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A family of Hamilton type methods for congressional apportionments

A FAMILY OF HAMILTON TYPE METHODS FOR CONGRESSIONAL APPORTIONMENTS (*)

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Abstract. — *Several principles have been proposed for determining whether the apportionment of seats of a legislative house by a given method is fair. In this work a certain family of methods that satisfy the "fair share" criterion is introduced and it is shown that such methods also satisfy other criteria like the "near fair share" and the "partial population monotonicity". The family includes the traditional Hamilton's method which is the only one in such a family that satisfies a certain type of "independence".*

Keywords : Hamilton's method; apportionment; quota, fair share.

Résumé. — *De nombreux principes ont été proposés pour déterminer si la répartition des sièges d'un parlement obtenue par une méthode donnée est juste. Dans ce travail, une famille de méthodes, qui vérifie le critère « entre quotas », est introduit et vérifie aussi d'autres critères importants tels que « presque entre quotas » et la « monotonie partielle de population ». Cette famille de méthodes contient la méthode classique de Hamilton, qui est la seule de cette famille, qui vérifie un certain type d'indépendance.*

Mots clés : Méthode de Hamilton; répartition proportionnelle; quota; entre quota.

1. INTRODUCTION

In connection with proportional representation (PR) schemes for assigning the seats of a legislature, a number of principles have been proposed for determining whether the apportionment as performed by a given method is fair. Among such principles, population monotonicity, house monotonicity and the fair share properties appear to be the most desirable (e. g., [2] and [8]). Besides these, certain criteria have been proposed for comparing methods and justifying the superiority of some with respect to others, e. g., binary consistency [3], called near fair share in [2], and Droop minimum [7].

In this work, the importance of such principles is discussed, particularly the population monotonicity and the near fair share in relation with the fair property. As a result, a certain family of methods, that includes Hamilton's is defined by a particular optimization problem. We show that Hamilton's method is the unique in that family which satisfies a certain condition of "independence".

2. PROPORTIONAL REPRESENTATION PROBLEM

Given h, the number of seats of a legislature; n, the number of parties participating in the allocation; and p_j, the number of votes cast in favor of party j, $j = 1, \ldots, n$, the *proportional representation* problem consists of apportioning the h seats among the n parties, or, specifically, finding non-negative integers a_j such that $\sum a_j = h$ and a_j is porportional to the vote of party j. Further, $q_j = p_j h/p$ with $p = \sum p_j$, is called the *quota* associated with party j, and represents the "ideal" solution, *i.e.*, the one involving "exact proportionality", when all the q_j are integers. The rarity of that event, has motivated the development of methods yielding solutions as approximate to exact proportionality as possible. Among them, Hamilton's method and divisor methods are the most well-known [2].

Hamilton's method perfoms the apportionment in two steps: first, each party j is assigned its *lower quota* $\lfloor q_j \rfloor$, and then each one of the r parties having the r largest *remainders* $r_j = q_j - \lfloor q_j \rfloor$ receives one of the $r = h - \sum \lfloor q_j \rfloor$ remaining seats, *i.e.*, is assigned its *upper quota* $\lceil q_j \rceil$. If $r = 0$, the distribution is exact ($\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and $\lceil x \rceil = \lfloor x \rfloor + 1$ if x is not integer).

A divisor method based on a given *divisor* criterion d, which is an increasing function defined for every integer $m \geq 0$ and satisfies $m \leq d(m) \leq m + 1$ and $d(m')/(m' + 1) < d(m)/m$ for all $m \geq 1$ and $m' \geq 0$, allocates the seats in n steps (one seat in each step) according to the following rule: if $s_j(k)$ is the number of seats assigned to party j at the end of the k-th step, then the next seat is given to the party with the highest value $p_j/d(s_j(k))$. The following are the divisor methods that have been used in most of the cases to solve the problem **PR** [2]:

SD Smallest divisor (Adams): $d(m) = m$

HM Harmonic mean (Dean): $d(m) = m(m+1)/(m+1/2)$

GM Geometric mean or equal proportions (Hill): $d(m) = \sqrt{m(m+1)}$

AM Arithmetic mean (Webster, Saint-Lague): $d(m) = m + 1/2$

GD Greatest divisor (Jefferson, d'Hont): $d(m) = m + 1$.

Constrained optimization provides a basis for a general approach to the problem **PR**. In fact, if we let $f(\underline{a}, \underline{q})$ denote the function that measures the "error" or "deviation" between the distribution $\underline{a}$ and the quota vector $\underline{q}$, then problem **PR** can be formulated as follows:

(**PR**): $\text{Min}\{f(\underline{a}, \underline{q}): \sum a_j = h, a_j \geqq 0 \text{ and integer}, j = 1, \ldots, n\}$.

It can easily be seen that any solution $\underline{a}$ of Hamilton's method minimizes $\sum |a_j - q_j|$, $\sum (a_j - q_j)^2$ and, generally, any norm of $\underline{a} - \underline{q}$.

Huntington [5], established that each of the above five divisor methods is optimal regarding certain criteria that measure some "amount of inequality" in pairwise rankings of parties. Further, AM minimizes $\sum p_j (a_j/p_j - h/p)^2$, GM minimizes $\sum a_j (p_j/a_j - p/h)^2$, whereas SD and GD optimize $\min_a \max_j (p_j/a_j)$ and $\min_a \max_j (a_j/p_j)$, respectively ([2], [5]).

3. PRINCIPLE RANKING

The main goal of an apportionment scheme for a legislature is that representation be proportionally as fair as possible. However, since the **PR** problem is intrinsically related to political power, its importance and implications go beyond the purely mathematical aim of finding a suitable, just (fair) solution. History shows that the problem has been widely addressed not only by politicians but also by mathematicians, who have sought to support some methods and disfavor others [2]. The discussion has given rise to some basic properties that appear to be the most desirable and that any fair method ought to have. Among such principles we consider the following three:

(i) *Fair share (satisfying quota)*. The number of seats a_j assigned to party j should satisfy $\lfloor q_j \rfloor \leq a_j \leq \lceil q_j \rceil$, for all j.

(ii) *House monotonicity*. The vote of each party remaining constant, the number of seats assigned to each party should not decrease as the house size is augmented.

(iii) *Population monotonicity*. No party that increases its vote should lose a seat to another that decreases its.

Hamilton's method fulfills the fair share principle. The divisor methods satisfy both monotonicity criteria, whereas a certain type of methods, called Quota methods ([1], [6]), were designed to satisfy, specifically, the fair share and the house monotonicity principles. However, it was proved that no method simultaneously satisfies all three of the above principles [2]. This

implies that the distribution resulting from any method will only be "partially fair" and, hence, the importance ascribed to a principle vis-a-vis the others will give grounds to determine whether a method could outrank another.

First, it can be noticed that the fair share criterion tends to neither undervalue nor overprize each party's vote: no party should receive fewer seats than it deserves (*i. e.*, it should receive at least its lower quota), nor should be assigned more seats than it merits (*i. e.*, it should not go over its upper quota).

The principle of house monotonicity was originated in the situation known as the *Alabama paradox:* after the 1880 Census in the United States, the fact was detected that if the size of the House of Representatives were increased from 299 to 300, the number of seats assigned to the state of Alabama by the Hamilton apportionment scheme would decrease from 8 to 7. While this type of occurrence should be avoided, we agree with Birchoff [3] in the sense that when the size of a legislature is set in advance, it is not necessary to require that the apportionment satisfies this principle. Further, the assertion holds true in practice, for changes in the process of an election, and in the few cases were they might, it is legitimate to state that it is a different poll that is being dealt with.

Unlike those two principles, population monotonicity does not have an unique conception. Regardless of its interpretation, however, its importance is self-evident, because within the parameters that define the problem, viz. number of seats, number of parties, and the vote of each party, it is the latter that regularly changes, and so it deserves first consideration. In particular, it is desirable for an allocation to avoid situations like the "more-is-less" paradox [4], which consits of a party receiving fewer seats after increasing its vote. This, among other real life situations, is far from being "...curiosities and nothing more..." (Still [6]), and therefore some type of population monotonicity must be required to any acceptable method of apportionment.

A method will be said "partially population monotonous" if no party that increases its *relative vote* p_j/p should lose a seat to another that decreases its.

This concept effectively accounts for real-life situations (e. g., in two consecutive elections of a legislature, some parties increase their votes, others decrease theirs, whereas the total vote, usually, increases). Besides, it precludes undesirable occurrences like the population paradoxes, and, in particular, satisfies the following properties:

— A party that increases its vote whereas the rest remains constant, will not lose any seats.

— If one and only party increases its vote whereas the total population remains constant, then it will not lose any seats.

In addition to the above principles, there is one introduced in [3] and [2], which is related to the fair share property. It is called the *near fair share* criterion which rules out the possibility that the transfer of a seat from one party to another takes simultaneously both parties closer to their quotas. This means, in particular, that if a party receives more than its upper quota, no other party can receive less than its lower quota, and similarly, if a party receives less than tis lower quota no other party can get more than its upper quota.

In the next section we introduce a certain family of methods, that includes the Hamilton method, using the fair share, near fair share and partial population monotonicity principles, and show that the apportionments given for such methods correspond to the optimal solutions of problems that admit a formulation of the type defined by (**PR**), *i. e.*, are based on the optimization of certain functions.

4. HAMILTONIAN TYPE METHODS

We start showing that, despite the fact that both principles fair share and near fair share are defined only in term of the quotas, none of them is implied by the other, except for the cases of two and three parties in which near fair share implies fair share.

Divisor methods are partially population monotonous (actually, they satisfy a stronger population monotonicity criterion [2]). However, only one of them fulfils the near fair share principle.

THEOREM 1 [2]: *Webster procedure is the only divisor method satisfying the near fair share principle.*

This result implies, in particular, that there exist near fair share methods which do not satisfy the fair share property.

For the case of two parties, every divisor method is fair share, whereas for the three-party case Webster's is the unique divisor method that is fair share. For $n \geq 4$ no divisor method is fair share [1].

PROPOSITION 1: *The Hamilton method satisfies the near fair share and the partial population monotonicity principles.*

Proof: Let (p_j) and (p'_j) be two vectors of the parties' vote, and assume that $p'_k/p' \geqq p_k/p$ and $p'_m/p' \leqq p_m/p$. Then $q'_k \geqq q_k$ and $q'_m \leqq q_m$ (h is mantained constant).

If $\lfloor q'_k \rfloor \geqq \lfloor q_k \rfloor + 1$, then $a'_k \geqq \lfloor q'_k \rfloor \geqq a_k$ and so party k can not lose any seat. Similarly, if $\lfloor q'_m \rfloor \leqq \lfloor q_m \rfloor - 1$, then $a'_m \leqq \lfloor q_m \rfloor \leqq a_m$ and party m can not get an extra seat. So, the other possible case is $\lfloor q'_k \rfloor = \lfloor q_k \rfloor$ and $\lfloor q'_m \rfloor = \lfloor q_m \rfloor$, which implies $r'_k \geqq r_k$ and $r'_m \leqq r_m$. But, since a seat transfer requires that $r_k > r_m$ and $r'_m > r'_k$, the Hamilton's solutions satisfy $a'_k \geqq a_k$ and $a'_m \leqq a_m$. Thus, Hamilton's method satisfies the partial population monotonicity principle.

Now assume that Hamilton's method is not near fair share, and let $(h, p_1, \ldots, p_n)$ be a problem for which the Hamilton's solution satisfies $a_i = \lfloor q_i \rfloor$, $a_j = \lceil q_j \rceil$, $q_j - (a_j - 1) < a_j - q_j$ and $a_i + 1 - q_i < q_i - a_i$ for some parties i and j. But this implies $q_i - a_i < 1/2$ and $q_j - a_j < 1/2$ and hence $q_i - \lfloor q_i \rfloor > 1/2$ and $q_j - \lfloor q_j \rfloor < 1/2$ which contradicts the outcome of Hamilton's method, completing the proof. □

The next result, though it can not be generalized for more than three parties, confirms the importance of the near fair share principle.

PROPOSITION 2: *For two and three parties, the near fair share property implies the fair share property.*

Proof: For the two-party case if (a_1, a_2) is near fair share and is not fair share, say $a_1 \geqq \lceil q_1 \rceil + 1$, then $h - a_2 \geqq h - \lfloor q_2 \rfloor + 1$, and so $a_2 \leqq \lfloor q_2 \rfloor - 1$. This implies that (a_1, a_2) is not near fair share, a contradiction.

We now consider the three-party case. Let (a_1, a_2, a_3) be the near fair share apportionment for (p_1, p_2, p_3, h) and assume it is not fair share. Then, at least one party does not satisfy quota, say 1. If $a_1 < \lfloor q_1 \rfloor$, then it is not possible that $a_2 > \lceil q_2 \rceil$ or $a_3 > \lceil q_3 \rceil$ because of the near fair share principle, and thus, neither $a_2 \leqq \lfloor q_2 \rfloor$ nor $a_3 \leqq \lfloor q_3 \rfloor$ because $a_1 + a_2 + a_3 = h$ and $\lfloor q_1 \rfloor + \lfloor q_2 \rfloor + \lfloor q_3 \rfloor < h$. Hence, $a_2 = \lceil q_2 \rceil$ and $a_3 = \lceil q_3 \rceil$. This implies $q_2 - \lfloor q_2 \rfloor > 1/2$ and $q_3 - \lfloor q_3 \rfloor > 1/2$ because of the near fair share principle and therefore, $\sum [(q_j - \lfloor q_j \rfloor) : j = 1, 2, 3] > 1$. From this it follows that

$$\sum \lceil q_j \rceil + 1 < \sum q_j = h = \sum a_j = a_1 + \lfloor q_2 \rfloor + \lfloor q_3 \rfloor + 2$$

which implies $\lfloor q_1 \rfloor < a_1 + 1$, i.e., $a_1 \geqq \lfloor q_1 \rfloor$, a contradiction. If there are two parties which are not satisfying quota, say parties 1 and 2, then either $a_1 < \lfloor q_1 \rfloor$ and $a_2 < \lfloor q_2 \rfloor$ or $a_1 > \lceil q_1 \rceil$ and $a_2 > \lceil q_2 \rceil$ because (a_1, a_2, a_3) is near fair share. But none of these cases is possible because the former one implies $a_3 \geqq \lceil q_2 \rceil + 2$ and the latter implies $a_3 \geqq \lfloor q_2 \rfloor - 2$, contradicting the near fair share principle, which completes the proof. □

The reciprocal of the above result does not hold even for the two-party case, *i.e.*, there exist fair share methods which are not near fair share. In fact, consider the "smallest remainders" method, which we define in a way similar to Hamilton's method but assigning the remaining r seats to the r parties associated with the r *smallest* remainders $r_j = q_j - \lfloor q_j \rfloor$. The resulting method is fair share but is not near fair share (e. g., if $h = 10$, and $p_1 = 13$ and $p_2 = 107$, then $q_1 = 1.083$ and $q_2 = 8.917$, and the method gives the solution (2,8) which is not near fair share because the transfer of one seat from party 1 to party 2 get both parties closer to their respective quotas).

The two methods, Hamilton's and Webster's have one principle in common, namely the near fair share. This property is shared by all methods that optimize a certain type of criterion, as deduced from the next result, whose proof is evident.

PROPOSITION 3: *Every apportionment method that minimizes* $\sum \alpha_j |a_j - q_j|^s$ *with* $\alpha_j > 0$ *and* $s > 0$, *satisfies the near fair share principle.*

COROLLARY: For the two-party case, all apportionment schemes that minimize $\alpha_1 |a_1 - q_1|^s + \alpha_2 |a_2 - q_2|^s$ with $\alpha_1 > 0$, $\alpha_2 > 0$ and $s > 0$, yield the same solutions when $r_1 \neq 1/2$.

Proof: Let (a_1', a_2') and (a_1'', a_2'') be the optimal solutions obtained by two given methods **M'** and **M''** associated with $(\alpha_1', \alpha_2', s')$ and $(\alpha_1'', \alpha_2'', s'')$, respectively. By Proposition 3, **M'** is near fair share, and hence the following inequalities must hold $|a_1' - q_1| \leq |a_1'' - q_1|$ and $|a_2' - q_2| \leq |a_2'' - q_2|$. Since the same argument holds for **M''**, it follows that $|a_1'' - q_1| \leq |a_1' - q_1|$ and $|a_2'' - q_2| \leq |a_2' - q_2|$, and therefore $|a_1' - q_1| = |a_1'' - q_1|$ and $|a_2' - q_2| = |a_2'' - q_2|$. Furthermore, since $r_1 + r_2 = 1$, these two equalities imply that $a_1' = a_1''$ and $a_2' = a_2''$, when $r_1 \neq 1/2$ (note that if $r_1 = 1/2$, then $a_1'' = q_1 \pm 1/2 = a_1' \pm 1$ and $a_2'' = q_2 \pm 1/2 = a_2' \pm 1$). □

From this Corollary it follows that for the two-party case, the Webster's and Hamilton's methods coincide.

We end this work by considering the set of all the apportionment methods whose solutions satisfy Proposition 3 and the condition that they also have to be fair share, *i.e.*, those methods that solve

$$\mathbf{P}(\mathbf{\alpha}, s): \operatorname{Min} \sum \left\{ \alpha_j |a_j - q_j|^s : \sum a_j = h, \lfloor q_j \rfloor \leq a_j \leq \lceil q_j \rceil, a_j \text{ integer}, j = 1, \ldots, n \right\}$$

where $s > 0$ and $\alpha_j > 0$ for $j = 1, \ldots, n$. $\mathbf{a}^*(\mathbf{\alpha}, s)$ will denote the optimal solution of $\mathbf{P}(\mathbf{\alpha}, s)$.

First, we note that by letting $x_j = a_j - \lfloor q_j \rfloor$ and $r_j = q_j - \lfloor q_j \rfloor$, the objective function of problem $P(\underline{\alpha}, s)$, denoted by $F(\underline{\alpha}, s)$, associated with the parameters $\underline{\alpha} = (\alpha_j)$ and s, becomes

$$F(\underline{\alpha}, s) = \sum \alpha_j \left\{ (1 - r_j)^s - r_j^s \right\} x_j + \sum \alpha_j r_j^s$$

and therefore, problem $P(\underline{\alpha}, s)$ is equivalent to the following:

$$(P): \text{Min} \sum \left\{ \beta_j x_j : \sum x_j = r, \; x_j = 0 \text{ or } 1 \right\}$$

where $\beta_j = \alpha_j \left\{ (1 - r_j)^s - r_j^s \right\}$ and $r = h - \sum \lfloor q_j \rfloor$.

(P) corresponds to a particular Knapsack problem, whose solution $\underline{x} = (x_j)$ can easily be obtained by giving the value 1 to the r variables x_j having the r smallest coefficient β_j. If $\alpha_j = 1$ for all j, then the resulting optimal solution of (P) corresponds to the solution given by the method of Hamilton.

Given $\underline{\alpha} > 0$ and $s > 0$, we denote by $M(\underline{\alpha}, s)$ the apportionment method whose solution for the problem $(p_1, \ldots, p_n; h)$ is $\underline{a}^*(\underline{\alpha}, s)$, and by M_H the set of all such methods.

A method $M(\underline{\alpha}, s)$ in M_H is called *independent* of s if and only if for any problem $(p_1, \ldots, p_n; h)$ it satisfies $\underline{a}^*(\underline{\alpha}, s') = \underline{a}^*(\alpha, s'')$ for all $s' > 0$ and $s'' > 0$.

It is apparent that the method of Hamilton is independent of s. Moreover, we have the following result.

THEOREM 2: *Hamilton's is the unique method in M_H that is independent of the parameter s.*

Proof: Let $M(\underline{\alpha}, s)$ be any method in M_H different from Hamilton's. We will show that there exists a problem $(p_1, \ldots, p_n, h)$ and $s' > 0$, $s'' > 0$ for which $\underline{a}(\alpha, s') \neq \underline{a}(\underline{\alpha}, s'')$.

Assume that $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \ldots \geq \alpha_n$ and let $(p_1, \ldots, p_n; h)$ be a problem whose remainders r_j are defined as follows

$$r_1 = (1 - \varepsilon)/2; \qquad r_k = 1/2(n - 1), \quad \text{for} \quad k = 2, \ldots, n - 1; \quad \text{and} \quad r_n = r_{n-1} + \frac{\varepsilon}{2}$$

where $0 < \varepsilon < 1/2$.

The definition of the remainders r_j implies that $r = h - \sum \lfloor q_j \rfloor = 1$, and so, method $M(\underline{\alpha}, s)$ assigns $\lfloor q_{j_0} \rfloor + 1$ seats to some one party j_0 and $\lfloor \bar{q}_j \rfloor$ seats to each party $j \neq j_0$.

We claim that there exist positive values of ε ($\varepsilon < 1/2$) for which (a) $j_0 = n$ if $s' = 2$, and (b) $j_0 = 1$ if $s'' = 3$. The definition of the objective function of (P) implies that the former case is obtained when $\alpha_n(1 - 2r_n) < \alpha_k(1 - 2r_k)$ for all $k \neq n$, whereas the latter holds when

$$\alpha_1(1 - 2r_1)(1 - r_1 + r_1^2) < \alpha_k(1 - 2r_k)(1 - r_k + r_k^2)$$

for all $k \neq 1$. Further, the assumption on the α_k's together with the definition of the r_k's imply that case (a) holds for all values of ε satisfying $\alpha_n(1 - 2r_n) < \alpha_1(1 - 2r_1)$, i.e., for $\varepsilon > \varepsilon_0 = \alpha_n \beta/(\alpha_1 + \alpha_n)$, where $\beta = (n-2)/(n-1)$. Furthermore, the same argument implies that case (b) is equivalent to

(i) $\varepsilon(3 + \varepsilon^2) < \alpha_{n-1}\beta(3 + \beta^2)/\alpha_n$, and

(ii) $\varepsilon(3 + \varepsilon^2) < \alpha_n(\beta - \varepsilon)(3 + (\beta - \varepsilon)^2)/\alpha$.

Since $f(x) = x(3 + x^2)$ ($x \in \mathbf{R}$) is a continuous and strictly increasing function (and so is one-to-one), there exists an unique $\bar{\varepsilon} > 0$ satisfying $\bar{\varepsilon}$ $(3 + \bar{\varepsilon}^2) = \alpha_{n-1}\beta(3 + \beta^2)/\alpha_1$. But, $\alpha_n \leq \alpha_{n-1}$ implies $f(\varepsilon_0) < f(\bar{\varepsilon})$ and hence, $\varepsilon_0 < \bar{\varepsilon}$. Therefore, condition (i) is satisfied for $\varepsilon_0 \leq \varepsilon \leq \bar{\varepsilon}$. Moreover, since $\alpha_n < \alpha_1$ and $\varepsilon_0 < \beta - \varepsilon_0 < 1/2$, it follows that ε_0 also satisfies (ii), and by continuity, there exists $\delta > 0$ such that condition (ii) holds for $\varepsilon^* = \varepsilon_0 + \delta$ with $\varepsilon^* < \bar{\varepsilon}$, completing the proof. $\quad\square$

5. CONCLUSIONS

In this paper we have considered some criteria in the context of the apportionment problem which allow us to introduce a certain family of methods that work similarly to Hamilton's which is the only one in that family satisfying some type of independence.

REFERENCES

1. M. L. Balinski and H. P. Young, The Quota Method of Apportionment, *Amer. Math. Monthly*, 1975, *82*, pp. 701-730.

2. M. L. Balinski and H. P. Young, Fair Representation, New Haven and London, Yale University, Press, 1982.

3. G. Birkhoff, House monotone apportionment schemes, *Proc. Nat. Acad. Sci. U.S.A.* 1976, *73*, pp. 684-686.

4. P. C. Fishburn and S. J. Brams, Paradoxes of Preferntial Voting, *Math. Mag.*, 1983, *56*, pp. 207-214.

5. E. V. Huntington, The Apportionment of Representatives in Congress, *Trans. Am. Math. Soc.* 1968, *80*, pp. 85-110.

6. J. W. STILL, A Class of New Methods for Congressional Apportionment, *S.I.A.M. J. Appl. Math.*, 1979, *37*, pp. 401-418.
7. D. R. WOODALL, How Proportional is Proportional Representation, *Math. Intelligencer*, 1986, *8*, pp. 36-46.
8. H. P. YOUNG, Fair Allocation, *Procceding of Symposia in Applied Mathematics*, *33*, AMS Providence, Rhode Island, 1985.