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ACHIEVABLE POTENTIAL REDUCTIONS IN THE METHOD OF KOJIMA *et al.* IN THE CASE OF LINEAR PROGRAMMING (*)

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Abstract. – *Kojima et al. proposed a primal-dual interior point method, based on potential reduction for solving the linear complementarity problem, and they derived an $O(\sqrt{n}\,L)$ iteration bound. In the derivation of this bound they used equal step sizes in the primal and the dual space, and they showed that taking the step size equal to 0.4 the potential reduction, is at least 0.2. In this paper we specialize to the case of linear programming and we show that the latter constant can be improved to 0.267. Further, admitting different step sizes in both spaces, we show that a further improvement of this constant to 0.345 is possible. Furthermore, for the case of equal step sizes it is shown that the duality gap is monotically decreasing.*

Keywords: Interior point method, linear complementarity problem, linear programming, potential reduction, polynomial algorithm.

Résumé. – *Kojima et al. ont proposé une méthode primale-duale de points intérieurs, fondée sur une technique de réduction de potentiel, pour la résolution de problèmes de complémentarité linéaire ; ils en ont dérivé une borne d'ordre $O(\sqrt{n}\,L)$ sur le nombre d'itérations. Dans la dérivation de cette borne ils ont utilisé le même pas dans l'espace et l'espace dual ; ils ont démontré qu'un pas de longueur 0,4 donnait une réduction du potentiel d'au moins 0,2. Dans cet article, nous particularisons l'analyse au cas de programmation linéaire et nous montrons que la constante de décroissance du potentiel peut être portée à 0,267. De plus, nous démontrons qu'avec des pas de longueurs différentes dans le primal et dans le dual une amélioration supplémentaire de cette constante au niveau de 0,345 est possible. Finalement, nous prouvons que, dans le cas de pas de même longueur dans le primal et le dual, le saut de dualité décroît de manière monotone.*

Mots clés : Méthode de points intérieurs, problème de complémentarité linéaire, problème de programmation linéaire, réduction de potentiel, algorithme polynomial.

1. INTRODUCTION

In [2], Kojima *et al.* considered a primal-dual interior point method for solving the linear complementarity problem, and they obtained an $O(\sqrt{n}\,L)$

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bound for the number of the iterations. This iteration bound is obtained by showing that in each iteration the so-called primal-dual potential function decreases with (at least) some constant. In the derivation of this bound they use equal step sizes in the primal and the dual space, and they show that taking the step size equal to 0.4 the potential reduction is at least 0.2.

In this paper we specialize to the case of linear programming. We consider the primal-dual pair of problems

$$(P) \quad \min_{x \in \mathbb{R}^n} (c^T x : A x = b,\ x \geq 0)$$

$$(D) \quad \max_{y \in \mathbb{R}^m} (b^T y : A^T y + s = c,\ s \geq 0).$$

Just an in [2] the search directions are derived from the primal-dual potential function

$$f(x, s) = (n + \nu) \ln x^T s - \sum_{j=1}^{n} \ln x_j s_j,$$

where ν is some constant. $\nu \geq 1$. We firstly show that by a little more accurate analysis than in [2], for the case of equal step sizes, the potential reduction constant can be improved to 0.267.

It is well known that in practical implementations of the primal-dual method different step sizes for the primal and the dual space are used. Therefore, in the second part of this note we admit different step sizes in both spaces, and we show that the potential function decrease in each iteration can be further improved to at least 0.345.

Furthermore, for the case of equal step sizes it will be shown that the duality gap is monotically decreasing.

Notation. As far as notations are concerned, e shall denote the vector of all ones, and I the identity matrix. Given an n-dimensional vector x we denote by X then $n \times n$ diagonal matrix whose diagonal entries are the coordinates x_j of x; x^{-1} denotes the vector obtained by inverting the entries of x, and x^T is the transpose of the vector x. Finally $\|x\|$ denotes the l_2 norm of x.

2. POTENTIAL FUNCTION AND SEARCH DIRECTIONS

We shall consider search directions derived from the potential function

$$f(x, s) = (n + \nu) \ln x^T s - \sum_{j=1}^{n} \ln x_j s_j.$$

The gradients with respect to x and s respectively, are given by

$$\nabla_x f = (n+\nu)\,\frac{s}{x^T s} - x^{-1}$$

$$\nabla_s f = (n+\nu)\,\frac{x}{x^T s} - s^{-1}.$$

Let us introduce the matrices

$$D = (XS^{-1})^{1/2} \qquad \text{and} \qquad V = (XS)^{1/2}.$$

We have the useful relationship:

$$D^{-1} X = V = DS.$$

LEMMA 1: $D\,\nabla_x f = D^{-1}\,\nabla_s f = \dfrac{n+\nu}{\|v\|^2}\, v - v^{-1}$.

Let Δx and Δs be the candidate displacements in the x and s spaces. To maintain feasibility, we impose

$$\Delta x \in \mathcal{N}(A) \qquad \text{and} \qquad \Delta s \in \mathcal{R}(A^T),$$

where $\mathcal{N}$ and $\mathcal{R}$ respectively denote the null space and the range space of a matrix. If we choose for Δx and Δs to be the vectors of steepest descent with respect to the norms induced by D^{-1} and D respectively, we obtain the conditions

$$\begin{aligned} D^{-2}\,\Delta x + \nabla_x f \in \mathcal{R}(A^T), &\quad \Delta x \in \mathcal{N}(A)\\ D^2\,\Delta s + \nabla_s f \in \mathcal{N}(A), &\quad \Delta s \in \mathcal{R}(A^T). \end{aligned}$$

Changing variables Δx and Δs for $D^{-1}\,\Delta x$ and $D\,\Delta s$ yields

$$\begin{aligned} D^{-1}\,\Delta x + D\,\nabla_x f \in \mathcal{R}(DA^T), &\quad D^{-1}\,\Delta x \in \mathcal{N}(AD)\\ D\,\Delta s + D^{-1}\,\nabla_s f \in \mathcal{N}(AD), &\quad D\,\Delta s \in \mathcal{R}(DA^T). \end{aligned}$$

Hence $D^{-1}\,\Delta x$ and $D\,\Delta s$ form the orthogonal decomposition of $-D\,\nabla_x f = -D^{-1}\,\nabla_s f = v^{-1} - (n+\nu)/(\|v\|^2)\,v$ over $\mathcal{N}(AD)$ and $\mathcal{R}(DA^T)$ respectively. Consequently,

$$0 = (D^{-1}\,\Delta x)^T\, D\,\Delta s = \Delta x^T\,\Delta s$$

$$D^{-1}\,\Delta x + D\,\Delta s = -D\,\nabla_x f = v^{-1} - \frac{n+\nu}{\|v\|^2}\,v$$

$$\|D^{-1}\,\Delta x\|^2 + \|D\,\Delta s\|^2 = \|D\,\nabla_x f\|^2 = \left\| v - 1\,\frac{n+\nu}{\|v\|^2}\,v \right\|^2.$$

We also have the simple relation

$$V\,(X^{-1}\,\Delta x + S^{-1}\,\Delta s) = D^{-1}\,\Delta x + D\,\Delta s = V^{-1}\,(S\,\Delta x + X\,\Delta s).$$

A full steepest descent step may not be feasible. We shall use a safeguarded line search with step lengths α_x and α_s. A sufficient condition to maintain strict feasibility is

$$\alpha_x\,\|X^{-1}\,\Delta x\|_\infty < 1 \qquad \text{and} \qquad \alpha_s\,\|S^{-1}\,\Delta s\|_\infty < 1.$$

The condition is fulfilled by taking α_x and α_s according to following lemma.

LEMMA 2: *Let* $0 < \tau < 1$ *be some arbitrary number. If*

$$\alpha_x = \alpha_s = \frac{\tau}{\|v^{-1}\|_\infty\,\|D\,\nabla_x\,f\|}$$

then

$$\alpha_x\,\|X^{-1}\,\Delta x\|_\infty \leq \tau < 1 \qquad \textit{and} \qquad \alpha_s\,\|S^{-1}\,\Delta s\|_\infty \leq \tau < 1.$$

Proof:

$$\begin{aligned}\|X^{-1}\,\Delta x\| &= \|V^{-1}\,D^{-1}\,\Delta x\|\\ &\leq \|v^{-1}\|_\infty\,\|D^{-1}\,\Delta x\|\\ &\leq \|v^{-1}\|_\infty\,\|D\,\nabla_x\,f\|.\end{aligned}$$

Therefore

$$\alpha_x\,\|X^{-1}\,\Delta x\|_\infty \leq \tau.$$

A similar reasoning yields the result for $\|S^{-1}\,\Delta s\|$.

3. POTENTIAL AND DUALITY GAP REDUCTIONS

Let us study first the duality gap reduction. In this section we assume throughout that $\alpha_x = \alpha_s = \alpha$, where α satisfies the condition of Lemma 2. We first establish that the duality gap is strictly decreasing for all values of $\nu > 0$.

LEMMA 3: $\Delta(x^T s) = -\alpha\nu$.

Proof:

$$\begin{aligned}
&(x + \alpha\,\Delta x)^T (s + \alpha\,\Delta s) - x^T s \\
&\quad = \alpha\,(s^T \Delta x + x^T \Delta s) \quad (\text{since } \Delta x^T \Delta s = 0) \\
&\quad = \alpha\, e^T V\,(D^{-1}\Delta x + D\,\Delta s) \\
&\quad = \alpha\, e^T V \left(v^{-1} - \frac{n+\nu}{\|v\|^2}\, v \right) \\
&\quad = -\alpha\nu. \quad \square
\end{aligned}$$

The potential variation is given by

$$\begin{aligned}
\Delta f = (n+\nu) \ln &\left(1 + \alpha\, \frac{s^T \Delta x + x^T \Delta s}{x^T s} \right) \\
&- \sum_{j=1}^{n} \ln \left(1 + \alpha\, \frac{\Delta x_j}{x_j} \right) \left(1 + \alpha\, \frac{\Delta s_j}{s_j} \right).
\end{aligned}$$

Using the standard inequalities with logarithms (see, *e.g.* [1] [3]) we get

$$\begin{aligned}
\Delta f \le \alpha\, e^T &\left(\frac{n+\nu}{x^T s} (S\,\Delta x + X\Delta s) - (X^{-1}\Delta x + S^{-1}\Delta s) \right) \\
&+ \frac{\alpha^2\,(\|X^{-1}\Delta x\|^2 + \|S^{-1}\Delta s\|^2)}{2\,(1-\tau)}.
\end{aligned}$$

We can now bound the various terms in the right-hand side of the above inequality. The coefficient of the linear part is

$$\begin{aligned}
e^T &\left(\frac{n+\nu}{x^T s} (S\,\Delta x + X\,\Delta s) - (X^{-1}\Delta x + S^{-1}\Delta s) \right) \\
&\qquad = e^T \left(\frac{n+\nu}{\|v\|^2}\, V\,(D^{-1}\Delta x + D\,\Delta s) \right. \\
&\qquad\qquad \left. - V^{-1}\,(D^{-1}\Delta x + D\,\Delta s) \right) \\
&\qquad = -e^T \left(\frac{n+\nu}{\|v\|^2}\, V - V^{-1} \right)^2 e \\
&\qquad = - \left\| \frac{n+\nu}{\|v\|^2}\, v - v^{-1} \right\|^2.
\end{aligned}$$

Since α is taken as in Lemma 2,

$$\alpha\, e^T \left(\frac{n+\nu}{x^T\, s} \,(S\,\Delta x + X\,\Delta s) - (X^{-1}\,\Delta x + S^{-1}\,\Delta s) \right) = -\frac{\|(n+\nu)/(\|v\|^2)\, v - v^{-1}\|\, \tau}{\|v^{-1}\|_\infty}.$$

To bound the right-hand side we use the following lemma, which improves the result of Lemma 2.5 in [2] if $\nu \geq \sqrt{2\,n - 1}$.

LEMMA 4: *For all* $v > 0$

$$\left\| \frac{n+\nu}{\|v\|^2}\, v - v^{-1} \right\| \geq \|v^{-1}\|_\infty \qquad \text{if} \quad \nu \geq \sqrt{2\,n-1}$$

and

$$\left\| \frac{n+\nu}{\|v\|^2}\, v - v^{-1} \right\| \geq \frac{\sqrt{3}}{2}\, \|v^{-1}\|_\infty \qquad \text{if} \quad \nu \geq \sqrt{2\,n-1} > \nu \geq \sqrt{n}.$$

Proof: Let $w = \|v^{-1}\|_\infty\, v$. Since $\|v^{-1}\|_\infty^{-1} = \min_i \{v_i\}$, then $w_i \geq 1$, $i = 1, \ldots, n$ and $w_j = 1$ for some j. Without loss of generality, assume that $w_n = 1$. Then

$$\|v^{-1}\|_\infty^{-1} \left\| v^{-1} - \frac{n+\nu}{\|v\|^2}\, v \right\| = \left\| w - 1\, \frac{n+\nu}{\|w\|^2}\, w \right\|^2 = \|w^{-1}\|^2 + \frac{\nu^2 - n^2}{\|w\|^2}.$$

To bound this quantity we consider the family of problems

$$\min \{\|w^{-1}\| : \|w\| = \sigma,\ w_i \geq 1,\ i = 1, \ldots, n,\ w_n = 1\}$$

for all $\sigma \geq \sqrt{n}$. Eliminating the last variable $w_n = 1$ we obtain the equivalent problem

$$\min \{1 + \|u^{-1}\|^2 : \|u\|^2 = \sigma^2 - 1,\ u_i \geq 1,\ i = 1, \ldots, n-1\}.$$

Neglecting the inequality constraints, we obtain from the optimality conditions:

$$-u_i^{-3} = 2\,\lambda\, u_i, \qquad i = 1, \ldots, n-1.$$

This implies that the u_i's are mutually equal. So we have

$$u_i = \sqrt{\frac{\sigma^2 - 1}{n - 1}}, \qquad i = 1, \ldots, n - 1.$$

Since $\sigma \geq \sqrt{n}$, we have $u_i \geq 1$, $i = 1, \ldots, n - 1$, and hence the unconstrained solution is also the solution for the constrained problem. Replacing u, and hence w, by these values we get

$$\begin{aligned} \|v^{-1}\|_{\infty}^{-1} \left\| v^{-1} - \frac{n+\nu}{\|v\|^2} v \right\| &= \|w^{-1}\|^2 + \frac{\nu^2 - n^2}{\sigma^2} \\ &\geq 1 + (n-1)\frac{n-1}{\sigma^2 - 1} + \frac{\nu^2 - n^2}{\sigma^2} \\ &\geq 1 + \frac{\nu^2 - (2n-1)}{\sigma^2} \geq 1. \end{aligned}$$

This concludes the proof of Lemma 4 for $\nu = \sqrt{2n - 1}$. The proof with $\nu = \sqrt{n}$ can be given in just the same way, although for this case a shorter proof is given in [2]. □

Lemma 4 implies that the linear therm is bounded by $-\tau$. For the quadratic term we have from Lemma 2

$$\frac{\alpha^2 (\|X^{-1} \Delta x\|^2 + \|S^{-1} \Delta s\|^2)}{2(1-\tau)} \leq \frac{\tau^2}{2(1-\tau)}.$$

So we have shown that

$$\Delta f \leq -\tau + \frac{\tau^2}{2(1-\tau)}.$$

The right hand side expression achieves its minimal value on $\tau > 0$ at $\tau = 0.42$. Substitution of this value yields a potential decrease of at least -0.267.

Remark: If ν is taken very large we get the affine scaling direction for the primal dual. The argument on the reduction of the potential is still valid. Unfortunately a large value of ν requires a big reduction of the potential before convergence is reached, hence more iterations. There is an obvious tradeoff between the one step potential reduction and the total reduction which is required. However it is not clear where the borderline is.

4. FURTHER IMPROVEMENT

In this section we consider the case where the step sizes in the primal and the dual space are not necessarily the same, *i.e.* $\alpha_x \neq \alpha_s$. Then the potential variation is given by

$$\begin{aligned}
\Delta f &= (n+\nu)\ln\left(1+\alpha_x \frac{s^T \Delta x}{s^T x} + \alpha_s \frac{x^T \Delta s}{x^T s}\right) \\
&\quad - \sum_{j=1}^{n} \ln\left(1+\alpha_x \frac{\Delta x_j}{x_j}\right) - \sum_{j=1}^{n} \ln\left(1+\alpha_s \frac{\Delta s_j}{s_j}\right) \\
&\leq (n+\nu)\left(\alpha_x \frac{s^T \Delta x}{s^T x} + \alpha_s \frac{x^T \Delta s}{x^T s}\right) \\
&\quad - \sum_{j=1}^{n} \ln\left(1+\alpha_x \frac{\Delta x_j}{x_j}\right) - \sum_{j=1}^{n} \ln\left(1+\alpha_s \frac{\Delta s_j}{s_j}\right) \\
&= (n+\nu)\,\alpha_x \frac{s^T \Delta x}{s^T x} - \sum_{j=1}^{n} \ln\left(1+\alpha_x \frac{\Delta x_j}{x_j}\right) \\
&\quad + (n+\nu)\,\alpha_s \frac{x^T \Delta s}{x^T s} - \sum_{j=1}^{n} \ln\left(1+\alpha_s \frac{\Delta s_j}{s_j}\right).
\end{aligned}$$

LEMMA 5: *If* $z := (z_1, \ldots, z_n)$, *with* $z_i + 1 > 0$, $1 \leq i \leq n$, *then one has*

$$-\sum_{j=1}^{n} \ln(1+z_j) \leq e^T z - \|z\| - \ln(1-\|z\|).$$

Proof:

$$\begin{aligned}
-\sum_{j=1}^{n} \ln(1+z_j) &= -\sum_{j=1}^{n}\left(z_j - \frac{z_j^2}{2} + \frac{z_j^3}{3} - \ldots\right) \\
&= -e^T z + \frac{\|z\|^2}{2} - \sum_{j=1}^{n}\left(\frac{z_j^3}{3} - \frac{z_j^4}{4} + \ldots\right) \\
&\leq -e^T z + \frac{\|z\|^2}{2} + \frac{\|z\|^3}{3} + \frac{\|z\|^4}{4} + \ldots \\
&\leq -e^T z - \|z\| - \ln(1-\|z\|). \quad \square
\end{aligned}$$

Using this lemma we obtain

$$\begin{aligned}\Delta f \leq\, & (n+\nu)\,\alpha_x \frac{s^T \Delta x}{s^T x} - \alpha_x\, e^T X^{-1} \Delta x \\ & - \alpha_x \|X^{-1} \Delta x\| - \ln(1 - \alpha_x \|X^{-1} \Delta x\|) \\ & + (n+\nu)\,\alpha_s \frac{x^T \Delta s}{x^T s} - \alpha_s\, e^T S^{-1} \Delta s \\ & - \alpha_s \|S^{-1} \Delta s\| - \ln(1 - \alpha_s \|S^{-1} \Delta s\|).\end{aligned}$$

At this stage we note that

$$\begin{aligned}(n+\nu)\frac{s^T \Delta x}{s^T x} - e^T X^{-1} \Delta x &= e^T \left((n+\nu)\frac{S}{s^T x} - X^{-1}\right)\Delta x \\ &= (\nabla_x f)^T \Delta x \\ &= (D \nabla_x f)^T D^{-1} \Delta x \\ &= -\|D^{-1} \Delta x\|^2,\end{aligned}$$

and similarly

$$(n+\nu)\frac{x^T \Delta s}{x^T s} - e^T S^{-1} \Delta s = -\|D \Delta s\|^2.$$

Using this we obtain the following inequality:

$$\begin{aligned}\Delta f \leq & -\alpha_x (\|D^{-1} \Delta x\|^2 + \|X^{-1} \Delta x\|) - \ln(1 - \alpha_x \|X^{-1} \Delta x\|) \\ & - \alpha_s (\|D \Delta s\|^2 + \|S^{-1} \Delta s\|) - \ln(1 - \alpha_s \|S^{-1} \Delta s\|).\end{aligned}$$

The proof of the following lemma is straightforward.

LEMMA 6: *Let A, $B > 0$ be given numbers. Consider $g(x) := -A x - \ln(1 - B x)$. Then*

$$g(x) \leq 1 - \frac{A}{B} + \ln \frac{A}{B},$$

and $g(x)$ attains this value for $x = 1/B - 1/A$.

Giving α_x and α_s the values

$$\alpha_x = \frac{1}{\|X^{-1} \Delta x\|} - \frac{1}{\|D^{-1} \Delta x\|^2 + \|X^{-1} \Delta x\|}$$

$$\alpha_s = \frac{1}{\|S^{-1} \Delta s\|} - \frac{1}{\|D \Delta s\|^2 + \|S^{-1} \Delta s\|}$$

and applying Lemma 6 twice, yields

$$\Delta f \leq -\frac{\|D^{-1}\,\Delta x\|^2}{\|X^{-1}\,\Delta x\|} + \ln\left(1 + \frac{\|D^{-1}\,\Delta x\|^2}{\|X^{-1}\,\Delta x\|}\right) - \frac{\|D\,\Delta s\|^2}{\|S^{-1}\,\Delta s\|} + \ln\left(1 + \frac{\|D\,\Delta s\|^2}{\|S^{-1}\,\Delta s\|}\right).$$

Clearly the chosen values of α_x and α_s guarantee that $x + \alpha_x\,\Delta x$ is primal feasible and $s + \alpha_s\,\Delta s$ is dual feasible.

We proceed by noting that

$$\frac{\|D^{-1}\,\Delta x\|^2}{\|X^{-1}\,\Delta x\|} = \frac{\|D^{-1}\,\Delta x\|^2}{\|V^{-1}\,D^{-1}\,\Delta x\|} \geq \|v^{-1}\|_\infty^{-1}\,\|D^{-1}\,\Delta x\|$$

and

$$\frac{\|D\,\Delta s\|^2}{\|S^{-1}\,\Delta s\|} = \frac{\|D\,\Delta s\|^2}{\|V^{-1}\,D\,\Delta x\|} \geq \|v^{-1}\|_\infty^{-1}\,\|D\,\Delta s\|.$$

Thus we have

$$\Delta f \leq -\|v^{-1}\|_\infty^{-1}\,\|D^{-1}\,\Delta x\| + \ln\left(1 + \|v^{-1}\|_\infty^{-1}\,\|D^{-1}\,\Delta x\|\right) - \|v^{-1}\|_\infty^{-1}\,\|D\,\Delta s\| + \ln\left(1 + \|v^{-1}\|_\infty^{-1}\,\|D\,\Delta s\|\right).$$

Recall that

$$\|D^{-1}\,\Delta x\|^2 + \|D\,\Delta s\|^2 = \|D\,\nabla_x\,f\|^2.$$

So $\|D^{-1}\,\Delta x\|$ and $\|D\,\Delta s\|$ satisfy the hypotheses of the following simple lemma with $r = \|D\,\nabla_x\,f\|$.

LEMMA 7: *Let* $h(x, y) := -x + \ln(1 + x) - y + \ln(1 + y)$, *and* $x^2 + y^2 = r^2$. *Then*

$$h(x, y) \leq 2\left(-\frac{r}{\sqrt{2}} + \ln\left(1 + \frac{r}{\sqrt{2}}\right)\right).$$

Thus we obtain that

$$\Delta f \leq 2\left(-\frac{1}{\sqrt{2}}\,\|v^{-1}\|_\infty^{-1}\,\|D\,\nabla_x\,f\| + \ln\left(1 + \frac{1}{\sqrt{2}}\,\|v^{-1}\|_\infty^{-1}\,\|D\,\nabla_x\,f\|\right)\right).$$

By Lemma 4, with $\nu \geq \sqrt{2\,n - 1}$,

$$\|v^{-1}\|_\infty^{-1}\,\|D\,\nabla_x\,f\| = \|v^{-1}\|_\infty^{-1}\left\|\frac{n + \nu}{\|v\|^2}\,v - v^{-1}\right\| \geq 1.$$

Using this we obtain that

$$-\Delta f \geq -2\left(-\frac{1}{\sqrt{2}} + \ln\left(1 + \frac{1}{\sqrt{2}}\right)\right) \approx 0.345.$$

The above analysis also applies if $\nu \geq \sqrt{n}$. In that case Lemma 4 yields that

$$\|v^{-1}\|_{\infty}^{-1} \|D\nabla_x f\| = \|v^{-1}\|_{\infty}^{-1} \left\| \frac{n+\nu}{\|v\|^2} v - v^{-1} \right\| \geq \frac{\sqrt{3}}{2}.$$

So we then have the looser bound

$$-\Delta f \geq -2\left(-\frac{\sqrt{3}}{2\sqrt{2}} + \ln\left(1 + \frac{\sqrt{3}}{2\sqrt{2}}\right)\right) \approx 0.269.$$

REFERENCES

1. N. KARMARKAR, A New Polynomial-Time Algorithm for Linear Programming, *Combinatorica, 4*, 1984, pp. 373-395.
2. M. KOJIMA, S. MIZUNO, A. YOSHISE, *An $O(\sqrt{n}\,L)$ Potential Reduction Algorithm for Linear Complementary Problems.*
3. Y. YE, A Class of Projective Transformations for Linear Programming, *SIAM Journal on Computing, 19*, 1990, pp. 457-466.