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The weighted minimax location problem with set-up costs and extensions

THE WEIGHTED MINIMAX LOCATION PROBLEM WITH SET-UP COSTS AND EXTENSIONS (*)

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Abstract — *We consider the problem of locating several facilities in an n-dimensional space For each demand point we calculate the sum of weighted distances of the new facilities plus possibly a set-up cost The maximal value of this sum for all demand points is to be minimized This is a generalization of the single facility minimax problem (also called the 1-center problem) The problem reduces to the weighted minimax problem with a set-up cost if only one facility need to be located We present theorems and algorithms for the general problem but mainly deal with the single facility case*

Keywords Facility location, minimax, 1-center

Résumé — *Nous considerons le problème consistant a localiser plusieurs services dans un espace a n dimensions Pour chaque point de demande, nous calculons la somme des distances pondérées des nouveaux services plus, peut-être, un coût (fixe) de mise en route On doit minimiser la valeur maximale de cette somme etendue à tous les points de demande Ceci est la généralisation du problème d'un seul service au sens du minimax (appelé aussi le probleme du 1-centre) Le problème se réduit a un problème de minimax pondéré avec coût de mise en route si un seul service doit être localisé Nous présentons des theoremes et des algorithmes pour le problème général, mais traitons principalement le cas d'un seul service*

INTRODUCTION

The formulation is based on m points given in R^n. Find the locations of k new facilities that minimize the maximum, over all demand points, of the sum of weighted distances from the demand point to all new facilities, plus a set-up cost.

Let

P_i for $i = 1, \ldots, m$ be the location of demand point i ;

X_j for $j = 1, \ldots, k$ be the site of new facility j;

$X = (X_1, \ldots, X_k)$ be the unknown vector;

w_{ij} for $i = 1, \ldots, m$; $j = 1, \ldots, k$ be a positive weight associated with demand point i and new facility j;

g_i for $i = 1, \ldots, m$ be a set-up cost associated with demand point i;

d_{ij} be the Euclidean distance between demand point i and new facility j,

$$F_i(X) = \sum_{j=1}^{k} w_{ij} d_{ij} + g_i.$$

The problem is to select X to minimize $F(X)$, where

$$F(X) = \max_{1 \leq i \leq m} \{ F_i(X) \} \tag{1}$$

The special case where $k = 1$, $n = 2$, and all $g_i = 0$ is the single-facility minimax location problem on the plane [11, 14, 19, 21, 22] (also known as the 1-center problem). The simpler case where all the weights are equal is discussed in [15] and [26]. In [15] set-up costs are considered. The case $k = 1$ and general n is discussed in [10] and [16]. Other generalizations of the single-facility minimax location problem that don't fit into our extension are the multi-facility minimax location problem [13, 17, 18, 20, 25], and the p-center [5, 6, 8, 27] problems. *See* [3] and [24] for a survey of location papers.

Consider the following application to the problem. Emergency hospital services usually involve dispatching an ambulance (or emergency helicopter service) and bringing the patient to the hospital. The total time from the emergency call until the patient arrives at the hospital consists of the travel time of the ambulance, some set-up time, and travelling back to the hospital. In most cases the ambulance station is located on the hospital premises so that the total traval time is twice the travel time from the hospital plus a set-up time. Organizing the facility this way may not be optimal. If we select different sites for the hospital and the ambulance station we may be able to shorten the response time for the farthest customer. We cannot lose by such a proposition because if the best strategy is to locate both the ambulance station and the hospital at the same site, then the procedure will select such an option. When one station (hospital or ambulance station) is already in place, the problem is defined with $k = 1$ (locating the other station) and the set-up time is the driving time to the existing facility plus possibly and additional set-up time. The model may include different weights for the new facilities. The time needed for the ambulance to get to the patient is more crucial than the time it takes to get to the hospital because some medical care may be provided once the ambulance reached the patient.

The set-up costs can help to model demand coming from areas in the plane when $k = 1$. We can approximate each area by a union of discs (not necessarily disjoint) thus converting the areas problem into a discs problem. The weights and set-ups are assumed to be the same for all the points inside each disc. A demand point is assigned to the center of each disc, and it can be easily verified that we need to add to the set-up time the product of the weight and the radius of the disc in order to accommodate the farthest point of the disc. Many applications assume a demand point rather than a demand area for the sake of simplified analysis. This generalization allows us to better model practical applications. Discussion of area demand is given in [1], [7], [12], [23] and [29].

A THEOREM

The following theorem can be used in many proofs involved with the analysis of the single facility minimax location problem and is necessary for some proofs given here.

THEOREM 1: *Consider problem* (1) *with* $k = 1$, *and* X^* *as the optimal solution. Given a bounded set* B, *there exists a constant* $K > 0$ *such that for all* X *in* B,
$$F(X) \geqq F(X^*) + K \| X - X^* \|^2.$$

Proof : Consider any X in B. We can assume $X \neq X^*$. Define
$$U = (X - X^*)/\| X - X^* \|, \qquad I = \{ i \mid F_i(X^*) = F(X^*) \},$$

and $dF_i(r) = F_i(X^* + rU) - F_i(X^*)$ for $r \geqq 0$. There must exist $v \in I$ such that $dF_v(r) \geqq 0$ for a small r, for otherwise F could be decreased by moving from X^* in direction U, contradicting the optimality of X^*. If $d_v(X^*) \neq 0$, we can define $U_1 = (X^* - P_v)/d_v(X^*)$. Note that $\| U \| = \| U_1 \| = 1$. This yields
$$dF_v(r) = w_v \{ [d_v^2(X^*) + r^2 + 2rd_v(X^*) \langle U, U_1 \rangle]^{1/2} - d_v(X^*) \},$$

where $\langle U, U_1 \rangle$ is the scalar product between U and U_1. Since $dF_v(r) \geqq 0$ for a small r, $\langle U, U_1 \rangle \geqq 0$, and thus
$$dF_v(r) \geqq w_v \{ [d_v^2(X^*) + r^2]^{1/2} - d_v(X^*) \}, \quad \text{for all } r \geqq 0,$$

a result which also holds when $d_v(X^*) = 0$. Therefore,
$$F(X) \geqq F_v(X) = F_v(X^*) + dF_v(\| X - X^* \|)$$
$$= F(X^*) + dF_v(\| X - X^* \|)$$

$$\geqq F(X^*) + w_v \left\{ [d_v^2(X^*) + \|X - X^*\|^2]^{1/2} - d_v(X^*) \right\}$$
$$= F(X^*) + w_v \|X - X^*\|^2 / \left\{ [d_v^2(X^*) + \|X - X^*\|^2]^{1/2} + d_v(X^*) \right\}. \quad (2)$$

Since B is bounded, there exists $M > 0$ independent of the choice of X in B, such that $\|X - X^*\| \leq M$ and $d_i(X^*) \leq M$ for all i (hence, in particular, for $i = v$). The preceeding inequality (2) yields

$$F(X) \geqq F(X^*) + w_v \|X - X^*\|^2 / (\sqrt{2} + 1) M.$$

Q.E.D.

THE EXCHANGE ALGORITHM

Since Euclidean distances are convex functions, so is $F_i(X)$. Therefore, a local minimum to problem (1) is the global minimum. The following theorem is a corollary of Theorem 1.

THEOREM 2: *For $k = 1$, the solution point is unique.*

The following is an example with $k = 2$ (it can be easily generalized to $k \geqq 2$) for which the solution point is not unique. The problem is defined with $k = 2$, $n = 1$, $m = 2$, $w_{ij} = 1$, and $g_i = 0$. The two demand points are located on a line at 0 and 1. When two new facilities are located at t and $1 - t$ for $0 \leq t \leq 1$, the value of the objective function is equal to 1. These points are all the optimal solutions. To show this observe that for any point x, $d_1(x) + d_2(x) \geqq 1$, and therefore for any two points x and y: $d_1(x) + d_1(y) + d_2(x) + d_2(y) \geqq 2$.   Thus $\max \left\{ d_1(x) + d_1(y), d_2(x) + d_2(y) \right\} \geqq 1$.

The following exchange algorithm is a specific case of the exchange algorithm presented in [9].

The exchange algorithm

1. Select starting points $X^{(0)} = [X_1^{(0)}, \ldots, X_k^{(0)}]$. A possible selection is to locate all of them at the center of gravity of points P_i with weights w_i. Set the iteration number $r = 1$. Find the set $E^{(1)}$ of $nk + 1$ demand points [4] corresponding to the $nk + 1$ largest values of $F_i(X^{(0)})$. Solve problem (1) based on the demand points in $E^{(1)}$ getting a solution vector $X^{(1)}$ with value of objective function (for the sub-problem) of $F^{(1)}$.

2. Let j be an index i for which $F_i(X^{(r)})$ is maximal.

3. If $F(X^{(r)}) = F^{(r)}$ then $X^{(r)}$ is optimal.

4. Otherwise, for each $i \in E^{(r)}$ a problem based on $E^{(r)} - \{i\} \cup \{j\}$ is solved until a value of the objective function greater than $F^{(r)}$ is found. Set $E^{(r+1)}$ to $E^{(r)} - \{i\} \cup \{j\}$ for that i, $X^{(r+1)}$ to the solution vector to this problem with value of objective function $F^{(r+1)}$. Set $r = r + 1$ and go to step 2.

Such an exchange algorithm is especially efficient when $m \gg nk + 1$ and instead of solving a problem with m demand points several problems based on $nk + 1$ demand points must be solved. In [4] it is shown that if there is only one solution point to the problems in step 4, then the algorithm converges to the optimal solution.

In the next section we give an iterative algorithm for solving the problem of $nk + 1$ demand points for $k = 1$ and $n = 2$. Such an algorithm can be used in the exchange algorithm.

THE TWO AND THREE POINTS PROBLEMS

When there are only two points in the problem, the solution must be either at one of the points or on the line between them where $F_1(X) = F_2(X)$ [28]. Let the two points be P_1 and P_2 on the x-axis with weights w_1 and w_2 respectively. The point P^* where $F_1(X) = F_2(X)$ is by straightforward calculations (assuming $P_1 < P_2$):

$$P^* = (w_1 P_1 + w_2 P_2 + g_2 - g_1)/(w_1 + w_2) \tag{3}$$

with a cost of:

$$F^* = (w_1 w_2 |P_1 - P_2| + w_1 g_2 + w_2 g_1)/(w_1 + w_2) \tag{4}$$

The optimal value is the *maximum* among g_1, g_2, F^* with the solution point at P_1, P_2, or P^* respectively.

Now consider a problem with three demand points in the plane located at $P_i = (x_i, y_i)$ for $i = 1, 2, 3$. Three weights $w_i > 0$ and three set-up cost g_i for $i = 1, 2, 3$ are associated with the demand points. The optimal solution must lie in the closed triangle defined by the three demand points [28]. The solution is on vertex P_i if and only if $g_i \geq F_j(P_i)$ for all $j \neq i$. The solution is on the side connecting P_i and P_j if for the third vertex $k F_k(X') \leq F_i(X')$ where X' is the solution point for the two point problem based on P_i and P_j by (3).

If the optimum is not on the boundary of the triangle, it must be in its interior. In such a case, X^* must fulfill [4]:

$$F_1(X^*) = F_2(X^*) = F_3(X^*). \tag{5}$$

In the following we assume that the solution is not on the boundary of the triangle and therefore (5) is fulfilled. Also, this entails that for every X, $F(X) > g_i$ for $i = 1, 2, 3$.

A transformation H from R^2 to R^2 is defined as follows.

$$w_i'(X) = w_i / [F(X) - g_i] \qquad \text{for} \quad i = 1, 2, 3 \tag{6}$$

$$F'(X) = \min_U \left\{ \max_i \left\{ w_i'(X) d_i(U) \right\} \right\} \tag{7}$$

$$H(X) = U^* \tag{8}$$

where U^* is the solution point of (7). An explicit formula for U^* is given in Appendix A of [11]. U^* is unique by Theorem 2 and thus well defined. The algorithm is to iteratively replace X with $H(X)$ until convergence.

Algorithm for the three point problem

1. Check if a vertex is optimal or if the optimum lies on a side of the triangle. If the optimum has been found, stop.

2. If the optimum lies in the interior of the triangle, select a starting point $X^{(0)}$ (for example the center of gravity), a tolerance $\varepsilon > 0$, and set the iteration counter r to 0.

3. Set $X^{(r+1)} = H(X^{(r)})$ by (8).

4. If $\| X^{(r+1)} - X^{(r)} \| < \varepsilon$ stop and accept $X^{(r+1)}$ as the solution.

5. Otherwise, set $r = r + 1$ and go to step 3.

We now prove that $\lim_{r \to \infty} (X^{(r)})$ exists, and that it is the optimal solution to the three point problem.

LEMMA 1: $F'(X^*) = 1$ *and* $H(X^*) = X^*$.

Proof: By (5) $w_i d_i(X^*) + g_i = F(X^*)$ for $i = 1, 2, 3$. By (6) $w_i'(X^*) d_i(X^*) = 1$ for $i = 1, 2, 3$, which means that X^* is weighted-equidistant from the vertices, and therefore $F'(X^*) = 1$. Since the solution to (7) is unique by Theorem 2, $U^* = H(X^*) = X^*$.

<div align="right">Q.E.D.</div>

LEMMA 2: $F'(X) < 1$ *for* $X \neq X^*$.

Proof: Since $F(X) > F(X^*)$ by Theorem 1, then $w_i'(X) < w_i'(X^*)$ for $i = 1, 2, 3$ and therefore $F'(X) < F'(X^*) = 1$ Lemma 1.

<div align="right">Q.E.D.</div>

LEMMA 3: *For* $X \neq X^*$: $F[H(X)] < F(X)$.

Proof: By (8) and Lemma 2: $d_i[H(X)] \leq F'(X)/w_i'(X) < 1/w_i'(X)$ for all i. By (6): $F[H(X)] < \max_i \{w_i/w_i'(X) + g_i\} = F(X)$.

<div align="right">Q.E.D.</div>

Clearly $F(X)$ is a continuous function of X, and $w_i'(X)$ is a continuous function of $F(X)$. Also, by Theorem 1, the solution point U^* is a continuous function of the weights, and therefore $H(X)$ is a continuous function of $F(X)$.

THEOREM 3: $\lim\limits_{r \to \infty} \{X^{(r)}\} = X^*$.

Proof : By Lemma 3 $F(X^{(r)})$ is monotonically decreasing and bounded below by zero, therefore $\lim\limits_{r \to \infty} \{F(X^{(r)})\} = F'$ exists. Since $H(X)$ is a continuous function of $F(X)$ $\lim\limits_{r \to \infty} \{H(X^{(r)})\} = \lim\limits_{r \to \infty} \{X^{(r+1)}\} = X'$ exists. Since $H(X)$ is a continuous function of X, $H(X') = X'$ and thus $F[H(X')] = F(X')$. By Lemma 3 $X' = X^*$.

<div align="right">Q.E.D.</div>

We tested the algorithm on 10,000 randomly generated problems. The data x_i, y_i, w_i, g_i were uniformly generated in $(0,1)$. For these particular data, the optimal solution was on a vertex in 17.16% of the cases, and was on a side of the triangle (and not on a vertex) in 80.49% of the cases. Only 235 problems required the iterative procedure. An $\varepsilon = 10^{-5}$ was used. The number of iterations was between 4 and 64, with an average of 12 and a median of 10. All 10,000 problems were solved in a total time of 1.66 seconds on the Amdahl 470/V7 computer at the University of Michigan.

A DISCUSSION ON THE GENERAL PROBLEM

We first present a heuristic approach, show that it always terminates, but give an example where a limit point is not optimal. Then we briefly discuss the construction of a rigorous approach.

Since the single facility case is relatively simple (especially for $n = 2$) it is tempting to apply a univariate search to the multi-facility problem. This means: select a starting point; go over all k new facilities in turn; for each new facility obtain the new optimum of the single facility location problem, holding the other facilities rooted to their place; repeat until convergence is identified. The single facility location problem simply consists of a set-up cost for demand point i which is the sum of g_i added to all the weighted

distances to the rooted facilities. If facility r is the only one not rooted, then:

$$F_i(X) = w_{ir} d_{ir} + \left\{ g_i + \sum_{j \neq r} w_{ij} d_{ij} \right\}$$

where the term in the braces is the set-up cost.

We develop evidence that the univariate search will usually converge, and give an example where the limit point is not optimal.

Let the transformation $T_s(X)$ be the resulting location vector when we optimize facility s while holding the rest rooted to their sites. Let the transformation $T(X)$ be the convolution of all transformations $T_s(X)$ for $s = 1, \ldots, k$ in order. In other words, $T(X)$ is the resulting vector when we optimize once all new facilities one at a time. Note that $F[T_s(X)] \leq F(X)$, and consequently $F[T(X)] \leq F(X)$. Consider the sequence defined by a given $X^{(0)}$, and $X^{(r+1)} = T(X^{(r)})$. Also, $F^{(r)} = F(X^{(r)})$.

THEOREM 4: $\lim\limits_{r \to \infty} \| X^{(r+1)} - X^{(r)} \| = 0$.

Proof: Since $F^{(r+1)} \leq F^{(r)}$ and the sequence is bounded from below by zero, $\lim\limits_{r \to \infty} \{ F^{(r)} \}$ exists. Since $X^{(r)}$ are in the convex hull of demand points, then by Theorem 1 there exist $K > 0$ independent of the g_i, such that $\| X_i^{(r+1)} - X_i^{(r)} \|^2 \leq |F_i^{(r+1)} - F_i^{(r)}|/K$ where $F_i^{(r)}$ is the value of the objective function for the single facility problem when all demand points except X_i are rooted to their place. Since $F_i^{(r+1)} - F_i^{(r)}$ are going to zero, so do $X_i^{(r+1)} - X_i^{(r)}$.

Q.E.D.

Although Theorem 4 does not necessarily imply convergence of the sequence $X^{(r)}$, it rules out the "usual" cases of non-convergence. An example of nonconvergence satisfying Theorem 4 suggested by a referee is the sequence $\sin \alpha_r$ with $\alpha_r = \sum\limits_{i=1}^{r} \pi/i$ for $r = 1, 2, \ldots$ Since $X^{(r)}$ is in the convex hull of all P_i, it cannot diverge to infinity. A "loop" where we can identify two cyclical sub-sequences that converge to two different points of accumulation, is also impossible. The distance between two different points of accumulation is finite so the distance between successive points cannot go to zero. In conclusion, I believe it is very difficult to construct an example, should one exist, for which the univariate search does not converge.

Note that if we stop the univariate search when the distance between two consecutive solution points is less than a given tolerance, then by Theorem 4 the procedure will always terminate. Since a limit point, as is shown below,

is not necessarily optimal, the question of convergence (in the rigorous sense) is much less important.

An example for which a limit point to $X^{(r)}$ is not optimal: $k=2$, $n=2$, $m=4$, $w_{ij}=1$, and $g_i=0$. Demand points are located at the vertices of a rectangle. Assume that $X^{(0)}$ consists of two points located at the midpoints of two opposite sides of the rectangle. It is easy to verify that $T(X^{(0)})=X^{(0)}$ but the objective function is lower at the center of the rectangle.

A general approach for solving convex minimax problems was presented by Demjanov [2]. In [13] and [20] this general approach was adapted to solving specific minimax location problems. Essentially it is a complicated iterative steepest descent method. When the general approach is implemented here, it is very similar to the formulation in [13]. We believe that repeating the details here is redundant. Please consult the analysis in [13] if such a rigorous approach is desired.

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