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BAYESIAN TESTING OF FUZZY PARAMETRIC HYPOTHESES FROM FUZZY INFORMATION (*)

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Abstract. – *In this paper, we perform the extension of the problem of testing parametric hypotheses when the available experimental information and the hypotheses considered are fuzzy subsets of the sample space and the parameter space respectively. Such an extension will be characterized as a special fuzzy decision problem when we assume the Bayesian framework. Furthermore, the Bayes principle of testing will be extended and we shall analyze some of its properties.*

Keywords: *Bayes test, fuzzy information system, fuzzy set, fuzzy decision problem, simple fuzzy hypothesis, test of fuzzy hypothesis from fuzzy information, Zadeh's probabilistic definition.*

Résumé. – *Dans cet article, nous étendons le problème du test des paramètres au cas où l'information disponible et les hypothèses considérées sont des sous-ensembles flous de l'espace d'échantillon et de l'espace des paramètres respectivement. Une telle extension sera caractérisée comme un problème spécial de décision floue lorsque nous nous plaçons dans le cadre Bayésien. En outre, le principe bayésien des tests sera étendu et nous analyserons certaines de ses propriétés.*

Mots clés : *Tests Bayésiens, système d'information flou, ensemble flou, problème de décision floue, hypothèse simple floue, test d'hypothèses floues à partir d'informations floues, définition probabiliste de Zadeh.*

1. INTRODUCTION

To test a statistical hypothesis is to perform an experiment concerning this hypothesis and, on the basis of the outcome of the experiment, to conclude whether the hypothesis can be considered as correct.

An approach to the Bayesian optimality criterion of testing statistical hypotheses concerning a random experiment when the available experimental observations are imprecise has been studied in [1] and [2]. This extension has been carried out under the assumption that these imprecise experimental observations may be assimilated with *fuzzy information associated with*

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the experiment (Tanaka *et al.* [10], [12] and Zadeh [17]), and the set of all available imprecise observations determines a *fuzzy information system* (Tanaka *et al.*, [12]). On the basis of these requirements, the use of Zadeh's probabilistic definition (Zadeh, [16]) allowed us to extend immediately the Bayesian optimality criterion.

In the same way, we are now going to extend the Bayesian optimality criterion of testing fuzzy parametric hypotheses, when the available experimental information is fuzzy.

2. PRELIMINARIES

Now we remind some definitions which will be needed in the sequel.

Let $\mathbf{X}=(X, \beta_X, P_\theta)$ be an experiment, where (X, β_X) is a measurable space and P_θ belongs to a family of probability measures $\mathbf{P}=\{P_\theta/\theta\in\Theta\}$ associated with the experiment $\mathbf{X}$.

DEFINITION 2.1: *A simple fuzzy parametric hypothesis* with respect to experiment $\mathbf{X}$ is a fuzzy subset of the parameter space Θ associated with $\mathbf{X}$.

DEFINITION 2.2: A fuzzy event $\mathcal{x}$ on X, characterized by a Borel-measurable membership function $\mu_{\mathcal{x}}$ from X to $[0, 1]$, where $\mu_{\mathcal{x}}(x)$ represents the "grade of membership" of x to $\mathcal{x}$ (or degree of compatibility of x with $\mathcal{x}$), is called *fuzzy information* associated with the experiment $\mathbf{X}$.

In practice, the grade of membership $\mu_{\mathcal{x}}(x)$ is often regarded as a kind of "probability with which the observer gets the fuzzy information $\mathcal{x}$ when he really has obtained the exact outcome x (*see* [12])". This interpretation cannot be rigourously formalized within the probabilistic framework, since $\mathcal{x}$ is not identified with an ordinary subset of X. Nevertheless, it justifies intuitively the assumption of orthogonality for the set of all observable events from $\mathbf{X}$, that is, to consider that this set is a fuzzy information system associated with $\mathbf{X}$, defined as follows.

DEFINITION 2.3: *A fuzzy information system* $\mathcal{X}$ associated with the experiment $\mathbf{X}$ is a fuzzy partition with fuzzy events on X, that is, a finite set of fuzzy events on X satisfying the orthogonality condition $\sum_{\mathcal{x}\in\mathcal{X}} \mu_{\mathcal{x}}(x)=1$, for all $x\in X$.

If a simple random sample of size n from the experiment $\mathbf{X}$ is considered and the ability to observe does not permit one to perceive exactly the experimental outcomes, the following notions supply an operative model to express the available sample observations with fuzzy imprecision (Gil *et al.* [6]).

Let $\mathbf{X}^{(n)}=(X^n, \beta_{X^n}, P_\theta)$ be a simple random sample of size n from $\mathbf{X}$, and let $\mathcal{X}$ be a fuzzy information system associated with $\mathbf{X}$.

DEFINITION 2.4: An n-tuple of elements in $\mathcal{X}$, $(\mathcal{x}_1, \ldots, \mathcal{x}_n)$, representing the algebraic product of $\mathcal{x}_1, \ldots, \mathcal{x}_n$, is called *sample fuzzy information of size n* from $\mathcal{X}$.

DEFINITION 2.5: A *fuzzy random sample of size n from* $\mathcal{X}$, $\mathcal{X}^{(n)}$ (associated with the random sample $\mathbf{X}^{(n)}$) is the set consisting of all algebraic products of n elements in $\mathcal{X}$.

The probability measure P_θ on (X^n, β_{X^n}) induces a probability distribution $\mathcal{P}_\theta$ on $\mathcal{X}$ defined (Zadeh, [16]) as follows.

DEFINITION 2.6: The *probability distribution on* $\mathcal{X}$ *induced by* P_θ is the mapping $\mathcal{P}_\theta$ from $\mathcal{X}^{(n)}$ to [0, 1] given by

$$\mathcal{P}_\theta(\mathcal{x}^n) = \int_{X^n} \mu_{\mathcal{x}^n}(x^n)\, dP_\theta(x^n), \quad \text{for all } \mathcal{x}^n \in \mathcal{X}^{(n)}$$

(the integral being the Lebesgue-Stieltjes integral).

Remark 2.1: It is worth emphasizing that we could use a more general definition for the concepts in Definitions 2.4 and 2.5, so that the membership function of each n-tuple $\mathcal{x}^n \in \mathcal{X}^{(n)}$ would be given by the expression $\mu_{\mathcal{x}^n}(x^n)=f(\mu_{\mathcal{x}_1}(x_1), \ldots, \mu_{\mathcal{x}_n}(x_n), \mathcal{x}_1, \ldots, \mathcal{x}_n)$ for all $x^n=(x_1, \ldots, x_n)\in X^n$, f being a function taking on values in the unit interval [0, 1] and satisfying some natural conditions. However, in practice, when we consider examples involving probabilities, one of the most operative and suitable functions f is the product of the first n components. This suitability is confirmed by the fact that the probabilistic independence of the experimental performances implies that (in the Zadeh's sense [16]) of the fuzzy observations from them, whenever f is the product above.

3. AN EXTENSION OF BAYES OPTIMALITY CRITERION

If we consider the problem of testing a concrete fuzzy parametric hypothesis about θ on the basis of sample fuzzy information, a natural way to approach this problem is testing the considered fuzzy hypothesis (denoted by the *null fuzzy hypothesis*) against the fuzzy parametric hypothesis determined by the complementary fuzzy subset of the considered fuzzy hypothesis (*alternative fuzzy hypothesis*).

This problem of testing the null fuzzy hypothesis $\tilde{\theta}_0$ against the alternative fuzzy hypothesis $\tilde{\theta}_1=\bar{\tilde{\theta}}_0$ from fuzzy information (denoted by $(\tilde{\theta}_0 \mid \mathcal{X}^{(n)})$) may be regarded as a special case of the *fuzzy decision problem* (Tanaka *et al.*, [12]), the following are the four essential elements of the problem:

– The *fuzzy state space*, $\tilde{\Theta}=\{\tilde{\theta}_0, \tilde{\theta}_1\}$.

– The *action space* $\mathcal{A}=\{a_0, a_1\}$,

$$a_0=\text{“to accept the null fuzy hypothesis } \tilde{\theta}_0\text{”}$$
$$a_1=\text{“to reject the null fuzzy hypothesis } \tilde{\theta}_0\text{”}$$

– A *loss function*, L: $\tilde{\Theta}\times\mathcal{A}\to[0, +\infty)$, defined by

$$L(\tilde{\theta}_0, a_0)=L(\tilde{\theta}_1, a_1)=0$$
$$L(\tilde{\theta}_1, a_0)=a>0, \qquad L(\tilde{\theta}_0, a_1)=b>0$$

– The information supplied by a fuzzy information system $\mathcal{X}$ associated with $\mathbf{X}$ (or, more precisely, the information supplied by a fuzzy random sample $\mathcal{X}^{(n)}$ from $\mathcal{X}$).

The considered problem of testing may be intuitively described as follows: Given a sample fuzzy information we wish to find a reasonable decision rule (perhaps a randomized decision rule) that leads to the decision of rejecting or not the null hypothesis. This choice will be carried out by means of a *test function of fuzzy hypotheses from fuzzy information*, κ: $\mathcal{X}^{(n)}\to[0, 1]$. From now on, $\mathcal{K}$ will denote the class of all tests functions for the problem $(\tilde{\theta}_0 \mid \mathcal{X}^{(n)})$.

The problem we have just described is essentially a probabilistic-possibilistic extension of the problem of testing statistical hypotheses based on exact information. However, the first problem can be theoretically regarded as a particularization of the second one (this fact is one of the main advantages of the present formulation of the fuzzy problem and the considered approach). Consequently, the principles and procedures for testing statistical hypotheses may be immediately established for the fuzzy case as follows:

In the statistical decision problem a reasonable decision rule should minimize the expected loss, risk function, in a certain sense. Thus, Bayes procedure require the minimization of some expression depending on the risk function.

Similarly, we assume that a reasonable decision rule in the fuzzy decision problem can be characterized by a minimization in a certain sense of the "risk function", which for a test function $\kappa \in \mathcal{K}$ is defined as follows:

DEFINITION 3.1: The mapping $\mathcal{R}$ from $\tilde{\Theta} \times \mathcal{K}$ to $[0, +\infty)$ defined by

$$\mathcal{R}(\tilde{\theta}, \kappa) = \sum_{\mathcal{X}^n \in \mathcal{X}^{(n)}} [L(\tilde{\theta}, a_0)[1 - \kappa(\mathcal{X}^n)] + L(\tilde{\theta}, a_1)\kappa(\mathcal{X}^n)] \mathcal{L}(\mathcal{X}^n; \tilde{\theta})$$

for all $\tilde{\theta} \in \tilde{\Theta}$ and $\kappa \in \mathcal{K}$, is called the *risk function* associated with κ at $\tilde{\theta}$. ($\mathcal{L}$ being the conditional probability distribution on $\mathcal{X}^{(n)}$ given $\tilde{\theta} \in \tilde{\Theta}$).

In order to extend the Bayesian principle of reasonability to the fuzzy framework we must assume that Θ can be endowed with the structure of a probability space, so that there exist *a prior* distribution $g(\theta)$, $\theta \in \Theta$, expressing the additional information about the experimental distribution. On the other hand, when the fuzzy hypothesis $\tilde{\theta}_0$ is a fuzzy event on Θ (that is, the membership function of $\tilde{\theta}_0$, $\mu_{\tilde{\theta}_0}$, is Borel measurable), Zadeh's probabilistic definition, (Zadeh, [16]), let us establish the prior probability distribution on $\tilde{\Theta}$, $\mathcal{G}(\tilde{\theta}) = \int_{\Theta} \mu_{\tilde{\theta}}(\theta)\, dg(\theta)$. This assumptions will permit us to define the prior risk of a test function as

DEFINITION 3.2: The *prior risk* of a test $\kappa \in \mathcal{K}$, with respect to the prior distribution g on Θ, is defined by:

$$\mathcal{R}(\mathcal{G}, \kappa) = \mathcal{G}(\tilde{\theta}_0)\mathcal{R}(\tilde{\theta}_0, \kappa) + \mathcal{G}(\tilde{\theta}_1)\mathcal{R}(\tilde{\theta}_1, \kappa)$$

On the basis of the preceding concepts we can formally describe the proposed extension of the Bayes criterion as follows:

DEFINITION 3.3: Given a prior distribution g on Θ, a test $\kappa^* \in \mathcal{K}$ is said to be an *optimal test* with respect to g, or a *Bayes test for fuzzy hypotheses from fuzzy observations* with respect to g, if: $\mathcal{R}(\mathcal{G}, \kappa^*) \leqq \mathcal{R}(\mathcal{G}, \kappa)$ for all tests $\kappa \in \mathcal{K}$.

Then, we set up two properties of the proposed extension which guarantees that given any finite set of fuzzy events on X we can improve a fuzzy partition of X, so that for any *fuzzy* event in the first set there exists one element in the fuzzy partition leading to the same inference, when an optimal test is applied. We conclude that the orthogonality constraint assumed for the f.i.s. associated with the experiment does not entail a relevant loss of generality for the Bayes criterion established in this paper. In order to set up such properties, we next introduce a previous concept:

DEFINITION 3.4: We say that $\kappa \in \mathcal{K}$ is a *scale invariant test with respect to the membership function of the sample fuzzy information*, whenever there exist a positive constant α such that: $\kappa(\mathcal{X}_1^n)=\kappa(\mathcal{X}_2^n)$ if $\mu_{\mathcal{X}_1^n}(x^n)=\alpha \cdot \mu_{\mathcal{X}_2^n}(x^n)$, for all $x^n \in X^n$.

THEOREM 3.1: *Given a test $\kappa \in \mathcal{K}$ there exists an almost sure scale invariant test with respect to the membership function of the sample fuzzy information, $\tilde{\kappa} \in \mathcal{K}$, such that its risk function equals that of the former test.*

Proof: Let $\mathcal{X}_i^n$ be a sample fuzzy information, $i=1, \ldots, r$, whose membership functions differ from each other almost sure in positive constants $\alpha_1, \ldots, \alpha_{r-1}$, that is: $\mu_{\mathcal{X}_i^n}(x^n)=\alpha_i \cdot \mu_{\mathcal{X}_r^n}(x^n)$ for almost all $x^n \in X^n$, $i=1, \ldots, r-1$ and $\kappa(\mathcal{X}_i^n) \neq \kappa(\mathcal{X}_j^n)$, $i, j=1, \ldots, r$, $i \neq j$.

We can define the test $\tilde{\kappa}$ as follows:

$$\tilde{\kappa}(\mathcal{X}_1^n)= \ldots =\tilde{\kappa}(\mathcal{X}_r^n)=\left[\sum_{i=1}^{r-1} \alpha_i\, \kappa(\mathcal{X}_i^n)+\kappa(\mathcal{X}_r^n)\right] \Big/ \left[1+\sum_{j=1}^{r-1} \alpha_j\right],$$

and $\tilde{\kappa}=\kappa$, otherwise.

Then:

a)

$$\mathcal{L}(\mathcal{X}_i^n; \tilde{\theta})=\begin{cases} \alpha_i\, \mathcal{L}(\mathcal{X}_r^n; \tilde{\theta}), & i=1, \ldots, r-1 \\ \dfrac{\alpha_i}{\alpha_j}\, \mathcal{L}(\mathcal{X}_j^n; \tilde{\theta}), & i, j=1, \ldots, r-1; \quad i \neq j \end{cases}$$

(a.s.) for all $\tilde{\theta} \in \tilde{\Theta}$.

b)

$$\tilde{\kappa}(\mathcal{X}_j^n)\, \mathcal{L}(\mathcal{X}_j^n; \tilde{\theta})=\begin{cases} \displaystyle\sum_{i=1}^{r}\left[\alpha_j \Big/ \left(1+\sum_{l=1}^{r-1} \alpha_l\right)\right] \kappa(\mathcal{X}_i^n)\, \mathcal{L}(\mathcal{X}_i^n; \tilde{\theta}), \\ \qquad j=1, \ldots, r-1 \\ \displaystyle\sum_{i=1}^{r}\left[1 \Big/ \left(1+\sum_{l=1}^{r-1} \alpha_l\right)\right] \kappa(\mathcal{X}_i^n)\, \mathcal{L}(\mathcal{X}_i^n; \tilde{\theta}), \\ \qquad j=r \end{cases}$$

(a.s.) for all $\tilde{\theta} \in \tilde{\Theta}$.

and consequently: $\mathcal{R}(\tilde{\theta}_0, \tilde{\kappa})=\mathcal{R}(\tilde{\theta}_0, \kappa)$.

Analogously, the relation $\mathcal{R}(\tilde{\theta}_1, \tilde{\kappa})=\mathcal{R}(\tilde{\theta}_1, \kappa)$ is proved. ■

THEOREM 3.2: *Given a prior distribution g on* Θ, *an optimal test, if it exists, is almost sure scale invariant with respect to the membership function of the sample fuzzy information.*

Proof: This theorem is obvious in virtue of Theorem 3.1. and the optimality criterion in Definition 3.3. ■

The following theorems connect the optimal tests based on exact information with the optimal test based on a fuzzy information system.

THEOREM 3.3: *Given a test function based on a f.i.s.* $\mathcal{X}$, *there exist a test function of the same fuzzy hypothesis, based on exact information, for which the risk function coincides with the risk function of the first test function.*

Proof: Indeed, if $\mathcal{K}$ denotes the test function of an arbitrary test of a fuzzy hypothesis, which is based on the fuzzy random sample $\mathcal{X}^{(n)}$, we now define a test of this fuzzy hypothesis, based on the random sample $\mathbf{X}^{(n)}$, where the test function is

$$k_{\mathcal{K}}(x^n) = \sum_{\mathcal{X}^n \in \mathcal{X}^{(n)}} \mathcal{K}(\mathcal{X}^n)\, \mu_{\mathcal{X}^n}(x^n) \quad \text{for all } x^n \in X^n$$

Then, for all $\tilde{\theta}$ in $\tilde{\Theta}$,

$$\mathcal{R}(\tilde{\theta}, k_{\mathcal{K}}) = \int_{X^n} [L(\tilde{\theta}, a_0)[1 - k_{\mathcal{K}}(x^n)] + L(\tilde{\theta}, a_1)\, k_{\mathcal{K}}(x^n)]\, dL(x^n; \tilde{\theta}) = \mathcal{R}(\tilde{\theta}, \mathcal{K}) \quad ■$$

THEOREM 3.4: *Given a prior distribution g on* Θ, *if an optimal test for fuzzy hypotheses based on exact information and an optimal test of the same fuzzy hypotheses based on a f.i.s. exist with respect to g, then the first test does not provide more prior risk than the second test.*

Proof: Indeed, Theorem 3.3 implies that the optimality criterion proposed in this section is a restriction of the Bayesian criterion in the subset of the tests of fuzzy hypotheses based on exact information with tests functions $k_{\mathcal{K}}$. Then, minimizing $\mathcal{R}(\mathcal{G}, \mathcal{K})$ is equivalent to minimizing $\mathcal{R}(\mathcal{G}, k_{\mathcal{K}})$, and if the minimum of $\mathcal{R}(\mathcal{G}, k_{\mathcal{K}})$ is achieved it is greater than the minimum of $\mathcal{R}(\mathcal{G}, k)$ over all the tests based on exact information with test function k. ■

4. TESTING OF SIMPLE FUZZY HYPOTHESES FROM FUZZY INFORMATION

Next, there will be stated a theorem providing the structure of the optimal test given by the criterion proposed above.

Theorem 4.1: *Given a prior distribution g on Θ, for testing the null fuzzy hypotheses $\tilde{\theta}_0$ against the alternative fuzzy hypothesis $\tilde{\theta}_1$ from the fuzzy random sample $\mathcal{X}^{(n)}$, there exists a Bayes test with respect to g given by:*

$$\mathcal{K}^*(x^n) = \begin{cases} 1 & \text{if } \mathcal{P}(x^n, \tilde{\theta}_1) > c^* \mathcal{P}(x^n, \tilde{\theta}_0) \\ 0 & \text{otherwise} \end{cases}$$

where $c^* = b/a$.

($\mathcal{P}$ being the joint probability distribution on $\mathcal{X}^{(n)}$ and $\tilde{\Theta}$).

Proof: Of course, whatever the test function $\mathcal{K} \in \mathcal{K}$ may be, we have

$$\mathcal{R}(\mathcal{G}, \mathcal{K}) = \sum_{x^n \in \mathcal{X}^{(n)}} [b\,\mathcal{P}(x^n, \tilde{\theta}_0) - a\,\mathcal{P}(x^n, \tilde{\theta}_1)]\,\mathcal{K}(x^n) + \sum_{x^n \in \mathcal{X}^{(n)}} a\,\mathcal{P}(x^n, \tilde{\theta}_1)$$

Then, the minimum of $\mathcal{R}(\mathcal{G}, \mathcal{K})$ over all $\mathcal{K} \in \mathcal{K}$ is achieved by the test given by

$$\mathcal{K}^*(x^n) = \begin{cases} 1 & \text{if } b\,\mathcal{P}(x^n, \tilde{\theta}_0) - a\,\mathcal{P}(x^n, \tilde{\theta}_1) < 0 \\ 0 & \text{otherwise} \quad \blacksquare \end{cases}$$

The following theorems establish other properties of the Bayes tests.

Theorem 4.2: *Let $\mathcal{X}$ be a f.i.s. associated with* **X**, *and let g be a prior distribution on Θ, and assume that the null hypothesis $\tilde{\theta}_0$ is a completely fuzzy event (fuzzy event with constant membership function). Then, the Bayes test function with respect to g is independent of the available sample fuzzy information.*

Proof: If $\mu_{\tilde{\theta}_0}(\theta) = c(\tilde{\theta}_0)$, $\forall\, \theta \in \Theta$, we have $\mathcal{P}(x^n, \tilde{\theta}_0) = c(\tilde{\theta}_0) \,.\, \mathcal{P}(x^n)$ and $\mathcal{P}(x^n, \tilde{\theta}_1) = (1 - c(\tilde{\theta}_0)) \,.\, \mathcal{P}(x^n)$, $\forall\, x^n \in \mathcal{X}^{(n)}$. Then (Theorem 4.1), the Bayes test function is given by (a.s. ($\mathcal{P}$))

$$\mathcal{K}^*(x^n) = \begin{cases} 1 & \text{if } c(\tilde{\theta}_0) < (1 + c^*)^{-1} \\ 0 & \text{otherwise} \quad \blacksquare \end{cases}$$

Theorem 4.3: *Let g be a prior distribution on Θ, and let $\mathcal{X}^{(n)}$ be a fuzzy random sample on X^n that only provides completely fuzzy information (or uniformly imprecise information). Then, the Bayes test function for fuzzy hypotheses with respect to g is independent of the available sample fuzzy information.*

Proof: If $\mu_{\mathcal{X}^n}(x^n) = c(\mathcal{X}^n) > 0$ for all $x^n \in X^n$, $\mathcal{X}^n \in \mathcal{X}^{(n)}$, we have $\mathcal{P}(\mathcal{X}^n, \tilde{\theta}) = c(\mathcal{X}^n) \, . \, \mathcal{G}(\tilde{\theta})$, $\forall \mathcal{X}^n \in \mathcal{X}^{(n)}$ and $\tilde{\theta} \in \tilde{\Theta}$. Then (Theorem 4.1), the Bayes test function is given by

$$\mathcal{K}^*(\mathcal{X}^n) = \begin{cases} 1 & \text{if } \mathcal{G}(\tilde{\theta}_0) < (1+c^*)^{-1} \\ 0 & \text{otherwise} \end{cases} \blacksquare$$

COROLLARY 4.1: *a) Let* $\mathcal{X}^{(n)}$, $\mathcal{Y}^{(n)}$ *be fuzzy random samples on* X^n *that only provide completely fuzzy information, and let* $\mathcal{K}^*(\mathcal{X}^{(n)})$, $\mathcal{K}^*(\mathcal{Y}^{(n)})$ *be Bayes test functions for testing* $\tilde{\theta}_0$ *against* $\tilde{\theta}_1$ *based on* $\mathcal{X}^{(n)}$ *and* $\mathcal{Y}^{(n)}$ *respectively, with respect to a prior distribution g on* Θ. *Then,* $\mathcal{K}^*(\mathcal{X}^{(n)}) \equiv \mathcal{K}^*(\mathcal{Y}^{(n)})$.

b) Let $\mathcal{K}^*(\mathcal{X}^{(n)})$, $\mathcal{K}_0^*$, *be Bayes test functions for testing* $\tilde{\theta}_0$ *against* $\tilde{\theta}_1$ *with respect to a prior distribution g on* Θ, *based on a fuzzy random sample* $\mathcal{X}^{(n)}$ *that only provides completely fuzzy information and without sampling, respectively. Then,* $\mathcal{K}^*(\mathcal{X}^{(n)}) \equiv \mathcal{K}_0^*$.

c) Given a prior distribution on Θ, *a Bayes test function from an arbitrary fuzzy random sample provides less Bayes risk than a Bayes test function from a fuzzy random sample that only provides completely fuzzy information.*

Proof: This corollary is obvious, in virtue of Theorem 4.3 and the optimality criterion in Definition 3.3. ■

5. EXAMPLES

EXAMPLE 5.1: Consider a large population of insects, a proportion p of which is infected with a given virus. From further information we assume that we know that the parameter p follows a uniform distribution on [0, 1].

In order to test the null hypothesis "p is small" (identified with a fuzzy event $\mathcal{P}_0$ on $\mathbf{P}$=[0, 1] whose membership function is defined by $\mu_{\mathcal{P}_0}(p) = 1 - p$, $p \in \mathbf{P}$), against the alternative fuzzy hypothesis "p is not small" (identified with a fuzzy event $\mathcal{P}_1$ on $\mathbf{P}$ whose membership function is defined by $\mu_{\mathcal{P}_1}(p) = 1 - \mu_{\mathcal{P}_0}(p)$, $p \in \mathbf{P}$), we take a sample of 20 insects an examine each insect independently for presence of virus. Suppose we do not have a precise mechanism for an exact discrimination between the presence and the absence of virus, but rather they can inform us whether $\mathcal{X}_1$="with much certainty the insect presents infection" or else $\mathcal{X}_2$="with much certainty the insect does not present infection".

A proper mathematical model for this problem takes up the Bernoulli experiment $\mathbf{X}$ associated with the presence of virus an identifies the information $\mathcal{X}_1$ with a fuzzy set on $X=\{0, 1\}$ (for instance, one where the membership function is $\mu_{\mathcal{X}_1}(0)=0.1$, $\mu_{\mathcal{X}_1}(1)=0.9$) and the information $\mathcal{X}_2$ with another fuzzy set on $X=\{0, 1\}$ (for instance, one where the membership is $\mu_{\mathcal{X}_2}(x) = 1 - \mu_{\mathcal{X}_1}(x)$, $x=0, 1$).

If we consider the loss function with $a=1$, $b=2$ and the membership function of each sample fuzzy information of the type $\mu_{\mathcal{X}^{20}}(x^{20}) = \prod_{i=1}^{20} \mu_{\mathcal{X}_i}(x_i)$ then, Theorem 4.1 leads to the following test function

$$\kappa^*(\mathcal{X}^{20}) = \begin{cases} 1 & \text{if } v=13, \ldots, 20 \\ 0 & \text{otherwise} \end{cases}$$

(v=the observed frequencie of $\mathcal{X}_1$).

Example 5.2: Consider the example 5.1 and we assume that the experimental observation is exact, that is, $\mathcal{X}_1 \equiv 1$ ($\mu_{\mathcal{X}_1}(0)=0$, $\mu_{\mathcal{X}_1}(1)=1$) and $\mathcal{X}_2 \equiv 0$ ($\mu_{\mathcal{X}_2}(0)=1$, $\mu_{\mathcal{X}_2}(1)=0$).

Then, the particularization of theorem 4.1 leads to the test function

$$\kappa^*(\mathcal{X}^{20}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{20} \mathcal{X}_i > 41/3 \\ 0 & \text{otherwise} \end{cases}$$

6. CONCLUDING REMARKS

An alternative different way (*see* [4]) for solving a Bayesian testing fuzzy hypothesis when the information supplied by the experimental sampling is exact, may be to establish an optimal *fuzzy acceptance region.*

The particularization of results in this paper to the case when the available information is exact, leads to an optimal *nonfuzzy acceptance region* (*see* example 5.2), and consequently procedures and results for both approaches cannot be compared.

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