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AN EFFICIENT METHOD FOR OBTAINING SHARP BOUNDS FOR NONLINEAR BOOLEAN PROGAMMING PROBLEMS (*)

by P. F. KÖRNER ([1])

Abstract. — *It is well-known that the integrality condition for variables in Boolean programming problems can be formulated by quadratic equality constraints. The Lagrangian dual to the nonlinear programming problem thus obtained is examined. We state an efficient method for obtaining sharp bounds for the branch and bound process in solving the integer problem. The obtained results are discussed on a Boolean quadratic optimization problem. Numerical results are presented.*

Keywords : Boolean quadratic programming; Lagrange duality, branch and bound.

Résumé. — *Il est bien connu que les conditions d'intégrité des variables dans les problèmes de programmation mathématiques booléens peuvent être formulés à l'aide de contraintes d'égalité quadratiques. Nous examinons le dual lagrangien du problème de programmation non-linéaire ainsi obtenu. Nous donnons une méthode efficace pour obtenir des bornes précises pour la résolution du problème en nombres entiers par la méthode « branch and bound ». Nous analysons les résultats obtenus sur un problème d'optimisation booléen quadratique, et présentons des résultats numériques.*

1. INTRODUCTION

In the present paper problems of the following form are considered:

(Q) $f_0(x) \to \min$ subject to

$$f_j(x) \leqq 0, \qquad j=1, \ldots, m, \quad x \in R^n,$$

with

$$f_j(x) := x^T C^j x + x^T p^j + b_j, \qquad j=0, 1, \ldots, m,$$

where the matrices C^j are symmetric for all j.

The problem (Q) is denoted by (Q_B) if we require, in addition, the variables satisfy the condition $x \in \{0, 1\}^n$.

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We use the branch and bound method for solving (Q_B). A good description of this method is contained in [6]. In this method it is favourable to use sharp and simple computable bounds. In order to determine the bounds, the so-called real imbedding will be applied, *i. e.* a problem of the form (Q) has to be solved at every node of the branch and bound tree. The optimal objective function value of this problem will then be applied as a bound.

If the functions f_j are convex for all j, then there exist efficient algorithms for solving (Q) (*cf.* [2]). In the nonconvex case many difficulties arise. However, the following relevant statement can be found in [6]:

"Linear 0-1 programs are often solved efficiently by linear-programming based branch and bound algorithms. When such an approach is extended to the quadratic case a difficulty arises as the continuous relaxation obtained by replacing $x_i \in \{0, 1\}$ by $x_i \in [0, 1]$ will usually be a nonconvex quadratic programming problem. However, P. L. Hammer and A. A. Rubin (1970) have shown that to any quadratic 0-1 program could be associated an equivalent quadratic 0-1 program in 0-1 variables with a convex continuous relaxation."

In this paper the idea of P. L. Hammer and A. A. Rubin will be generalized.

2. ANALYTIC INVESTIGATIONS

The following trivial equation describes a relationship between Boolean and quadratic programming:

$$\left.\begin{array}{c} x_i \in \{0, 1\} \quad \Leftrightarrow \quad x_i(x_i - 1) = x_i^2 - x_i = 0, \\ i = 1, \ldots, n. \end{array}\right\} \tag{1}$$

Problem (Q) with the additional constraints (1) is denoted by (Q').

We obtain the following Lagrange function for (Q'):

$$L(x, u, v) : = x^T \left(\sum_{j=0}^{m} v_j C^j + U \right) x + \left(\sum_{j=0}^{m} v_j p^j - u \right)^T x + v^T b,$$

with $U : = \operatorname{diag}(u_i)$, $b : = (b_0, \ldots, b_m)^T$ and

$$v \in V : = \{ (v_0, \ldots, v_m)^T : v_0 = 1, v_j \geqq 0, j = 1, \ldots, m \}.$$

The dual problem takes the form:

(D) $$\varphi(u, v) \to \max, \qquad u \in \mathrm{R}^n, \quad v \in V,$$

with

$$\varphi(u, v) : = \inf\{L(x, u, v) : x \in R^n\}.$$

Let

$$\operatorname{dom} \varphi : = \{(u, v) : v \in V, \varphi(u, v) > -\infty\}$$

and

$$W(u, v) : = \{x : \varphi(u, v) = L(x, u, v)\} \quad \text{for} \quad (u, v) \in \operatorname{dom} \varphi.$$

LEMMA 1: *The following three conditions are equivalent:*

a) $(u, v) \in \operatorname{dom} \varphi$.

b) $W(u, v) \neq \varnothing$.

c) The matrix $\left(\sum_{j=0}^{m} v_j C^j + U\right)$ *is positive semidefinite, and the system:*

$$2\left(\sum_{j=0}^{m} v_j C^j + U\right) x + \left(\sum_{j=0}^{m} v_j p^j - u\right) = 0$$

is solvable with respect to x.

The proof of this lemma is obvious and may be omitted. Now we maximize the special function φ with known algorithms (*cf.* e. g. [1] and [2]. Let, for $u^* \in R^n$ and $v^* \in V$ the inequality be satisfied

$$\varphi(u, v) \leqq \varphi(u^*, v^*) \quad \text{for all } u \text{ and } v \in V. \tag{2}$$

If there exists a vector $x' \in \{0, 1\}^n$ with $f_j(x') \leqq 0$, $j = 1, \ldots, m$, then there exists a bound r with

$$\varphi(u, v) \leqq r \quad \text{for all } u \text{ and } v \in V.$$

We now consider the following problem:

$$(\mathrm{Q}_j) \qquad \begin{cases} f_j(x) \to \min \text{ subject to the constraints (1),} \\ \qquad j = 1, \ldots, m. \end{cases}$$

The Lagrange function takes the form

$$L_j(x, u) : = f_j(x) + x^T U x - u^T x = : f_j^u(x) \tag{3}$$

and we obtain the following known result:

LEMMA 2: *a)* $f_j(x) = f_j^u(x)$ *for all* u *and for all* $x \in \{0, 1\}^n$.

There always exists a vector u'' *in such a way that the following statements are true:*

b) $f_j^{u''}$ *is convex.*

c) $f_j^{u''}((1/2)e) < 0$ $(e := (1, \ldots, 1)^T)$.

That means, with the help of transformation (3) we can transform every problem (Q_B) into an equivalent problem (Q_B'') with convex functions, which fulfils the Slater condition.

3. ON THE LAGRANGE DUAL PROBLEM

Now we examine the problem (D).

THEOREM 3: *Let* u^* *and* v^* *be defined according to* (2). *Then the following statements are true:*

a) There exists always a $x' \in W(u^*, v^*)$ *with* $0 \leqq x' \leqq e$.

b) If the functions f_j, $j = 0, \ldots, m$, *are convex and* (Q) *fulfils fulfils the Slater condition (with a* $x \in (0, 1)^n$*), then there exists a* $x' \in W(u^*, v^*)$ *with* $f_j(x') \leqq 0$, $j = 1, \ldots, m$, *and* $0 \leqq x' \leqq e$.

Proof: We give the proof only for statement *a*). Statement *b*) can be proved in an analogous manner.

The function h defined via

$$h(x) := L(x, u^*, v^*)$$

is convex. Now we consider the problem

$$(\mathrm{H}) \qquad \begin{cases} h(x) \to \min \text{ subject to } x_i^2 - x_i \leqq 0, \\ \qquad i = 1, \ldots, n. \end{cases}$$

The constraints of (H) are convex and satisfy the Slater condition. Thus in problem (H) no duality gap occurs. The objective function of the dual problem to (H) is denoted by q. From the optimality of (u^*, v^*) we obtain:

$$q(u) \leqq q(0) = \varphi(u^*, v^*) \quad \text{for all } u \geqq 0.$$

No duality gap occurs. Therefore, we obtain the result.

Q.E.D.

The main result of the paper follows:

COROLLARY 4: *Let* (u^*, v^*) *be defined according to* (2). *Under the assumption of the positive semidefiniteness of the matrix:*

$$\left(\sum_{j=0}^{m} v_j^* C^j + U^*\right)$$

the value $\varphi(u^*, v^*)$ *is the best possible bound for the optimal objective function value of* (Q_B).

If we have computed the bound $\varphi(u^*, v^*)$ then, with the assumption $v_j^* > 0$ for all j, it is in general impossible to split the vector u^* $(u^* = u^0 + \ldots + u^m)$ in such a way that the matrices

$$C^j + \frac{1}{v_j^*} U^j \, (U^j := \operatorname{diag}(u_i^j)), \qquad j = 1, \ldots, m$$

and $C^0 + U^0$ are positive semidefinite.

The next example proves this fact. Let $m = 1$, $v_1^* = 1$,

$$C^0 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \qquad C^1 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \qquad \text{and} \qquad u^* = \begin{pmatrix} -2 \\ -2 \end{pmatrix}.$$

That means, in order to obtain tight constraints, we have to solve the following problems:

$$(D_j) \qquad \varphi_j(u) \to \max$$

with

$$\varphi_j(u) := \inf\{ f_j^u(x) : x \in R^n \}$$

$[f_j^u$ via (3)], $j = 1, \ldots, m$.

Numerical problems arising in the solution process of (D_j) are discussed in [9].

4. A BRANCH AND BOUND ALGORITHM

We use the following branch and bound algorithm for solving (Q_B).

Algorithm

S0: Solve problem (D) and set $f(x) := L(x, u^*, v^*)$. We use this function for obtaining bounds.

S1: Solve the problems (D_j), $j=1, \ldots, m$, and set $f'_j(x) := L_j(x, (u^j)^*)$. We use these functions for checking feasibility.

S2: Sort the variables in an appropriate manner (*cf.* [8]).

S3: Solve the following problem with the branch and bound method:

$$f(x) \rightarrow \min \text{ subject to } x \in \{0, 1\}^n.$$

Check the feasibility of this point with respect to the constraints f'_j at every branch and bound node.

An efficient branch and bound algorithm for solving the nonrestricted quadratic integer programming problem is discussed in [8]. This algorithm uses the real imbedding, *i.e.* at every step a problem of the form:

$$f(x) \rightarrow \min$$

is to be solved. If we use a special Cholesky decomposition for solving, then we need this decomposition only at the beginning of the branch and bound process (*cf.* [8]).

The constraints are checked in the following way: We ask if there exist vectors x^j with $f'_j(x^j) \leqq 0$, $j=1, \ldots, m$. Using the Cholesky decomposition from [8], we need this decomposition only at the beginning of the branch and bound process.

5. NUMERICAL RESULTS

The elements of the matrices C^j and the vectors p^j for test have been randomly chosen in iniform distribution from the interval $(-5,10)$. We have put $b_j := 10n$ for all j. The tables contain the average value of 10 examples. The program is written in SIMULA and run on an IBM 3031 computer.

m	n	First bound	Optimal value	Number of b and b nodes	Time (sec.)
5	10	873.4	980.1	95.0	14.7
10	15	5,763.0	6,129.4	261.8	26.3
25	30	20,379.0	24,208.5	780.3	531.7

The same problems are solved with the algorithm described in [5].

m	n	Number of b and b nodes	Time
5	10	287.3	18.7
10	15	891.3	41.7
25	30	1,208.6	718.3

Now we have used the following algorithm. At every branch and bound node we have solved a problem of type (D), and the corresponding optimal value is used as a bound.

m	n	Number of b and b nodes	Time
5	10	51.3	20.1
10	15	143.7	35.0
25	30	511.9	735.1

We need further time for comparing the present method with other known methods.

6. CONCLUDING REMARKS ON THE GENERAL NONLINEAR CASE

Now we consider the nonlinear problem of type (Q), where all functions f_j are general nonlinear functions. In order to obtain sharp bounds for the corresponding Boolean problem, it is favourable to maximize the corresponding function φ. But unfortunately, most of the results for the quadratic case are not true for the nonlinear case. Especially, Theorem 3 does not hold in the nonlinear case, as the following example shows. Let

$$\text{(E)} \qquad f(x) \to \min \text{ subject to } x_i^2 - x_i = 0, \qquad i = 1, 2,$$

where $f(x^1) = f(x^2) < f(x)$ for all x with $x \neq x^1$ and $x \neq x^2$. Let φ be appropriately defined. If we set

$$x^1 := (0.5(1+\sqrt{2});\ 0.5) \qquad \text{and} \qquad x^2 := (0.5;\ 0.5(1+\sqrt{2}))$$

then we have

$$f(x^1)=f(x^2)=\varphi(0)\geqq\varphi(u) \quad \text{for all } u.$$

The optimality of 0 is easy to show. The set $W(0)$ contains only two elements. If we define two subgradients with these elements, then the sum of these subgradients is zero. But $W(0)=\{x^1, x^2\}$ does not contain any vector x' with $0\leqq x'\leqq e$. For the general nonlinear case we need further numerical investigations.

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