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TWO-POINT APPROXIMATIONS FOR ACTIVITY TIMES IN PERT NETWORKS (*)

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Abstract. — *This paper deals with a problem of determining the expected completion time in the PERT network. It is assumed that random activity durations are mutually independent and their continuous probability distributions are only characterized by two parameters: an expected value and a pessimistic time. It means that the exact forms of these distributions do not have to be known. A simple analytical method of deriving the upper bounds for the mean event occurrence times is presented. The errors of introduced approximations are analysed on some numerical examples.*

Keywords: PERT network; expected value; upper bound.

Résumé. — *Cet article traite du problème de détermination du temps moyen d'achèvement dans le réseau PERT. On suppose que les durées aléatoires d'activité sont mutuellement indépendantes et que leurs lois de probabilité continues sont seulement caractérisées par deux paramètres : une valeur moyenne et un temps pessimiste. Cela signifie que les formes exactes de ces lois n'ont pas à être connues. Une méthode analytique simple pour calculer les bornes supérieures des temps moyens d'occurence des évènements est présentée. Les erreurs dues aux approximations introduites sont analysées sur quelques exemples numériques.*

Mots clés : Réseau PERT; valeur moyenne; bornes supérieures.

1. INTRODUCTION

Recently J.-P. Melin [6] has shown the analytical method of deriving the mean of the greatest of several independent random variables with a special case of beta distribution. The method has been suggested for using in the PERT problem. In this paper we maintain the above suggestion presenting a reccurence procedure of evaluation the expected completion time of PERT network. It will be shown that the obtained results are the upper bounds for the exact values. The proposed method is numerically simple and the calculations can be performed manually even for more involved networks.

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2. ON A CERTAIN STOCHASTIC PRECEDENCE RELATION

We begin by introducing the necessary definitions and theorems which will be used extensively throughout the sequel.

DEFINITION 1 [1, 2]: Let X and Y be nonnegative random variables with the cumulative distribution functions (c. d. f.'s) F and G respectively. X is said to be stochastically smaller in mean than Y, written $X \prec Y$ or $F \prec G$, if for all $x \geqq 0$:

$$\int_x^\infty (1-F(t))\,dt \leqq \int_x^\infty (1-G(t))\,dt. \tag{2.1}$$

If F and G have finite means, then substituting $x=0$ in (2.1), we obtain:

$$E(X)=\int_0^\infty (1-F(t))\,dt \leqq \int_0^\infty (1-G(t))\,dt = E(Y). \tag{2.2}$$

THEOREM 1 (1, p. 121 and 125]: *Let $X=(X_i)$ and $Y=(Y_i)$ be nonnegative random n-vectors having independent coordinates. If $X_i \prec Y_i$ for $i=1, 2, \ldots, n$ then:*

(*a*) $\max(X_1, X_2, \ldots, X_n) \prec \max(Y_1, Y_2, \ldots, Y_n)$;

(*b*) $(X_1+X_2+\ldots+X_n) \prec (Y_1+Y_2+\ldots+Y_n)$.

DEFINITION 2 [1, 3, 7]: Two functions f and g cross at a point w if for all open sets W containing w there exist $w_1, w_2 \in W$ such that:

$$(f(w_1)-g(w_1)).f(w_2)<0. \tag{2.3}$$

THEOREM 2: *Let X and Y be nonnegative random variables (r. v.'s) with different c. d. f.'s F and G having the same finite mean.*

(*a*) *Criterion of Karlin-Novikoff* [3]: *If F crosses G exactly once and from below then $X \prec Y$.*

(*b*) *Criterion of Taylor for continuous r. v.'s* [7]: *If the densities F' and G' cross exactly twice and F' is heavier than G' in both tails (it means that at first F' crosses G' from below and next from above) then $X \prec Y$.*

Remark: Note that under the assumptions of theorem 2 F and G must cross at least once (*see* e. g. [1, p. 120]) and F' and G' must cross at least twice (e. g. [7]).

3. MODEL OF ACTIVITY TIME DISTRIBUTION

In the conventional PERT method it was assumed that all activity durations are beta-distributed. Although such a choice seems to be rather arbitrary it does have certain features that an actual activity time could be expected to possess. Namely, it was generally postulated that this distribution should be nonnegative, continuous, unimodal and have finite range. Taking it into account suppose that an activity duration X is described by a certain c. d. f. F with a density $f=F'$, which is unimodal on a range $[a, b]$, $0 \leqq a < b < \infty$ and $E(X)=m$. Furthermore it is assumed throughout the paper that only two parameters are given:

m, the expected value, and

b, the pessimistic time.

In other words the exact form of the activity time distribution do not have to be known.

Consider another r. v. Y with a c. d. f. G and a density g, where:

$$G(t)=\left\{\begin{array}{ccc} 0 & \text{for} & t \leqq 0, \\ (t/b)^{\alpha} & \text{for} & t \in [0, b] \text{ and } \alpha>0, \\ 1 & \text{for} & t \geqq b, \end{array}\right\} \tag{3.1}$$

and

$$g(t)=\left\{\begin{array}{cc} \alpha t^{\alpha-1}/b^{\alpha} & \text{for } t \in (0, b), \\ 0 & \text{elsewhere.} \end{array}\right\} \tag{3.2}$$

Assuming, in addition, that $E(Y)=m$, the parameter α is expressed as follows [6]:

$$\alpha=m/(b-m) \tag{3.3}$$

and conversely:

$$m=\alpha b/(1+\alpha). \tag{3.4}$$

For convenience we denote this two-parametric type of c. d. f. by $G(\alpha, m, b)$, remembering that α can be determined by (3.3).

The above defined densities f and g must cross at least twice. It is worth noticing that for the "moderately regular" function f these densities cross exactly twice and according to the value of m, five possible cases are illustrated in figure 1 (compare with figure 1 in [6]). For these examples the density f is heavier in both tails than g and in view of Taylor's criterion a relation $F \prec G(\alpha, m, b)$ is satisfied.

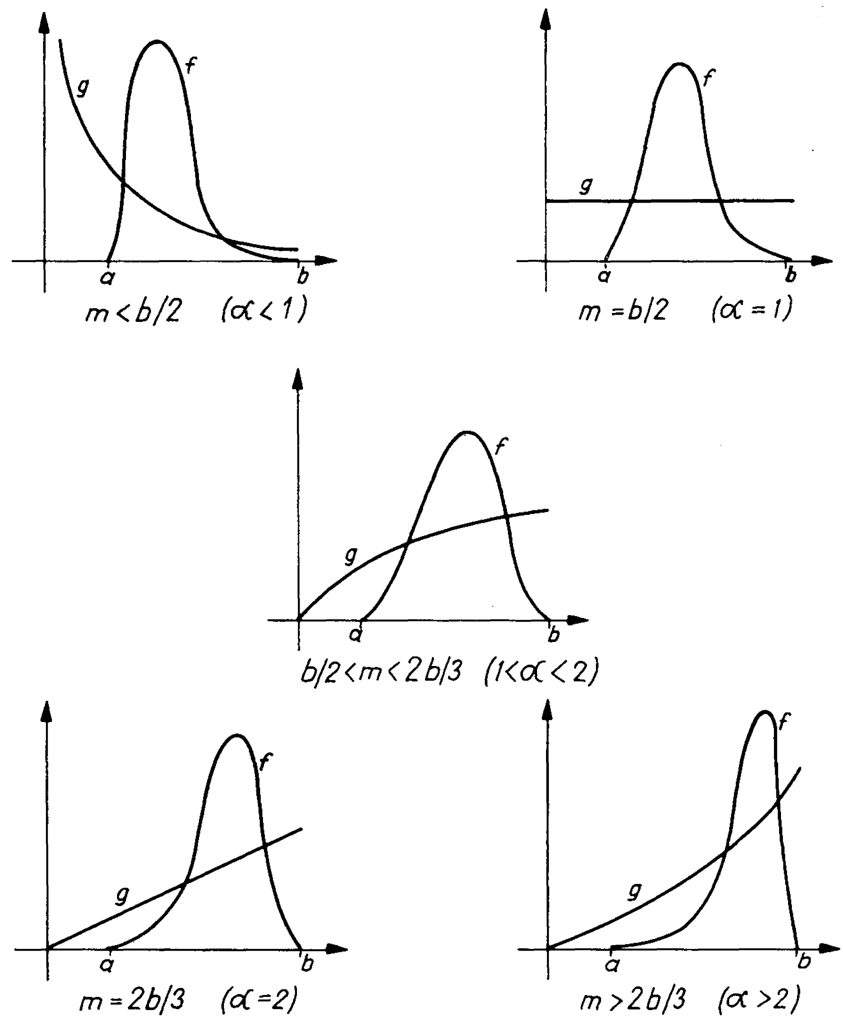

Figure 1. — The crossings of densities f and g.

Denote now by $\mathscr{F}(m, b)$ a class of all nonnegative distributions with the mean m, the finite right abscissa b and which are dominated by $G(\alpha, m, b)$ in the sense of the defined precedence relation, i. e.:

$$F \in \mathscr{F}(m, b) \quad \text{iff} \quad F \prec G(\alpha, m, b). \qquad (3.5)$$

It seems that $\mathscr{F}(m, b)$ comprises a very wide class of distributions which could be applied for the activity durations modelling. For example it is easy to show that the following distributions, which have been commonly used in the PERT problem, belong to $\mathscr{F}(m, b)$: triangular, truncated normal, uniform and beta with a density:

$$f(t)=(t-a)^{p-1}(b-t)^{q-1} \Big/ \int_a^b (t-a)^{p-1}(b-t)^{q-1}\,dt, \tag{3.6}$$

for $t \in [a, b]$ and $a \geqq 0$, $p>0$, $q \geqq 1$. Note that for $q=1$ and $a=0$ the beta distribution is equivalent to $G(p, m, b)$.

We will assume in the sequel that the activity time distributions concerning the PERT network belong to the class $\mathscr{F}(m, b)$. The author has not encountered in the literature any distribution F having unimodal, continuous density on a range $[a, b]$, $0 \leqq a < b < \infty$ for which $F \in \mathscr{F}(m, b)$ does not hold. However, let us remain the arising problem of a strict definition of the $\mathscr{F}(m, b)$ contents as a open problem.

4. SOME PROPERTIES OF $G(\alpha, m, b)$ DISTRIBUTION

Let X_i be independent r. v.'s with c. d. f.'s F_i, written $X_i \sim F_i$, such that $F_i \in \mathscr{F}(m_i, b_i)$ for $i=1, 2, \ldots, n$. In turn, assume that $Y_i \sim G(\alpha_i, m_i, b_i)$ and Y_i are also independent for $i=1, 2, \ldots, n$. Thus, by virtue of theorem 1 *a* we have:

$$\max_{1 \leqq i \leqq n}(X_i) \prec \max_{1 \leqq i \leqq n}(Y_i), \tag{4.1}$$

which yields:

$$E(\max(X_i)) \leqq E(\max(Y_i)). \tag{4.2}$$

The formula for deriving $E(\max(Y_i))$ was given in, [6] and is rewritten as follows:

$$E(\max(Y_i)) = h(\alpha_i, b_i; i=1, 2, \ldots, n)$$

$$= \frac{\alpha_n b_n}{1+\alpha_n} + \sum_{i=1}^{n-1} \frac{\alpha_i b_i^{1+\sum_{j=i+1}^{n} \alpha_j}}{\left(\prod_{j=i+1}^{n} b_j^{\alpha_j}\right)\left(1+\sum_{j=1}^{n} \alpha_j\right)\left(1+\sum_{j=1+1}^{n} \alpha_j\right)}, \tag{4.3}$$

where Y_i are renumerated to satisfy $b_1 \leqq b_2 \leqq \ldots \leqq b_n$.

In this way we have obtained the analytical method of determining the upper bound for $E(\max(X_i))$, where $F_i \in \mathscr{F}(m_i, b_i)$. On the other hand the relations $\Theta_{m_i} \prec F_i$ always hold, where:

$$\Theta_{m_i}(t)=\left\{\begin{matrix} 0 & \text{for} & t \leqq m_i, \\ 1 & \text{for} & t > m_i. \end{matrix}\right\} \tag{4.4}$$

Hence, a lower bound for $E(\max(X_i))$, which was applied in the PERT-calculated mean, can be given by:

$$\max(m_i) \leqq E(\max(X_i)). \tag{4.5}$$

Table I shows some of the obtained results for the above mentioned bounds. We have assumed that all X_i have the same distributions (on standardized interval [0, 1]). It is intuitively obvious that in this case the percentage errors of approximations should be the largest. The computations have been performed on the triangular and beta distributions for three arbitrary chosen values of mean ($m=1/3$, $m=1/2$, $m=2/3$). Since the beta distribution is associated with two parameters p and q [*see* (3.6)] we have considered the

TABLE I

The results of approximations

	Number of variables	2	3	4	5	6	7	8
$m=1/3$	Upper bound	.5000	.6000	.6667	.7143	.7500	.7778	.8000
	Triangular	.4667	.5429	.5936	.6306	.6559	.6788	.6977
	Error for upper bound	7.14	10.52	12.31	13.27	14.35	14.58	14.66
	Error for PERT bound	28.58	38.61	43.85	47.15	49.18	50.90	52.23
	Beta $p=2$, $q=4$	.4368	.4918	.5283	.5538	.5769	.5952	.6118
	Error for upper bound	14.47	22.00	26.20	28.98	30.01	30.68	30.76
	Error for PERT bound	22.70	32.23	36.91	39.82	42.23	44.00	45.52
$m=1/2$	Upper bound	.6667	.7500	.8000	.8333	.8571	.8750	.8888
	Triangular	.6167	.6750	.7121	.7386	.7588	.7749	.7882
	Error for upper bound	8.11	11.11	12.34	12.82	12.95	12.92	12.76
	Error for PERT bound	18.92	25.93	29.79	32.30	34.11	35.48	36.56
	Beta $p=q=3$	.6105	.6591	.6969	.7230	.7382	.7530	.7664
	Error for upper bound	9.21	13.79	14.79	15.26	16.11	16.20	15.97
	Error for PERT bound	18.10	24.14	28.25	30.84	32.27	33.60	34.76
$m=2/3$	Upper bound	.8000	.8571	.8889	.9091	.9231	.9333	.9412
	Triangular	–	–	–	–	–	–	–
	Error for PERT bound	16.66	22.21	25.00	26.87	27.78	28.57	29.16
	Beta $p=4$, $q=2$	.7688	.8149	.8396	.8592	.8730	.8833	.8929
	Error for upper bound	3.90	5.18	5.87	5.81	5.74	5.66	5.41
	Error for PERT bound	13.28	18.19	20.60	22.40	23.63	24.52	25.33

case for which $p+q=6$. It means that the density function is then closed to its expected value and the PERT bound should be more preferred. The results for beta distribution have been estimated by simulation solution, using 10000 independent trials. Other results have been derived in a analytical way. Table I shows only the results for $n=2, 3, \ldots, 8$ because the percentage errors of the proposed upper bound begin to decrease for greater n in all considered examples.

Although the mean of the maximum of $G(\alpha, m, b)$ – distributed independent r. v.'s could be derived analytically, many practical difficulties are encountered when we want to carry on the calculations on the PERT network. These difficulties arise from the fact that $G(\alpha, m, b)$ distribution is not preserved under sum and maximum of r. v.'s To overcome the above obstacles we will propose to use the other approximations resulting from the two following theorems:

THEOREM 3: *Let Y_i be independent* r. v.'s, $Y_i \sim G(\alpha_i, m_i, b_i)$ *for* $i=1, 2, \ldots, n$ *and* $Y \sim G(\alpha, m, b)$, *where* $b_1 \leqq b_2 \leqq \ldots \leqq b_n = b$ *and* m *is given by* (4.3). *Then* $\max(Y_i) \prec Y$, *which is equivalent to:*

$$\prod_{i=1}^{n} G(\alpha_i, m_i, b_i) \prec G(\alpha, m, b).$$

THEOREM 4: *Let Y_1, Y_2 be independent r. v.'s such that* $Y_i \sim G(\alpha_i, m_i, b_i)$ *for* $i=1, 2$, *and* $Y \sim G(\alpha, m, b)$, *where* $m=m_1+m_2$ *and* $b=b_1+b_2$. *Then* $Y_1+Y_2 \prec Y$, *which can be rewritten as:*

$$G(\alpha_1, m_1, b_1) * G(\alpha_2, m_2, b_2) \prec G(\alpha, m, b),$$

where "" indicates convolution.*

In other words the distributions of $\max(Y_1, Y_2, \ldots, Y_n)$ and Y_1+Y_2 can be approximated by the adequate $G(\alpha, m, b)$ distributions, preserving the same means. The proofs of these theorems are given in appendix.

5. THE USE OF $G(\alpha, m, b)$ DISTRIBUTION IN PERT NETWORK

Consider a directed, acyclic network $\langle N, A \rangle$, where $N=\{1, 2, \ldots, m\}$ is the set of nodes (events), 1 – the single start node, m – the single terminal node and $A \subset N \times N$ denotes the set of arcs (activities). With each activity (i, j) we connect a nonnegative r. v. X_{ij}, describing its duration, which is characterized by a certain c. d. f. F_{ij} with given parameters m_{ij} and b_{ij}. Next we assume that all X_{ij} are independent and $F_{ij} \in \mathscr{F}(m_{ij}, b_{ij})$ for $(i, j) \in A$.

If the numeration of nodes satisfies the condition:

$$(i, j) \in A \quad \Rightarrow \quad i<j, \tag{5.1}$$

then the earliest occurence times T_j are expressed by the following reccurence formulas:

$$\left.\begin{array}{ll} 1. & P(T_1=0)=1, \\ 2. & T_j=\max\limits_{i \in N_j}(T_i+X_{ij})=\max\limits_{i \in N_j}(Z_{ij}), \qquad j=2, 3, \ldots, m, \end{array}\right\} \tag{5.2}$$

where $N_j=\{i;\ (i, j) \in A\}$ and $Z_{ij}=T_i+X_{ij}$ is the earliest completion time of activity (i, j).

In the PERT-calculated mean, F_{ij} are replaced by the degenerated distributions $\Theta_{m_{ij}}$ [*see* formula (4.4)] and thus the method of estimating the mean project completion time is equivalent to CPM, i. e.:

$$\left.\begin{array}{ll} 1. & r_1=0, \\ 2. & r_j=\max\limits_{i \in N_j}(r_i+m_{ij}), \qquad j=2, 3, \ldots, m. \end{array}\right\} \tag{5.3}$$

Moreover, it is commonly known that $r_j \leqq E(T_j)$ for $j=1, 2, \ldots, m$.

In turn, a similar procedure can be shown to find the overestimates of $E(T_j)$. In this case we propose to replace F_{ij} by $G(\alpha_{ij}, m_{ij}, b_{ij})$ for all $(i, j) \in A$. By virtue of Theorems 1, 3 and 4 we suggest to apply the following algorithm:

1. $$s_1=0, \qquad b_1=0. \tag{5.4}$$

2. For the succesive events $j=2, 3, \ldots, m$:

$$\bar{m}_{ij}=s_i+m_{ij} \quad \text{for} \quad i \in N_j, \tag{5.5}$$

$$\bar{b}_{ij}=b_i+m_{ij} \quad \text{for} \quad i \in N_j, \tag{5.6}$$

$$\bar{\alpha}_{ij}=\bar{m}_{ij}/(\bar{b}_{ij}-\bar{m}_{ij}) \quad \text{for} \quad i \in N_j, \tag{5.7}$$

$$s_j=h(\bar{\alpha}_{ij}, \bar{b}_{ij};\ i \in N_j), \tag{5.8}$$

$$b_j=\max_{i \in N_j}(\bar{b}_{ij}), \tag{5.9}$$

where the function h is given by (4.3).

The parameters introduced above have their interpretations in the adequate distributions as follows:

$G(\alpha_j, s_j, b_j)$ — the approximation of a c. d. f. F_j of the earliest occurrence time T_j, $j=2, 3, \ldots, n$.

$G(\bar{a}_{ij}, \bar{m}_{ij}, \bar{b}ij)$ – the approximation of a distribution of the earliest completion time Z_{ij}, for $(i, j) \in A$.

The following theorem verifies the method.

Let us present the formal proof that the proposed method follows to deriving the upper bounds for the mean earliest occurrens times of the network events.

THEOREM 5: *Under the assumptions of the method* $F_j \prec G(\alpha_j, s_j, b_j)$, *which yields* $E(T_j) \leqq s_j$, *for* $j = 1, 2, \ldots, n$.

Proof (by induction):

Taking $j=1$ it is obvious that $F_1 = \Theta_0 \overset{df}{=} G(0, 0, 0)$.

Assume now that for any fixed j, $j \neq 1$, $F_i \prec G(\alpha_i, s_i, b_i)$ for all $i \leqq j-1$. We have to prove that $F_j \prec G(\alpha_j, s_j, b_j)$.

At first for all $i \in N_j$ we replace F_{ij} by $G(\alpha_{ij}, m_{ij}, b_{ij})$ having:

$$F_{ij} \prec G(\alpha_{ij}, m_{ij}, b_{ij}), \qquad i \in N_j. \tag{5.10}$$

From theorems 1 *b* and 4 we infer that:

$$F_i * F_{ij} \prec G(\alpha_i, s_i, b_i) * G(\alpha_{ij}, m_{ij}, b_{ij}) \prec G(\bar{\alpha}_{ij}, \bar{m}_{ij}, \bar{b}_{ij}). \tag{5.11}$$

It should be pointed out that in general the equation:

$$F_j(t) = \prod_{i \in N_j} (F_i * F_{ij}(t)) \tag{5.12}$$

is not correct because Z_{ij} for $i \in N_j$ can be dependent (*see* Z_{24} and Z_{34} in figure 2). However as it was shown (e. g. [4]) the inequality:

$$F_j(t) \geqq \prod_{i \in N_j} (F_i * F_{ij}(t)) \tag{5.13}$$

holds and in a consequence we obtain:

$$F_j \prec \prod_{i \in N_j} (F_i * F_{ij}). \tag{5.14}$$

Finally taking into account (5.11), (5.14), theorems 1 *a* and 3 we have:

$$F_j \prec \prod_{i \in N_j} (F_i * F_{ij}) \prec \prod_{i \in N_j} G(\bar{\alpha}_{ij}, \bar{m}_{ij}, \bar{b}_{ij}) \prec G(\alpha_j, s_j, b_j). \tag{5.15}$$

Thus, we have shown that s_j appears to be the upper bound of $E(T_j)$ for $j = 1, 2, \ldots, m$.

6. NUMERICAL EXAMPLE

Let us consider the simple crossed network in figure 2 and analyse in detail the proposed method. Assume that the duration X_{ij} of each activity (i, j) has the c. d. f. $F_{ij} \in \mathscr{F}(m_{ij}, b_{ij})$, where the values for parameters m_{ij}, b_{ij} are indicated above the network arcs in a form (m_{ij}, b_{ij}). For this network the PERT lower bound for $E(T_4)$ is equal to 3 and the expected lengths of three possible paths are the same. In such case we can expect that the errors concerning the PERT bound and our upper bound as well, should be serious.

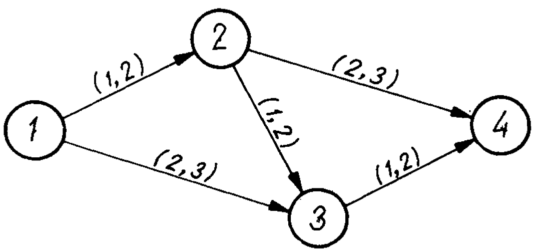

Figure 2. — Network for numerical example.

The use of the proposed algorithm gives:

$j=1$:

$$s_1=0, \qquad b_1=0,$$

$j=2$:

$$s_2=1, \qquad b_2=2,$$

$j=3$:

$$\bar{m}_{13}=2, \qquad \bar{b}_{13}=3, \qquad \bar{\alpha}_{13}=2, \qquad \bar{m}_{23}=2,$$
$$\bar{b}_{23}=4, \qquad \bar{\alpha}_{23}=1, \qquad s_3=h\,(2,\ 3,\ 1,\ 4)=41/16,$$

$j=4$:

$$\bar{m}_{24}=3, \qquad \bar{b}_{24}=5, \qquad \bar{\alpha}_{24}=3/2, \qquad \bar{m}_{34}=57/16,$$
$$\bar{b}_{34}=6, \qquad \bar{\alpha}_{34}=57/39, \qquad s_4=h\,(3/2,\ 5{,}57/39{,}6)=4.1517.$$

It is easily seen that we could estimate the upper bound for the expected completion time by the use of Monte Carlo procedure performed immediately on $G(\alpha_{ij}, m_{ij}, b_{ij})$ distributions. A total of 10000 independent simulated trials has produced a value 3.6854. The increase from 3.6854 to 4.1517 is very significant here. It results from the additional approximations connected with theorem 3 and 4 and particularly with formula (5.13). However these approximations have to be introduced to make the method simple in analytical and manual calculations. The percentage errors for some arbitrary chosen activity distributions are shown in table II. All results have been estimated by simulation with 10000 trials. We see that except the last case the percentage errors for the analytical upper bound are less than for the PERT bound. Considering the bound 3.6854 obtained through the use of simulation, the analogous errors would be much smaller.

TABLE II

Results of computations

Type of activity time distribution	Mean duration	Percentage errors	
		PERT bound	Upper bound
$G(\alpha, m, b)$	3.6854	18.60	12.65
Uniform on $(2m-b, b)$	3.6467	17.73	13.85
Beta on $(0, b)$ with $p+q=3$	3.6291	17,33	14.40
Triangular on $(0, b)$	3.5839	16.29	15.84
Beta on $(0, b)$ with $p+q=6$	3.4901	14.04	18.96

At the end it should be stressed that the above example was purposefully chosen to emphasize that the proposed upper bound can be characterized by the considerable bias. However for other examples analysed by us, the errors have not been so significant. For example we have used our method to evaluate the expected completion time of networks considered in [4, 5]. For the first network we have obtained 12.35 while the simulation result given by Kleindorfer is equal to 11.21. In turn for Martin's example the proposed bound is 21.193 and simulation solution equals 20.524.

APPENDIX

A. THE PROOF OF THEOREM 3

LEMMA 1: *Suppose that Y_1 and Y_2 are independent and $Y_i \sim G(\alpha_i, m_i, b)$, $i=1, 2$. Then a c.d.f. of* max (Y_1, Y_2) *is given by $G(\alpha_1+\alpha_2, m, b)$, where:*

$$m=E(\max(Y_1, Y_2))=b(\alpha_1+\alpha_2)/(1+\alpha_1+\alpha_2). \qquad (A.1)$$

Proof: obvious.

In view of lemma 1 we can further assume that in theorem 3:

$$0=b_0<b_1<b_2\ldots<b_n=b. \qquad (A.2)$$

LEMMA 2: $\alpha>\alpha_n$.

Proof: Since $m>\max(m_1, m_2, \ldots, m_n)$ so in particular $m>m_n$, which yields:

$$\alpha=m/(b-m)>m_n/(b-m_n)=m_n/(b_n-m_n)=\alpha_n. \qquad (A.3)$$

Denote now the c. d. f.'s of max $(Y_1, Y_2, \ldots, Y_n)$ and Y by F and G respectively, i. e.:

$$F=\prod_{i=1}^{n} G(\alpha_i, m_i, b_i) \quad \text{and} \quad G=G(\alpha, m, b). \qquad (A.4)$$

Therefore

$$G(t)=(t/b)^{\alpha} \quad \text{for} \quad t\in[0, b], \qquad (A.5)$$

and

$$F(t)=t^{\beta_i}/\prod_{j=i}^{n} b_j^{\alpha_j} \quad \text{for} \quad t\in(b_{i-1}, b_i]. \qquad (A.6)$$

where:

$$\beta_i=\sum_{j=i}^{n}\alpha_j \quad \text{for} \quad i=1, 2, \ldots, n.$$

Note that the sequence $\{\beta_i\}$ is decreasing, which means:

$$\beta_j>\beta_k \quad \text{for} \quad 1\leqq j<k\leqq n. \qquad (A.7)$$

It is easily seen [(A.5) and (A.6)] that F and G can cross on each interval $(b_{i-1}, b_i]$ at most once and these continuous functions can not cover themselves on any interval contained in their support $(0, b)$. On the other hand F and G have the same mean and therefore they must cross at least once on $(0, b)$ The use of lemma 2 implies:

$$F(t)>G(t) \quad \text{for} \quad t\in(b_{n-1}, b_n), \qquad (A.8)$$

which means that for the last crossing point, say y, F must cross G from below. Hence assuming that $y \in (b_{k-1}, b_k]$ we obtain:

$$\beta_k > \alpha \quad \text{and} \quad 1 \leqq k < n. \tag{A.9}$$

In view of Karlin-Novikoff's criterion it suffices to show now that y is the single crossing point on $(0, b)$.

Suppose that there exists another point $x \in (b_{j-1}, b_j]$ such that $F(x) = G(x)$, $0 < x < y$ and $1 \leqq j < k$. Since F crosses G at y from below, we can assume that F crosses G at x from above. It follows that:

$$\beta_j < \alpha \quad \text{and} \quad 1 \leqq j < k < n. \tag{A.10}$$

In a consequence (A.9) and (A.10) are contrary to (A.7).

B. THE PROOF OF THEOREM 4

Denote the c.d.f.'s of $Y_1 + Y_2$ and Y by F and G respectively, i.e. $F = G(\alpha_1, m_1, b_1) * G(\alpha_2, m_2, b_2)$ and $G = G(\alpha, m, b)$. The density F' is continuous at point b and $F'(b) = 0$. It follows that $F(t) > G(t)$ for all t sufficiently closed to b $(t < b)$. Thus, for the greatest crossing point, F must cross G from below. Moreover, it can be shown that if $F(x) = G(x)$ for $x \in (0, b)$ then $F'(x) > G'(x)$. It implies that if F crosses G at any point $x \in (0, b)$ then F must cross G from below. Therefore we infer that there exists only one crossing point of F and G and Karlin-Novikoff's criterion is applicable.

REFERENCES

1. R. E. Barlow and F. Proshan, *Statistical Theory of Reliability and Life Testing*, Holt, New York, 1975.
2. S. A. Bessler and A. F. Veinott, *Optimal Policy for a Dynamic Multi-Echelon Inventory Model*, Naval Res. Logist. Quart., Vol. 14, No. 4, 1966, p. 355-389.
3. S. Karlin and A. Novikoff, *Generalized Convex Inequalities*, Pacific J. Math, Vol. 13, 1963, p. 1251-1279.
4. G. B. Kleindorfer, *Bounding Distributions for a Stochastic Acyclic Network*, Opns. Res., Vol. 10, No. 7, 1971, p. 1586-1601.
5. J. J. Martin, *Distribution of the Time through a Directed Acyclic Network*, Opns. Res., Vol. 13, No. 1, 1965, p. 46-66.
6. J.-P. Melin, *Proposition d'une solution approchée pour l'étude du maximum de plusieurs variables aléatoires*, R.A.I.R.O., Rech. Op., Vol. 17, No. 2, 1983, p. 175-191.
7. J. M. Taylor, *Comparisons of Certain Distribution Functions*, Math. Operations forsch. Statist., Vol. 14, No. 3, 1983, p. 397-408.