RAIRO. RECHERCHE OPÉRATIONNELLE

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RAIRO. Recherche opérationnelle, tome 19, n° 1 (1985), p. 105-111
<http://www.numdam.org/item?id=RO_1985__19_1_105_0>

R.A.I.R.O. Recherche opérationnelle/Operations Research
(vol. 19, n° 1, février 1985, p. 105 à 111)

FUZZIFIED BLACKWELL'S METHOD TO COMPARE EXPERIMENTS (*)

by Angeles GIL [1] and Teófilo BREZMES [1]

Abstract. — *In this paper, we start by considering the decision problem in which experimentation is possible and the state space is finite.*

Then, we study the problem of selecting the best of two experiments (any experiments from a set of potential experiments which the decision maker wishes to take into account), when available information from each of them is "vague". More precisely, we assume that the possible experimentation only provides information belonging to a fuzzy information system (defined by H. Tanaka, T. Okuda and K. Asai).

The selection problem is approached by extending the Blackwell's method of comparing experiments to the described fuzzy case.

Keywords: statistical decision problem, loss vector, set of attainable loss vectors, fuzzy information system.

Résumé. — *Dans cet article, nous considérons en premier lieu le problème de décision où l'expérimentation peut être réalisée et l'ensemble des états de la nature est fini.*

Puis, nous envisageons le problème de choisir parmi deux expérimentations (dans l'ensemble des expérimentations dont dispose le décideur) alors que l'information qu'ils peuvent fournir soit « imprécise ». Ainsi, nous supposerons que l'expérimentation disponible apporte seulement information qui appartient à un système d'information flou (concept qui a été défini par H. Tanaka, T. Okuda et K. Asai).

Le problème de choix est résolu par l'extension au cas flou de la méthode de Blackwell pour la comparaison des expérimentations.

Mots clés : problème de décision statistique, vector de perte, ensemble des vectors de perte accessibles, système d'information flou.

1. INTRODUCTION

In this article, we shall be concerned with the problem of choice among certain potential experiments. In the Decision Theory, an *experiment* or *probabilistic information system* (*p. i. s.*), is a probability space $\mathbf{X}=(X, \mathscr{B}_X, P)$ where the probability measure P belongs to a specified family of probability

(*) Received in February 1984.

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R.A.I.R.O. Recherche opérationnelle/Operations Research, 0399-0559/85/01 105 07/$ 2.70

measures $\{P_\theta, \theta \in \Theta\}$. We assume that X is a set in the euclidean space $\mathbb{R}$ and $\mathscr{B}_X$ is the smallest Borel σ-field on X.

When $\Theta = \{\theta_1, \theta_2, \ldots \theta_N\}$ symbolizes the set of the possible states of nature, a *statistical decision problem with state space* Θ is characterized by a closed bounded convex subset **A** of $\mathbb{R}^N$, and a *loss function* so that the loss from $\mathbf{a} = (a_1, a_2, \ldots, a_N)$, when the state θ_i is arisen, is a_i ($i = 1, 2, \ldots, N$) [i. e., if **L** denotes the loss function, $a_i = L(\theta_i, \mathbf{a})$]. Consequently, the set **A** may be identified with the space of *terminal actions.* In addition, we suppose that to obtain information on Θ the decision maker may select a p. i. s. from the set of potential p. i. s. whose distributions depend on the state $\theta_i \in \Theta$ chosen by nature.

Blackwell, [1], has stated a preference relation between two p. i. s., assuming that the available information from them is exact. This preference relation is based in the following concepts: "loss vector" and "set of attainable loss vectors". The properties of this relation has been exhaustively studied in [2].

We now proceed to extend Blackwell's method to the more general case in which the available information from a potential experiment is "vague". Formally, we shall consider the situation where the available information from a p. i. s. $\mathbf{X} = (X, \mathscr{B}_X, P)$ belongs to a fuzzy information system on **X**. A *fuzzy information system* on **X** is a fuzzy partition $\mathscr{X} = \{X_1, X_2, \ldots, X_m\}$ on X, so that X_r ($r = 1, 2, \ldots, m$) is a fuzzy event (i. e., the membership function of X_r, μ_{X_r}, is a Borel measurable function from X to [0, 1]).

Remark: It is worth emphasizing that the preference relation above (which is called Blackwell's method) was gathered and exhaustively studied by Blackwell, but is was initially proposed (in a private communication) by Bohnenblust, Shapley and Sherman.

2. BASIC CONCEPTS

Suppose that the available information from the p. i. s. $\mathbf{X} = (X, \mathscr{B}_X, P)$ results in a f. i. s. $\mathscr{X} = \{X_1, X_2, \ldots, X_m\}$. Then, we define:

D. 2. 1. — Any function d that maps the f. i. s. $\mathscr{X}$ into the space of terminal actions **A** is called *decision function.* The class of all decision functions from $\mathscr{X}$ is denoted by $\mathscr{D}_{\mathscr{X}}$.

Following Zadeh, [13], we can define the probability distribution on $\mathscr{X}$, given by:

$$\mathscr{P}_{\theta_i}(X_r) = \int_X \mu_{X_r}(x)\, dP_{\theta_i}(x), \qquad \forall\, \theta_i \in \Theta, \quad r = 1, 2, \ldots, m \qquad (2.1)$$

(the integral being according to Lebesgue-Stieljes sense).

We next extend the concepts of "loss vector" and set of attainable "loss vectors", using Zadeh's probabilistic expression:

D.2.2. – The vector $\mathscr{R}(d)=(\mathscr{R}(\theta_1, d), \mathscr{R}(\theta_2, d), \ldots, \mathscr{R}(\theta_N, d))$, where:

$$\mathscr{R}(\theta_i, d)=\sum_{i=1}^{m} \mathscr{P}_{\theta_i}(X_r)\,\mathbf{L}(\theta_i, d(X_r)) \qquad \text{for} \quad i=1, 2, \ldots, N,$$

is called *loss vector associated with the decision function* $d \in \mathscr{D}_X$.

D.2.3. – The set of points $\mathscr{R}(d)$ above described, where d ranges through $\mathscr{D}_{\mathscr{X}}$, will be called *set of attainable loss vectors* in the decision problem. This set will be denoted by $\mathscr{B}(\mathscr{X}, \mathbf{A})$.

3. PREFERENCE RELATION IN THE FUZZY CASE

Let **X** and **Y** be two potential p. i. s. in the statistical decision problem. Suppose that the available information from **X** and **Y** results in the f. i. s.

$\mathscr{X}=\{X_1, X_2, \ldots, X_m\}$ and $\mathscr{Y}=\{Y_1, Y_2, \ldots, Y_{m'}\}$, respectively.

We say that $\mathscr{X}$ is *more fuzzy informative than* $\mathscr{Y}$, written $\mathscr{X} \overset{\mathbf{B}}{\gtrsim} \mathscr{Y}$, if and only if $\mathscr{B}(\mathscr{X}, \mathbf{A}) \supset \mathscr{B}(\mathscr{Y}, \mathbf{A})$, whatever the closed bounded convex set of terminal actions **A** may be.

We say that $\mathscr{X}$ is *as fuzzy informative as* $\mathscr{Y}$, written $\mathscr{X} \overset{\mathbf{B}}{\sim} \mathscr{Y}$, if and only if $\mathscr{B}(\mathscr{X}, \mathbf{A})=\mathscr{B}(\mathscr{Y}, \mathbf{A})$, whatever the closed bounded convex set **A** may be.

It is worth remarking that as in the non fuzzy case, [1], there are some equivalent conditions for the stated preference relation. Thus, in order to give it an intuitive interpretation, we note that $\mathscr{X} \overset{\mathbf{B}}{\gtrsim} \mathscr{Y}$ if and only if for every choice of $p_i \geqq 0$, $i=1, 2, \ldots N$, such that $\sum_{i=1}^{N} p_i=1$:

$$\min_{d \in \mathscr{D}_x} \sum_{i=1}^{N} p_i \mathscr{R}(\theta_i, d) \leqq \min_{d' \in \mathscr{D}_y} \sum_{i=1}^{N} p_i \mathscr{R}(\theta_i, d'),$$

for all closed bounded convex set of terminal actions.

Obviously, the relation $\underset{B}{\gtrsim}$ determines a partial preordering on the set of f. i. s. corresponding to the potential experiments.

4. PROPERTIES OF THE PREFERENCE RELATION

We now examine some properties of the extended method. First, we recall certain definitions:

D.4.1. — A p. i. s. **X** from which the exact information x can be obtained with a probability irrespective of the state, $\forall x \in X$, is called *null probability information system.*

D.4.2. — If $\mathbf{X}=(X, \mathscr{B}_X, P^1_{\theta_i})$ and $\mathbf{Y}=(Y, \mathscr{B}_Y, P^2_{\theta_i})$, $\theta_i \in \Theta$, are potential p. i. s., the *combined probabilistic information system* is a p. i. s.:

$$\mathbf{X}\times\mathbf{Y}=(X\times Y, \mathscr{B}_{X\times Y}, P_{\theta_i}), \qquad \theta_i \in \Theta,$$

such that $P^1_{\theta_i}$ and $P^2_{\theta_i}$ are the marginal probability measures from P_{θ_i}. Particularly, the p. i. s.

$$\mathbf{X}^{(n)}=\mathbf{X}\times\mathbf{X}\times\overset{(n)}{\dots}\times\mathbf{X}, \qquad n \in \mathbb{N},$$

is called *random sample of size n from* **X**.

D.4.3. — **X** and **Y** are said to be *independent p. i. s.* if and only if $\forall \theta_i \in \Theta$ the probability measure P_{θ_i} associated to $\mathbf{X}\times\mathbf{Y}$ is the product measure of the marginal probability measures $P^1_{\theta_i}$ and $P^2_{\theta_i}$.

D.4.4. — We say that the fuzzy event X_r belonging to a f. i. s. $\mathscr{X}$ is *quite fuzzy* if μ_{X_r} is constant.

In the fuzzy case we introduce following concepts:

D.4.5. — Let $\mathscr{X}$ and $\mathscr{Y}$ be f. i. s. on **X** and **Y**, respectively. We define the *combined fuzzy information system* as the f. i. s. $\mathscr{X}\times\mathscr{Y}$ on $\mathbf{X}\times\mathbf{Y}$ formed by pairs $(X_r, Y_{r'})$, $X_r \in \mathscr{X}$ and $Y_{r'} \in \mathscr{Y}$, so that (following Zadeh, [13]):

$$\mathscr{P}_{\theta_i}(X_r, Y_{r'})=\iint_{X\times Y} \mu_{X_r}(x).\,\mu_{Y_{r'}}(y)\,dP_{\theta_i}(x, y), \qquad \forall \theta_i \in \Theta.$$

Particularly, the f. i. s. $\mathscr{X}^{(n)}=\mathscr{X}\times\mathscr{X}\times\overset{(n)}{\dots}\times\mathscr{X}$, $n \in \mathbb{N}$, on $\mathbf{X}^{(n)}$, is called *fuzzy random sample of size n from* $\mathscr{X}$.

$D.4.6.$ — We define a *statistique from* $\mathscr{X}^{(n)}$ as a real function from $\mathscr{X}^{(n)}$. In addition, we shall usually suppose that a statistic $\mathbf{T}(\mathscr{X}^{(n)})$ from $\mathscr{X}^{(n)}$ is a p. i. s., so that in order to make "coherent" definition D.2.2 we propose to define the *probability distribution of* $\mathbf{T}(\mathscr{X}^{(n)})$ by:

$$Q_{\theta_i}(\mathbf{t}) = \mathscr{P}_{\theta_i}\Big(\bigcup_{\substack{X_r^{(n)} \in \mathscr{X}^{(n)} \\ \mathbf{T}(X_r^{(n)}) = \mathbf{t}}} X_r^{(n)}\Big) = \sum_{\substack{X_r^{(n)} \in \mathscr{X}^{(n)} \\ \mathbf{T}(X_r^{(n)}) = \mathbf{t}}} \mathscr{P}_{\theta_i}(X_r^{(n)}) \qquad (4.1)$$

(where $\bigcup$ denotes the "bold union", Giles [8], Zadeh [14]).

Remark: It should be recalled that a fuzzy partition is an ordinary partition in the sense of the "bold union", $\bigcup$, and the "bold intersection", $\bigcap$. Moreover, a statistic from $\mathbf{X}^{(n)}$ determines a partition on $X^{(n)}$ so that each statistic value may be identified with the set of sample points mapping into that value. In a similar way, the preceding probabilistic definition suggests that each statistic value $\mathbf{t}$ of $\mathbf{T}(\mathscr{X}^{(n)})$ may be identified with the bold union of fuzzy sample points mapping in $\mathbf{t}$, and thus $\mathbf{T}(\mathscr{X}^{(n)})$ determines a new fuzzy partition on $\mathscr{X}^{(n)}$.

The following are some interesting properties guaranteeing the suitability of the preference relation in section 3:

THEOREM 1: *If* $\mathbf{N}$ *is a null p. i. s. resulting in a f. i. s.* $\mathscr{N}$ *and* $\mathbf{X}$ *is a p. i. s. resulting in a f. i. s.* $\mathscr{X}$, $\mathscr{X} \overset{\mathbf{B}}{\gtrsim} \mathscr{N}$.

If Q *is a p. i. s. resulting in a f. i. s. where the available information is quite fuzzy,* $\mathscr{Q}$, *and* $\mathbf{X}$ *is a p. i. s. resulting in the f. i. s.* $\mathscr{X}$, $\mathscr{X} \overset{\mathbf{B}}{\gtrsim} \mathscr{Q}$.

THEOREM 2: *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two p. i. s. resulting in the f. i. s.* $\mathscr{X}$ *and* $\mathscr{Y}$, *respectively. Then:*

$$\mathscr{X} \times \mathscr{Y} \overset{\mathbf{B}}{\gtrsim} \mathscr{X}.$$

Furthemore, if:

(i) *the conditional distribution of* $\mathbf{Y}$ *given* $X_r \in \mathscr{X}$ *does not depend on the true state (that is,* $\mathbf{Y}$ *cannot add statistical information about* Θ *to what is contained in* X_r),

or:

(ii) $\mathscr{Y}$ *only provides fuzzy information quite fuzzy:*

$$\mathscr{X} \times \mathscr{Y} \overset{\mathbf{B}}{\sim} \mathscr{X}.$$

THEOREM 3: *A fuzzy random sample is more fuzzy informative that its size is greater.*

THEOREM 4: *Let* **X**, **Y**, **Z** *be p. i. s. resulting in the f. i. s.* $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$, *respectively. Suppose* **X** *and* **Y** *independent,* **Z** *and* **Y** *independent too. Then:* $\mathscr{X} \overset{\mathbf{B}}{\gtrsim} \mathscr{Z}$ *implies that:*

$$\mathscr{X} \times \mathscr{Y} \overset{\mathbf{B}}{\gtrsim} \mathscr{Z} \times \mathscr{Y},$$

THEOREM 5: *Let* $\mathscr{X} = \{X_{11}, \ldots, X_{1n_1}, X_{21}, \ldots, X_{2n_2}, \ldots, X_{m1}, \ldots, X_{mn_m}\}$ *be a f. i. s. on the p. i. s.* **X**, *and let* $\mathscr{X}^0 = \{X_1^0, X_2^0, \ldots, X_m^0\}$ *be the f. i. s. on* **X** *defined as follows:*

$$X_r^0 = \bigcup_{j=1}^{n_r} X_{rj}$$

(i. e., the elements in $\mathscr{X}^0$ *are obtained by grouping elements in* $\mathscr{X}$ *by means of the bold union). Then:*

$$\mathscr{X} \overset{\mathbf{B}}{\gtrsim} \mathscr{X}^0.$$

THEOREM 6: *Let* **X** *be a p. i. s. resulting in the f. i. s.* $\mathscr{X}$. *Let* $\mathscr{X}^{(n)}$ *be a fuzzy random sample from* $\mathscr{X}$. *Consider a statistic* $\mathbf{T}(\mathscr{X}^{(n)})$ *from* $\mathscr{X}^{(n)}$. *If condition* (4.1) *is assumed:*

$$\mathscr{X}^{(n)} \overset{\mathbf{B}}{\gtrsim} \mathbf{T}(\mathscr{X}^{(n)}).$$

[We wish to point out that a p. i. s. as $\mathbf{T}(\mathscr{X}^{(n)})$ is a very special case of f. i. s. on the subset of $\mathbb{R}$ formed by values from $\mathbf{T}(\mathscr{X}^{(n)})$.]

5. CONCLUDING REMARKS

The proposed method may be extended to the more general situation in which the states are fuzzy too. This extension can be accomplished in a way similar to what is followed in [5, 7]. Moreover, the methods [3, 4] may also extended to the fuzzy case.

Finally, we want to emphasize that this method is stronger than the criterion in [5].

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