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# ON THE COMPARISON OF PURE JUMP PROCESSES (*)

by B. BASSAN [1] and M. SCARSINI [2]

Abstract. — *We define some orderings on the class of pure jump processes and we study their mutual implications, in general and for some relevant subclasses of processes, such as compound Poisson. Some of these orderings are extensions of known orderings for point processes, while some others are new and take explicitly into account the variable magnitude of jumps. Preservation of these orderings under superposition and thinning is studied.*

Keywords : Stochastic ordering; Poisson processes; thinning, superposition.

Résumé. — *Nous définissons quelques ordres sur la classe des processus à sauts purs et nous étudions leurs implications mutuelles, en général aussi bien que pour des sous-classes intéressantes. Certains de ces ordres sont des extensions d'ordres connus par les processus ponctuels tandis que d'autres sont nouveaux et tiennent explicitement compte de la grandeur variable des sauts. On étudie la conservation de ces ordres sous diverses conditions.*

## 1. INTRODUCTION

Various applications of point processes require comparisons with respect to some ordering. Several of these orderings have been considered by different authors. *See* for example Deng (1985 *a*), Deng (1985 *b*), Whitt (1981), Rolski and Szekli (1989).

The key point of many of the orderings considered in the literature is the duality between a point process and the associated counting process. The use of point processes is not suitable for some applications, though, and general pure jump processes are required. Think, for instance, of the risk models of actuarial mathematics, where a jump represents a claim and the magnitude of the jump represents the monetary amount of the claim. *See*, for example, Gerber (1979), Grandell (1991) and references therein.

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Some of the orderings defined for point processes can be used also in this more general case, but the orderings that exploit the duality between point and counting processes can no longer be used and new orderings have to be defined.

Our analysis is restricted to right continuous processes with a finite number of jumps in any bounded interval. This allows us to represent a pure jump process as the difference of two increasing pure jump processes. Then we consider the counting process and the sequence of jumps associated to each of these increasing processes, and we define orderings for the original process in terms of these more tractable entities.

For simplicity, we begin our analysis considering increasing processes, and then we extend it to general pure jump processes. Finally, we provide some preservation results under thinning and superposition.

Related results can be found in Deng (1985*b*) who studied compound Poisson processes, and in Gal'chuk (1982) and Bassan, Çinlar and Scarsini (1989) who studied pure jump processes arising from stochastic differential equations of jump type.

## 2. INCREASING PROCESSES

Let $\mathcal{N}$ be the class of pure jump processes $N$ with a finite number of jumps in every bounded interval and such that $N(0)=0$, and let $\mathcal{N}_{\text{inc}}$ be the class of increasing processes in $\mathcal{N}$. It should be understood that the word "increasing" is used in the weak sense. Let also $\mathcal{A}\subset\mathcal{N}_{\text{inc}}$ be the class of counting processes.

For $N\in\mathcal{N}_{\text{inc}}$, let $A=\{A(t)\,|\,t\in\mathbb{R}_+\}$ be the counting process associated to $N$, *i. e.* the process such that

$$\begin{aligned} &A(0)=0 \\ &A(t)-A(t^-)=0 \quad \text{if} \quad N(t)-N(t^-)=0 \\ &A(t)-A(t^-)=1 \quad \text{if} \quad N(t)-N(t^-)\neq 0 \end{aligned}$$

where, clearly, $N(t^-)=\lim_{s\uparrow t} N(s)$. Let $T=\{T(k)\,|\,k\in\mathbb{N}\}$ be the associated sequence of event epochs, *i. e.*

$$T(n)=\inf\{s\geqq 0\,|\,A(s)\geqq n\};$$

furthermore, let $Y=\{Y(n) \mid n \in \mathbb{N}\}$ be the sequence of jump magnitudes, *i.e.*

$$Y(n)=N(T(n))-N(T(n)^-)$$

and let $W=\{W(n) \mid n \in \mathbb{N}\}$ be the sequence of interarrival times, *i.e.*

$$W(n)=T(n)-T(n-1)$$

with $T(0)=0$. A process $N \in \mathcal{N}$ will be written as the difference of two increasing pure jump processes $N^{\uparrow}$ and $N^{\downarrow}$, defined as follows:

$$N^{\uparrow}(t)=\sum_{n=1}^{A(t)}[N(T(n))-N(T(n)^-)]_+=\sum_{n=1}^{A(t)}[Y(n)]_+$$

$$N^{\downarrow}(t)=\sum_{n=1}^{A(t)}[N(T(n))-N(T(n)^-)]_-=\sum_{n=1}^{A(t)}[Y(n)]_-$$

where $a_+=\max(a, 0)$ and $a_-=-\min(a, 0)$.

Then we define $A^{\uparrow}(A^{\downarrow})$ as the counting process associated to $N^{\uparrow}(N^{\downarrow})$, and $T^{\uparrow}(T^{\downarrow})$ as the associated sequence of event epochs. Writing $Y^{\uparrow}(n)=N^{\uparrow}(T^{\uparrow}(n))-N^{\uparrow}(T^{\uparrow}(n)^-)$ and $Y^{\downarrow}(n)=N^{\downarrow}(T^{\downarrow}(n))-N^{\downarrow}(T^{\downarrow}(n)^-)$ we can express the above relations as

$$N^{\uparrow}(t)=\sum_{n=1}^{A^{\uparrow}(t)} Y^{\uparrow}(n)$$

and

$$N^{\downarrow}(t)=\sum_{n=1}^{A^{\downarrow}(t)} Y^{\downarrow}(n).$$

Now we define some orderings for processes in $\mathcal{N}_{\text{inc}}$. We denote by $\mathcal{L}(N)$ the law of the process $N$.

DEFINITION 2.1: For $N_1, N_2 \in \mathcal{N}_{\text{inc}}$, we write $N_1 \leqq_\alpha N_2$ if there exist processes $\tilde{N}_1$ and $\tilde{N}_2$ on a common probability space such that

$$\mathcal{L}(\tilde{N}_i)=\mathcal{L}(N_i), \qquad i=1, 2 \tag{2.1}$$

and

$$\tilde{N}_1(t)-\tilde{N}_1(t^-) \leqq \tilde{N}_2(t)-\tilde{N}_2(t^-) \quad \text{a.s.}, \qquad \forall t \in \mathbb{R}_+.$$

*Remark* 2.1: If $N_1 \leqq_\alpha N_2$ then

$$\{\tilde{T}_1(n) \mid n \in \mathbb{N}\} \subset \{\tilde{T}_2(n) \mid n \in \mathbb{N}\} \quad \text{a.s.}$$

where clearly $\{\tilde{T}_i(n) \mid n \in \mathbb{N}\}$ is the sequence of event epochs associated to $\tilde{N}_i$, $i=1, 2$.

*Remark* 2.2: For processes in $\mathscr{A}$, the ordering $\leqq_\alpha$ corresponds to ordering $\leqq_2$ of Whitt (1981) and $\leqq_{\text{inc}}$ of Deng (1985 *a*).

DEFINITION 2.2: For $N_1$, $N_2 \in \mathscr{N}_{\text{inc}}$, we write $N_1 \leqq_\beta N_2$ if there exist two processes $\tilde{N}_1$ and $\tilde{N}_2$ on a common probability space such that (2.1) holds and

$$\text{Prob}(\tilde{N}_1(t) \leqq \tilde{N}_2(t), \ \forall t \in \mathbb{R}_+) = 1.$$

*Remark* 2.3: If $N_1$, $N_2 \in \mathscr{A}$, this ordering coincides with $\leqq_4$ of Whitt (1981) and with $\leqq_n$ of Deng (1985 *a*).

*Remark* 2.4: By well known results [*see* Kamae, Krengel and O'Brien (1977)], $\leqq_\beta$ is equivalent to the usual stochastic ordering $\leqq_{\text{st}}$. Recall that, given two random variables $X_1$ and $X_2$ with values in a partially ordered Polish space $Z$, we say that $X_1 \leqq_{\text{st}} X_2$ if $E\varphi(X_1) \leqq E\varphi(X_2)$ for every increasing functional $\varphi : Z \to \mathbb{R}$. Notice that $X_1 \leqq_{\text{st}} X_2$ if and only if $\text{Prob}(X_1 \in B) \leqq \text{Prob}(X_2 \in B)$ for every upper set $B$, *i.e.* for every set $B$ such that

$$x \in B, \qquad x \leqq y \quad \Rightarrow \quad y \in B.$$

Thus, $N_1 \leqq_\beta N_2$ if and only if $\text{Prob}(N_1 \in B) \leqq \text{Prob}(N_2 \in B)$ for every $B \in \mathbb{R}^{\mathbb{R}_+}$ such that

$$f \in B, \qquad f(t) \leqq g(t), \qquad \forall t \in \mathbb{R}_+ \quad \Rightarrow \quad g \in B.$$

A generalization of ordering $\leqq_\beta$ can be given as follows: for $N_1$, $N_2 \in \mathscr{N}_{\text{inc}}$, write $N_1 \leqq_{\mathscr{F}} N_2$ if and only if $E\varphi(N_1) \leqq E\varphi(N_2)$ for every $\varphi \in \mathscr{F}$, where $\mathscr{F}$ is a suitable family of functionals defined on $\mathscr{N}_{\text{inc}}$. For example, $\mathscr{F}$ may be taken as the class of increasing convex functionals. Obviously, if $\mathscr{F} \subset \mathscr{G}$, then $\leqq_{\mathscr{G}}$ implies $\leqq_{\mathscr{F}}$.

Before we define the following order we need to introduce some notation. Let $\mathscr{H} = \{\mathscr{H}_t \mid t \in \mathbb{R}_+\}$ be a filtration and let

$$F^{\mathscr{H}}(u, t) = \text{Prob}([T(A(t)+1) - t] \leqq u \mid \mathscr{H}_t)$$

have conditional failure rate $\lambda^{\mathscr{H}}$ for each $u$ and $t$, a. s.:

$$\lambda^{\mathscr{H}}(u, t) = \frac{\partial F^{\mathscr{H}}(u, t)}{\partial u} \frac{1}{1 - F^{\mathscr{H}}(u, t)},$$

*i. e.*

$$\lambda^{\mathscr{H}}(u, t)\, du = \text{Prob}\left([T(A(t)+1)-t] \in (u, u+du) \mid [T(A(t)+1)-t] \geqq u, \mathscr{H}_t\right). \quad (2.2)$$

When $\mathscr{H}$ is the natural filtration, *i. e.* $\mathscr{H}_t = \sigma(\{A(s) \mid s \leqq t\})$ [or, shortly, $\mathscr{H} = \sigma(A)$], we omit for simplicity the superscript $\mathscr{H}$.

Whitt (1981) defined an ordering $\leqq_1$ for counting processes as follows: $A_1 \leqq_1 A_2$ if there exists a function $t \mapsto \rho(t)$ such that

$$\lambda_1(u, t) \leqq \rho(t) \leqq \lambda_2(u, t), \qquad \forall t \in \mathbb{R}_+, \quad \forall u \in \mathbb{R}_+.$$

This ordering was called $\leqq_f$ by Deng (1985 *a*).

As Whitt (1981) points out, regular conditional probabilities exist under the above assumptions, so that $\leqq_1$ is well defined.

Definition 2.3: For $N_1$, $N_2 \in \mathscr{N}_{\text{inc}}$, we write $N_1 \leqq_\gamma N_2$ if and only if $A_1 \leqq_1 A_2$ and $Y_1 \leqq_{\text{st}} Y_2$.

*Remark* 2.5: As Whitt (1981) shows, ordering $\leqq_1$ reflects the fact that the process $A_1$ can be obtained from $A_2$ via thinning, that is, if $A_2$ has a jump at time $t$, then $A_1$ has a jump at time $t$ with probability $1-p(t)$ linked to the structure of the processes, and $A_1$ jumps only when $A_2$ jumps. *See* also Miller (1979).

Definition 2.4: For $N_1$, $N_2 \in \mathscr{N}_{\text{inc}}$, we write $N_1 \leqq_\delta N_2$ if and only if there exist two processes $\tilde{N}_1$ and $\tilde{N}_2$ on a common probability space such that (2.1) holds and

$$\tilde{W}_1(n) \geqq \tilde{W}_2(n) \quad \text{a. s.}, \qquad \forall n \in \mathbb{N},$$
$$\tilde{Y}_1(n) \leqq \tilde{Y}_2(n) \quad \text{a. s.}, \qquad \forall n \in \mathbb{N}.$$

Definition 2.5: For $N_1$, $N_2 \in \mathscr{N}_{\text{inc}}$, we write $N_1 \leqq_\varepsilon N_2$ if and only if there exist two processes $\tilde{N}_1$ and $\tilde{N}_2$ on a common probability space such that (2.1) holds and

$$\tilde{A}_1(t) \leqq \tilde{A}_2(t) \quad \text{a. s.}, \qquad \forall t \in \mathbb{R}_+, \quad (2.3)$$

$$\tilde{Y}_1(n) \leqq \tilde{Y}_2(n) \quad \text{a.s.}, \qquad \forall n \in \mathbb{N}. \tag{2.4}$$

We will now prove some relationships among the orderings introduced so far. The simbol $\subset$ will denote implication between orderings; thus $\leqq_i \subset \leqq_j$ means: $N_1 \leqq_i N_2 \Rightarrow N_1 \leqq_j N_2$.

PROPOSITION 2.1: *The following implications hold:*

(i) $\leqq_\delta \subset \leqq_\varepsilon$;

(ii) *each of the orderings* $\leqq_\alpha$, $\leqq_\delta$ *and* $\leqq_\varepsilon$ *implies ordering* $\leqq_\beta$.

*Proof:* (i) The implication

$$\tilde{W}_1(n) \geqq \tilde{W}_2(n), \qquad \forall n \in \mathbb{N} \quad \Rightarrow \quad \tilde{A}_1(t) \leqq \tilde{A}_2(t), \qquad \forall t \in \mathbb{R}_+$$

is obtained easily from the relations $T(n) = \sum_{k=1}^{n-1} W(k)$ and $\{T(n) \leqq t\} = \{A(t) \geqq n\}$. *See* also Whitt (1981) and Deng (1985*a*).

(ii) The implications $\leqq_\alpha \subset \leqq_\beta$ and $\leqq_\varepsilon \subset \leqq_\beta$ follow easily from the relation

$$\tilde{N}_i(t) = \sum_{n=1}^{\tilde{A}_i(t)} [\tilde{N}_i(\tilde{T}_i(n)) - \tilde{N}_i(\tilde{T}_i(n)^-)] = \sum_{n=1}^{\tilde{A}_i(t)} \tilde{Y}_i(n). \tag{2.5}$$

The implication $\leqq_\delta \subset \leqq_\beta$ follows from what proved above. ■

*Remark* 2.6: Condition (2.1) plays a crucial role in the implication $\leqq_\varepsilon \subset \leqq_\beta$. For example, consider two processes $N_1$ and $N_2$ such that $A_1 = A_2$ are homogeneous Poisson processes, and

$$Y_1(n) = \begin{cases} 2, & \text{if } W_n > \text{median}(W); \\ 1, & \text{if } W_n \leqq \text{median}(W). \end{cases}$$

$$Y_2(n) = \begin{cases} 1, & \text{if } Y_1(n) = 2; \\ 2, & \text{if } Y_1(n) = 1. \end{cases}$$

*i.e.* $Y_2(n) = 3 - Y_1(n)$. In this case, (2.3) and (2.4) hold, since $Y_1 =_{st} Y_2$, but the two processes $N_1$ and $N_2$ are not ordered by $\leqq_\beta$.

This counterexample shows that the weaker order $\leqq_\gamma$ [whose definition does not involve (2.1)] does not imply $\leqq_\beta$.

## 3. THE GENERAL CASE

In this section we consider pure jump processes that are not necessarily increasing. Some of the orderings previously introduced, namely $\leqq_\alpha$, $\leqq_\beta$ and $\leqq_{\mathscr{F}}$, are well defined also on this larger class of processes. But all the orderings described in the previous section have a natural counterpart in this wider setup that can be constructed according to the following procedure: for $N_1$, $N_2 \in \mathscr{N}$ and $j \in \{a, \beta, \gamma, \delta, \varepsilon, \mathscr{F}\}$, write

$$N_1 \leqq_{j^*} N_2$$

if and only if

$$N_1^\uparrow \leqq_j N_2^\uparrow \qquad \text{and} \qquad N_2^\downarrow \leqq_j N_1^\downarrow .$$

PROPOSITION 3.1: $\leqq_\alpha \subset \leqq_{\alpha^*}$.

*Proof.* Suppose that $N_1 \leqq_\alpha N_2$, and consider the following three cases:
(*a*) $\tilde{N}_1(t) - \tilde{N}_1(t^-) \geqq 0$. Then, for $i = 1, 2$,

$$\tilde{N}_i(t) - \tilde{N}_i(t^-) = (\tilde{N}_i(t) - \tilde{N}_i(t^-))_+$$

and

$$(\tilde{N}_i(t) - \tilde{N}_i(t^-))_- = 0,$$

so that

$$(\tilde{N}_1(t) - \tilde{N}_1(t^-))_+ \leqq (\tilde{N}_2(t) - \tilde{N}_2(t^-))_+$$

*i.e.*

$$\tilde{N}_1^\uparrow(t) - \tilde{N}_1^\uparrow(t^-) \leqq \tilde{N}_2^\uparrow(t) - \tilde{N}_2^\uparrow(t^-)$$

and

$$\tilde{N}_1^\downarrow(t) - \tilde{N}_1^\downarrow(t^-) = \tilde{N}_2^\downarrow(t) - \tilde{N}_2^\downarrow(t^-) = 0.$$

(*b*) $\tilde{N}_2(t) - \tilde{N}_2(t^-) \leqq 0$. This implies that

$$(\tilde{N}_1(t) - \tilde{N}_1(t^-))_- \geqq (\tilde{N}_2(t) - \tilde{N}_2(t^-))_-,$$

*i.e.*

$$\tilde{N}_1^\downarrow(t) - \tilde{N}_1^\downarrow(t^-) \geqq \tilde{N}_2^\downarrow(t) - \tilde{N}_2^\downarrow(t^-)$$

and

$$\tilde{N}_1^{\uparrow}(t)-\tilde{N}_1^{\uparrow}(t^-)=\tilde{N}_2^{\uparrow}(t)-\tilde{N}_2^{\uparrow}(t^-)=0.$$

(*c*) $\tilde{N}_1(t)-\tilde{N}_1(t^-)\leqq 0\leqq \tilde{N}_2(t)-\tilde{N}_2(t^-)$. This implies that

$$\tilde{N}_1^{\uparrow}(t)-\tilde{N}_1^{\uparrow}(t^-)=0\leqq \tilde{N}_2^{\uparrow}(t)-\tilde{N}_2^{\uparrow}(t^-)$$

and

$$\tilde{N}_1^{\downarrow}(t)-\tilde{N}_1^{\downarrow}(t^-)\geqq 0=\tilde{N}_2^{\downarrow}(t)-\tilde{N}_2^{\downarrow}(t^-).$$

Therefore, in all these three cases, $N_1\leqq_{\alpha^*}N_2$. ■

PROPOSITION 3.2: *For $i, j\in\{\alpha, \beta, \gamma, \delta, \varepsilon\}$, $\leqq_j\subset\leqq_i$ on $\mathcal{N}_{\text{inc}}$ if and only if $\leqq_{j^*}\subset\leqq_{i^*}$ on $\mathcal{N}$.*

*Proof:* Assume that $\leqq_j\subset\leqq_i$ on $\mathcal{N}_{\text{inc}}$. If $N_1\leqq_{j^*}N_2$, then $N_1^{\uparrow}\leqq_j N_2^{\uparrow}$ and $N_2^{\downarrow}\leqq_j N_1^{\downarrow}$. It follows that $N_1^{\uparrow}\leqq_i N_2^{\uparrow}$ and $N_2^{\downarrow}\leqq_i N_1^{\downarrow}$, *i.e.* $N_1\leqq_{i^*}N_2$.

Conversely, assume that $\leqq_{j^*}\subset\leqq_{i^*}$ on $\mathcal{N}$; by restricting this implication to processes in $\mathcal{N}_{\text{inc}}$ we obtain the desired relation $\leqq_j\subset\leqq_i$. ■

We now consider special classes of processes for which more implications hold among the orderings previously introduced. Let $\mathcal{N}_{\text{hcP}}$ ($\mathcal{N}_{\text{icP}}$) be the class of homogeneous (inhomogeneous) compound Poisson processes; let also $\mathcal{N}_{\text{cr}}$ be the class of processes $\mathcal{N}\in\mathcal{N}$ whose associated counting processes $A$ are renewal processes and whose jumps are i. i. d. and independent of $A$.

PROPOSITION 3.3: *For processes in $\mathcal{N}_{\text{cr}}$, the following implications hold:*

(i) $\leqq_{\alpha^*}\subset\leqq_{\delta^*}$;

(ii) $\leqq_{\varepsilon^*}$ *is equivalent to* $\leqq_{\delta^*}$;

(iii) $\leqq_{\gamma^*}\subset\leqq_{\alpha^*}$;

(iv) $\leqq_{\gamma^*}\subset\leqq_{\varepsilon^*}$.

*Proof:* (i) If $N_1\leqq_{\alpha^*}N_2$, then $N_1^{\uparrow}\leqq_{\alpha}N_2^{\uparrow}$, *i.e.*

$$\tilde{N}_1^{\uparrow}(t)-\tilde{N}_1^{\uparrow}(t^-)\leqq \tilde{N}_2^{\uparrow}(t)-\tilde{N}_2^{\uparrow}(t^-),\qquad \forall t\in\mathbb{R}_+. \tag{3.1}$$

This implies

$$Y_1^{\uparrow}(n)\leqq_{\text{st}}Y_2^{\uparrow}(m) \tag{3.2}$$

for some $m, n\in\mathbb{N}$. Since the jumps are i. i. d., (3.2) implies

$$Y_1^{\uparrow}\leqq_{\text{st}}Y_2^{\uparrow}. \tag{3.3}$$

Moreover, (3.1) implies

$$A_1^{\uparrow} \leqq_{\alpha} A_2^{\uparrow}. \tag{3.4}$$

Now, $A_1^{\uparrow}$ and $A_2^{\uparrow}$ are again renewal processes, since they are obtained by $A_1$ and $A_2$ via independent thinning. In this case, (3.4) implies $A_1^{\uparrow} \leqq_{\delta} A_2^{\uparrow}$ [*see* Deng (1985 *a*)], which, together with (3.3) entails $N_1^{\uparrow} \leqq_{\delta} N_2^{\uparrow}$.

Repeating the argument for $N_1^{\downarrow}$ and $N_2^{\downarrow}$, we obtain $N_1 \leqq_{\delta^*} N_2$.

(ii) Since $\leqq_{\delta^*} \subset \leqq_{\varepsilon^*}$ is always true, all we need to show is that $\leqq_{\varepsilon^*} \subset \leqq_{\delta^*}$. But this is immediate, since for renewal processes $A_1 \leqq_{\varepsilon} A_2$ implies $A_1 \leqq_{\delta} A_2$ [*see* Deng (1985 *a*)].

(iii) If $N_1 \leqq_{\gamma^*} N_2$, then $A_1^{\uparrow} \leqq_1 A_2^{\uparrow}$ and $\bar{A}_2^{\downarrow} \leqq_1 A_1^{\downarrow}$. This implies $A_1^{\uparrow} \leqq_{\alpha} A_2^{\uparrow}$ and $A_2^{\downarrow} \leqq_{\alpha} A_1^{\downarrow}$ [*see* Deng (1985 *a*)]. Since the jumps are i.i.d., $Y_1^{\uparrow} \leqq_{st} Y_2^{\uparrow}$ and $Y_2^{\downarrow} \leqq_{st} Y_1^{\downarrow}$, then the result follows.

(iv) Whitt (1981) shows that $A_1 \leqq_1 A_2$ implies that $\tilde{A}_1 \leqq \tilde{A}_2$, $\forall t \in \mathbb{R}_+$ a.s. Furthermore (*see* Remark 2.4) $Y_1 \leqq_{st} Y_2$ is equivalent to $\tilde{Y}_1(n) \leqq \tilde{Y}_2(n)$, $\forall n \in \mathbb{N}$, where $\tilde{Y}_1$, $\tilde{Y}_2$, $\tilde{A}_1$ and $\tilde{A}_2$ can be defined on the same probability space, due to the independence of the counting processes and the sequences of jump magnitudes. ■

*Remark* 3.1: The implications $\leqq_{\gamma^*} \subset \leqq_{\alpha^*}$ and $\leqq_{\alpha^*} \subset \leqq_{\varepsilon^*}$ actually require only that the processes involved have i.i.d. jumps.

PROPOSITION 3.4: *For inhomogeneous compound Poisson processes, the following implications hold:*

(i) $\leqq_{\beta^*} \subset \leqq_{\varepsilon^*}$;

(ii) $\leqq_{\gamma^*} \subset \leqq_{\delta^*}$.

*Proof:* If $N \in \mathcal{N}_{icP}$, then $N^{\uparrow} \in \mathcal{N}_{icP} \cap \mathcal{N}_{inc}$, since $A^{\uparrow}$ is obtained from $A$ via independent thinning. In particular

$$\lambda^{\uparrow}(t) = \lambda(t) . [1 - \text{Prob}(\{Y_1(1) < 0\})].$$

Clearly, also $N^{\downarrow} \in \mathcal{N}_{icP} \cap \mathcal{N}_{inc}$. Therefore, it follows from arguments analogous to those used in Proposition 3.2 that it is enough to prove the desired implications only for processes in $\mathcal{N}_{icP} \cap \mathcal{N}_{inc}$.

(i) First, we show that, if $N_1 \leqq_{\beta} N_2$, then $Y_1 \leqq_{st} Y_2$. With probability one, $\tilde{T}_2(1) \leqq \tilde{T}_1(1)$. Furthermore, there is a positive probability that the process $\tilde{N}_2$ has no jumps on the interval $(\tilde{T}_2(1), \tilde{T}_1(1)]$. If the relation $Y_1 \leqq_{st} Y_2$ did not hold, then, for any choice of $\tilde{N}_1$, $\tilde{N}_2$, there would be a positive probability that $\tilde{Y}_1(1) > \tilde{Y}_2(1)$, and therefore there would be a positive probability that

$\tilde{N}_1(\tilde{T}_1(1)) > \tilde{N}_2(\tilde{T}_1(1))$, which would contradict the assumption that $N_1 \leqq_\beta N_2$.

Next, we show that $N_1 \leqq_\beta N_2$ implies that

$$\tilde{A}_1(t) \leqq \tilde{A}_2(t), \qquad \forall\, t \in \mathbb{R}_+ \quad \text{a.s.} \tag{3.5}$$

Suppose, by contradiction, that for any choice of $\tilde{N}_1$ and $\tilde{N}_2$,

$$\operatorname{Prob}(\exists\, t \in \mathbb{R}_+ \mid \tilde{A}_1(t) > \tilde{A}_2(t)) > 0. \tag{3.6}$$

Since the paths of counting processes are right continuous, (3.6) is equivalent to

$$\operatorname{Prob}\Big(\bigcup_{t \in \mathcal{Q}_+} \{\tilde{A}_1(t) > \tilde{A}_2(t)\}\Big) > 0$$

where $\mathcal{Q}_+$ is the set of nonnegative rational numbers. Then

$$\sum_{t \in \mathcal{Q}_+} \operatorname{Prob}(\tilde{A}_1(t) > \tilde{A}_2(t)) > 0$$

and hence there exists a positive rational $t_0$ such that

$$\operatorname{Prob}(\tilde{A}_1(t_0) > \tilde{A}_2(t_0)) > 0.$$

Since $\forall\, t \in \mathbb{R}_+$ one has Prob $(\tilde{A}_2(t) = 0) > 0$, then

$$\operatorname{Prob}(0 = \tilde{N}_2(t_0) < \tilde{N}_1(t_0)) > 0$$

which contradicts the assumption that $N_1 \leqq_\beta N_2$.

(ii) Trivial, since for inhomogeneous Poisson processes $A_1 \leqq_\gamma A_2$ implies $A_1 \leqq_\delta A_2$ [*see* Deng (1985 *a*)]. ■

Proposition 3.5: *For homogeneous compound Poisson processes the orderings* $\leqq_{\alpha^*}$, $\leqq_{\beta^*}$, $\leqq_{\gamma^*}$, $\leqq_{\delta^*}$ *and* $\leqq_{\varepsilon^*}$ *are equivalent.*

*Proof:* Since $\mathcal{N}_{\mathrm{hcP}} \subset \mathcal{N}_{\mathrm{icP}} \cap \mathcal{N}_{\mathrm{cr}}$, we have to prove only the following implications: $\leqq_{\alpha^*} \subset \leqq_{\gamma^*}$; $\leqq_{\delta^*} \subset \leqq_{\alpha^*}$.

(i) $\leqq_{\alpha^*} \subset \leqq_{\gamma^*}$. Proceeding as in the first part of the proof of Proposition 3.3, one shows that $N_1 \leqq_{\alpha^*} N_2$ implies $Y_1^\uparrow \leqq_{\mathrm{st}} Y_2^\uparrow$ and $A_1^\uparrow \leqq_\alpha A_2^\uparrow$. Now, $A_1^\uparrow$ and $A_2^\uparrow$ are again homogeneous Poisson processes with

$$\lambda_1^\uparrow = \lambda_1 \,.\, [1 - \operatorname{Prob}(\{ Y_1(1) < 0 \})]$$

and

$$\lambda_2^{\uparrow}=\lambda_2 . [1-\text{Prob}[\{ Y_2(1)<0 \})].$$

Notice that $\lambda_1^{\uparrow} \leqq \lambda_2^{\uparrow}$ since $\lambda_1 \leqq \lambda_2$ and $Y_1(1) \leqq_{st} Y_2(1)$.

In the case of homogeneous Poisson processes, $A_1^{\uparrow} \leqq_{\alpha} A_2^{\uparrow}$ implies $A_1^{\uparrow} \leqq_{\delta} A_2^{\uparrow}$ [*see* Deng (1985 *a*)], which in turn, together with $Y_1^{\uparrow} \leqq_{st} Y_2^{\uparrow}$, implies $N_1^{\uparrow} \leqq_{\delta} N_2^{\uparrow}$. Analogous considerations for $N_1^{\downarrow}$ and $N_2^{\downarrow}$ lead to the desired result.

(ii) $\leqq_{\delta^*} \subset \leqq_{\alpha^*}$. In fact $N_1 \leqq_{\delta^*} N_2$ implies that $A_1^{\uparrow} \leqq_{\alpha} A_2^{\uparrow}$ and $A_2^{\downarrow} \leqq_{\alpha} A_1^{\downarrow}$, since $A_1^{\uparrow}$, $A_2^{\uparrow}$, $A_1^{\downarrow}$ and $A_2^{\downarrow}$ are homogeneous Poisson processes [*see* Deng (1985 *a*)]. These relations, together with $Y_1^{\uparrow} \leqq_{st} Y_2^{\uparrow}$ and $Y_2^{\downarrow} \leqq_{st} Y_1^{\downarrow}$ entail that $N_1^{\uparrow} \leqq_{\alpha} N_2^{\uparrow}$ and $N_2^{\downarrow} \leqq_{\alpha} N_1^{\downarrow}$, *i. e.* $N_1 \leqq_{\alpha^*} N_2$. ■

## 4. PRESERVATION RESULTS

In this section we examine preservation of the orderings previously introduced under independent thinning [namely under the procedure described in Remark 2.5, with $p(t) \equiv p \in (0, 1)$] and under independent superposition. Given two independent processes $N$, $M \in \mathcal{N}$, we call $K=N+M$ the independent superposition of $N$ and $M$. To avoid ambiguity, we will sometimes write the parent process as a subscript: so, for example, we write $A_M$ to denote the counting process associated to the process $M$.

Sometimes we will make use of the following assumptions:

HYPOTHESIS (A): *$N_1$, $N_2$, $M_1$, $M_2$ are such that, for $i=1, 2$, the jumps of both $N_i$ and $M_i$ are i. i. d. and have the same distribution.*

HYPOTHESIS (B): *For $i=1, 2$, $N_i$ and $M_i$ are such that the probability of having a common jump time is zero*, i. e.

$$\text{Prob}(\exists t \in \mathbb{R}_+ \mid [M_i(t)-M_i(t^-)][N_i(t)-N_i(t^-)] \neq 0)=0.$$

PROPOSITION 4.1: *For processes in $\mathcal{N}$:*

(i) *ordering $\leqq_{\alpha^*}$ is preserved under independent superposition, provided Hypothesis* (B) *holds;*

(ii) *orderings $\leqq_{\alpha^*}$, $\leqq_{\delta^*}$ and $\leqq_{\varepsilon^*}$ are preserved under independent thinning.*

The following lemmata will be needed for the proof of Proposition 4.1. Their trivial proofs will be omitted.

LEMMA 4.1: *For* $n \in \mathbb{N}$, *let* $Y_1(n) \leqq Y_2(n)$ a.s., *and let* $Z_i(n) = \chi_n Y_i(n)$ $(i=1, 2)$, *where* $\{\chi_n \mid n \in \mathbb{N}\}$ *is a sequence of i. i. d. random variables such that*

$$\operatorname{Prob}(\chi_n = 0) = 1 - \operatorname{Prob}(\chi_n = 1) = p.$$

*Then* $Z_1(n) \leqq Z_2(n)$ a.s.

LEMMA 4.2: *If Hypothesis* (B) *is satisfied, then* $K^\uparrow = N^\uparrow + M^\uparrow$ *and* $K^\downarrow = N^\downarrow + M^\downarrow$, a.s.

*Proof of Proposition* 4.1: (i) Let $M_1, M_2, N_1, N_2 \in \mathcal{N}$. Then Hypothesis (B) guarantees that

$$\tilde{N}_1^\uparrow(t) - \tilde{N}_1^\uparrow(t^-) \leqq \tilde{N}_2^\uparrow(t) - \tilde{N}_2^\uparrow(t^-) \quad \text{a.s.} \qquad \forall t \in \mathbb{R}_+$$

and

$$\tilde{M}_1^\uparrow(t) - \tilde{M}_1^\uparrow(t^-) \leqq \tilde{M}_2^\uparrow(t) - \tilde{M}_2^\uparrow(t^-) \quad \text{a.s.,} \qquad \forall t \in \mathbb{R}_+$$

imply

$$\tilde{K}_1^\uparrow(t) - \tilde{K}_1^\uparrow(t^-) \leqq \tilde{K}_2^\uparrow(t) - \tilde{K}_2^\uparrow(t^-) \quad \text{a.s.,} \qquad \forall t \in \mathbb{R}_+.$$

Analogously for $\tilde{K}_1^\downarrow$ and $\tilde{K}_2^\downarrow$.

(ii) Let $N_1, N_2 \in \mathcal{N}$ and let $Q_1, Q_2$ be obtained from $N_1, N_2$ respectively via independent thinning with probability $p$. Let's begin with $\leqq_{\alpha^*}$. If $N_1 \leqq_{\alpha^*} N_2$, then there exist processes $\tilde{N}_1$ and $\tilde{N}_2$ such that

$$\tilde{N}_1^\uparrow(t) - \tilde{N}_1^\uparrow(t^-) \leqq \tilde{N}_2^\uparrow(t) - \tilde{N}_2^\uparrow(t^-)$$

and

$$\tilde{N}_2^\downarrow(t) - \tilde{N}_2^\downarrow(t^-) \leqq \tilde{N}_1^\downarrow(t) - \tilde{N}_1^\downarrow(t^-).$$

Consider the sequence $\{\tilde{T}_2^\uparrow(j) \mid j \in \mathbb{N}\} \cup \{\tilde{T}_1^\downarrow(k) \mid k \in \mathbb{N}\}$ and label it $\{\hat{T}(n) \mid n \in \mathbb{N}\}$. Delete the jumps of both $\tilde{N}_1$ and $\tilde{N}_2$ at time $\hat{T}(n)$ with probability $p$ (one of these jumps may be zero), and call $\tilde{Q}_1$ and $\tilde{Q}_2$ the resulting processes. Then $\tilde{Q}_1$ and $\tilde{Q}_2$ are copies of $Q_1$ and $Q_2$ respectively; furthermore,

$$\tilde{Q}_1^\uparrow(t) - \tilde{Q}_1^\uparrow(t^-) \leqq \tilde{Q}_2^\uparrow(t) - \tilde{Q}_2^\uparrow(t^-)$$

and

$$\tilde{Q}_2^\downarrow(t) - \tilde{Q}_2^\downarrow(t^-) \leqq \tilde{Q}_1^\downarrow(t) - \tilde{Q}_1^\downarrow(t^-).$$

It follows that $Q_1 \leqq_{\alpha^*} Q_2$.

Let us consider now ordering $\leqq_{\delta^*}$. If $N_1 \leqq_{\delta^*} N_2$, then $A^{\uparrow}_{N_1} \leqq_{\delta} A^{\uparrow}_{N_2}$ and $A^{\downarrow}_{N_2} \leqq_{\delta} A^{\downarrow}_{N_1}$. By Theorem 2.1 of Deng (1985 *a*) the corresponding thinned counting processes satisfy

$$A^{\uparrow}_{Q_1} \leqq_{\delta} A^{\uparrow}_{Q_2} \quad \text{and} \quad A^{\downarrow}_{Q_2} \leqq_{\delta} A^{\downarrow}_{Q_1}.$$

Combining this with Lemma 4.1 we get the result.

A similar proof applies for ordering $\leqq_{\varepsilon^*}$. ■

Stronger results hold for increasing processes.

PROPOSITION 4.2: *For processes in* $\mathcal{N}_{\text{inc}}$:

(i) *ordering* $\leqq_{\beta^*}$ *is preserved under independent superposition;*

(ii) *if Hypotheses* (A) *and* (B) *hold, ordering* $\leqq_{\varepsilon^*}$ *is also preserved under independent superposition.*

*Proof:* (i) It is well known that stochastic ordering is preserved under convolution [*see* Kamae, Krengel and O'Brien (1977)].

(ii) Since

$$\tilde{A}_{\tilde{N}_1}(t) \leqq \tilde{A}_{\tilde{N}_2}(t) \quad \text{and} \quad \tilde{A}_{\tilde{M}_1}(t) \leqq \tilde{A}_{\tilde{M}_2}(t), \qquad \forall t \in \mathbb{R}_+ \quad \text{a.s.}$$

and Hypothesis (B) is satisfied, we have

$$\tilde{A}_{\tilde{K}_1}(t) \leqq \tilde{A}_{\tilde{K}_2}(t), \qquad \forall t \in \mathbb{R}_+ \quad \text{a.s.}$$

Moreover, for $i=1, 2$, the distributions of the jumps of $K_i$, $N_i$ and $M_i$ are the same. Therefore the stochastic ordering for the jumps is preserved. ■

*Remark* 4.1: The following counterexamples show the necessity of Hypotheses (A) and (B).

– Let $N_1$ and $M_1$ be deterministic processes with jumps of unit magnitude occurring every three units of time. Let also $N_2 (M_2)$ be a deterministic process with unitary jumps occurring at every even (odd) integer. It is clear that $N_1 \leqq_{\varepsilon^*} N_2$ and $M_1 \leqq_{\varepsilon^*} M_2$, but the ordering is not preserved under superposition, since all the jumps of $K_2$ are unitary, while the jumps of $K_1$ are of size 2.

– Let

$$N_1(t) = 3 \sum_{n=1}^{\infty} I_{[3n, \infty)}(t)$$

$$N_2(t)=4\sum_{n=0}^{\infty} I_{[2n+1,\,\infty)}(t)$$

$$M_1(t)=\sum_{n=0}^{\infty} I_{[3n+2,\,\infty)}(t)$$

$$M_2(t)=2\sum_{n=1}^{\infty} I_{[2n,\,\infty)}(t)$$

Again, $N_1 \leqq_{\varepsilon^*} N_2$ and $M_1 \leqq_{\varepsilon^*} M_2$, but the order does not hold for $K_1$ and $K_2$, since $3=Y_{K_1}(2)>Y_{K_2}(2)=2$.

Notice that in each of the above counterexamples only one of the two hypotheses is removed.

Next proposition deals with the case of inhomogeneous compound Poisson processes.

PROPOSITION 4.3: *For processes in* $\mathcal{N}_{\mathrm{icP}}$:

(i) *ordering* $\leqq_{\beta^*}$ *is preserved under independent superposition;*

(ii) *if Hypothesis* (A) *is met, orderings* $\leqq_{\gamma^*}$ *and* $\leqq_{\varepsilon^*}$ *are also preserved under independent superposition;*

(iii) *ordering* $\leqq_{\beta^*}$ *is preserved under independent thinning.*

*Proof:* (i) For inhomogeneous compound Poisson processes Hypothesis (B) holds. Therefore, if $N_1^{\uparrow} \leqq_{\beta} N_2^{\uparrow}$ and $M_1^{\uparrow} \leqq_{\beta} M_2^{\uparrow}$, then $K_1^{\uparrow}=(N_1^{\uparrow}+M_1^{\uparrow}) \leqq_{\beta} K_2^{\uparrow}=(N_2^{\uparrow}+M_2^{\uparrow})$. Repeating the argument for the processes $N_i^{\downarrow}$, $M_i^{\downarrow}$ and $K_i^{\downarrow}$, $i=1, 2$, the result follows.

(ii) By Theorems 2.1 and 2.2 of Deng (1985*a*), orderings $\leqq_{\gamma^*}$ and $\leqq_{\varepsilon^*}$ are preserved under independent superposition of the associated counting processes. Moreover, for $i=1, 2$, the distributions of the jumps of $K_i$, $N_i$ and $M_i$ are the same. Therefore, the stochastic ordering for the jumps is preserved.

(iii) The result follows from the fact that $\mathcal{N}_{\mathrm{icP}}$ is closed under independent thinning, and orderings $\leqq_{\beta^*}$ and $\leqq_{\varepsilon^*}$ are equivalent for processes in $\mathcal{N}_{\mathrm{icP}}$. ■

Finally, we consider the case of homogeneous compound Poisson processes.

PROPOSITION 4.4: *For processes in* $\mathcal{N}_{\mathrm{hcP}}$:

(i) *if Hypothesis* (A) *is met, ordering* $\leqq_{\delta^*}$ *is preserved under independent superposition;*

(ii) *ordering* $\leqq_{\gamma^*}$ *is preserved under independent thinning.*

*Proof:* (i) Hypothesis (A) ensures that, for $i=1, 2$, $K_i \in \mathcal{N}_{\mathrm{hcP}}$ whenever $N_i$, $M_i \in \mathcal{N}_{\mathrm{hcP}}$. Therefore, the result follows easily from Propositions 4.1 and 3.5.

(ii) This follows from the fact that the class is closed under independent thinning and from Proposition 3.5. ■

*Remark* 4.2: Putting together all the preservation results proved in the previous propositions, it follows that, for homogeneous compound Poisson processes all the orderings are preserved under superposition, if Hypothesis (A) is satisfied, and all of them are preserved under thinning, anyway.

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## REFERENCES

1. B. Bassan, E. Çinlar and M. Scarsini, *Stochastic Comparison of Semimartingale Hunt Processes*, 1989 (unpublished manuscript).
2. Y. L. Deng, On the Comparison of Point Processes, *J. Appl. Probab.*, 1985 *a*, *22*, pp. 300-313.
3. Y. L. Deng, Comparison of Inhomogeneous Poisson Processes, *Chinese Ann. Math.*, 1985 *b*, Ser. *6B*, pp. 83-96.
4. L. I. Gal'chuk, A Comparison Theorem for Stochastic Equations with Integrals with Respect to Martingales and Random Measures, *Theory Probab. Appl.*, 1982, *27*, pp. 450-460.
5. H. U. Gerber, An Introduction to Mathematical Risk Theory, *Heubner*, Philadelphia, 1979.
6. J. Grandell, Aspects of Risk Theory, *Springer Verlag*, New York, 1991.
7. T. Kamae, U. Krengel and G. L. O'Brien, Stochastic Inequalities on Partially Ordered Spaces, *Ann. Probab.*, 1977, *5*, pp. 899-912.
8. D. R. Miller, Almost Sure Comparison of Renewal Processes and Poisson Processes, with Application to Reliability Theory, *Math. Oper. Res.*, 1979, *4*, pp. 406-413.
9. T. Rolski and R. Szekli, *Stochastic Ordering and Thinning of Point Processes*, Mathematical Institute, University of Wroclaw, 1989.
10. W. Whitt, Comparing Counting Processes and Queues, *Ad. in Appl. Probab.*, 1981, *13*, pp. 207-220.