beta=0} + \beta T(\tau_k^*(T))_{\alpha=0}}{\alpha + \beta T}$$

for any k. Note that the case $\gamma(t) = \alpha$ is dominant for small T while, for large T, it is the case $\gamma(t) = 2\beta t$ that is more important. This is expected as $\gamma(t) \to \alpha$ for $t \to 0$ while $\gamma(t) \to 2\beta t$ for $t \to \infty$. Of course, we must still prove that the choice we have done for $\tau_k^*(T)$ when both α and β are different from zero is appropriate for any k, not just for $k = 1$. This has been verified by various Monte Carlo simulations, as indicated in figure 7 of the text.

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