WILLIAM V. GEHRLEIN
THOMAS H. McINISH
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OPTIMAL SYSTEMATIC
WITHDRAWAL STRATEGIES (*)

by William V. Gehrlein (^{1}) and Thomas H. McInish (^{2})·(^{3})

Abstract. — *A systematic withdrawal plan is an investment program in which an investor makes an initial investment and, thereafter, receives payments from the income and principal. We consider a plan under which the investor withdraws all of his income and a constant percentage of his principal ($1/n$) each period. If an individual actually begins a systematic withdrawal plan, we show that the optimal withdrawal policy is to set n equal to his maximum horizon.*

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Résumé. — *Un programme de retrait systématique est un programme d'investissement dans lequel une personne place des fonds initiaux, et après, reçoit des paiements du revenu et du principal. Nous sommes en train de considérer un programme sous lequel la personne qui place ses fonds retire son revenu entier et un pourcentage fixe du principal chaque période. Si une personne commence un programme de retrait systématique, nous démontrons que le plan de retrait idéal est d'établir un équivalent à son horizon maximal.*

In a systematic withdrawal plan, an investor makes an initial investment and, thereafter, receives payments from the income and principal according to an agreed upon formula. In this paper, based on assumptions specified below, we develop criteria which can be used by an investor to decide whether to begin a specified type of systematic withdrawal program and to ascertain the optimal duration of that program. Investment programs similar to the one we describe could be designed using trusts, variable annuities, or unit trusts which are called "mutual funds" in the United States.

In the following section, we describe our assumption concerning (1) the type of systematic withdrawal plan available, (2) the goals of investors, and (3) the environment within which the plan operates. Then, we present our results beginning with a simplified case and, thereafter, considering a more complex case. Next, we discuss our results and, finally, we summarize our findings.

1. THE MODEL

In this paper we consider a systematic withdrawal plan in which an investor makes an initial investment k. In each period i the investment earns a return of R_i. The distribution of R_i has a constant mean and variance. The investor's horizon is n, and during each period the investor makes a withdrawal of k/n plus any investment income. There is a probability q that the investment will fail in any particular period. If the investment does fail in any particular period, there are no subsequent withdrawals. The investor's discount rate for this investment is d, which may be viewed as the investor's opportunity cost or hurdle rate. The probability q and the discount rate d are constant over n. Finally, the investor's goal is to maximize expected payoff.

A. The simplified case

Letting $q = 0$ and $d = 0$, the expected return of the systematic withdrawal plan described above is:

$$E\left\{ \sum_{i-1}^{n} \left[\left\{ k - \frac{(i-1)k}{n} \right\} R_i + \frac{k}{n} \right] \right\}. \tag{1}$$

Noting that for independent observations the expected value of a sum is the sum of the expected values, and using the rules for sums of powers of integers, we obtain:

$$k + k\left(\frac{n+1}{2}\right)r, \tag{2}$$

where r is the expected value of R. Thus, according to equation (2) the expected value of the withdraws equals the initial investment, plus the average return per period times the average amount of the investment per period. For this simplified case it is apparent that as long as $r > 0$ an investor whose goal is to maximize equation (2) will invest in a systematic withdrawal plan with a maximum horizon (n).

B. The unrestricted case

Letting $q \geq 0$ and $d \geq 0$, the expected payoff from a systematic withdrawal plan which lasts n periods (E_n) is:

$$E_n = \sum_{i-1}^{n} \left(\frac{1-q}{1+d}\right)^i \left\{ \left[k - \frac{(i-1)k}{n} \right] r + \frac{k}{n} \right\}, \tag{3}$$

which reduces to:

$$E_n = \frac{k}{n}\left(\frac{1-q}{1+d}\right)^{n+1} \sum_{i=1}^{n} \left\{\left(\frac{1+d}{1-q}\right)^i (ir+1)\right\}. \tag{4}$$

We can show $E_2 \geqq E_1$ if and only if:

$$\frac{1-q}{1+d} \geqq \frac{1}{1+r}. \tag{5}$$

We wish to show that, in general $E_{n+1} \geqq E_n$ if and only if equation (5) holds. The proof is straightforward. Assume that equation (5) holds so that $E_{n+1} \geqq E_n$ for some n. We know this is true since it holds when $n=1$. We now show that it is impossible for $E_{n+1} \geqq E_{n+2}$ if equation (5) holds so that the general relationship is valid. That is if:

$$\frac{1-q}{1+d} \geqq \frac{1}{1+r}, \tag{6}$$

$E_{n+1} \geqq E_n$ for all n.

To do this proof we begin by obtaining the recursion relation:

$$E_{n+1} - E_n = \frac{k(1-q)}{(n+1)(1+d)} E_n + \frac{k(1-q)}{(n+1)(1+d)}[(n+1)r+1], \tag{7}$$

which follows directly from equation (4). Furthermore:

$$E_{n+1} - E_n = \frac{k(1-q)}{(n+1)(1+d)}[(n+1)r+1] - \frac{(n+1)(1+d)-n(1-q)}{(n+1)(1+d)} E_n. \tag{8}$$

Thus:

$$E_{n+1} \geqq E_n$$

if and only if:

$$E_{n+1} - E_n \geqq 0$$

or:

$$\frac{k(1-q)[(n+1)r+1]}{(n+1)(1+d)-n(1-q)} \geqq E_n. \tag{9}$$

Using the recursion relation of equation (7) for E_n in equation (9) gives $E_n \geqq E_{n+1}$ if and only if:

$$\frac{k(1-q)[(n+1)r+1]}{(n+1)(1+d)-n(1-q)} \geqq E_{n+1}. \tag{10}$$

As we hypothesized, assume it is also true that $E_{n+1} \geqq E_{n+2}$. From equation (10) this is true if and only if:

$$\frac{k(1-q)[(n+2)r+1]}{(n+2)(1+d)-(n+1)(1-q)} \leqq E_{n+2}. \tag{11}$$

If $E_{n+1} \geqq E_{n+2}$, equations (10) and (11) require:

$$\frac{(n+1)r+1}{(n+1)(1+d)-n(1-q)} \geqq \frac{(n+2)r+1}{(n+2)(1+d)-(n+1)(1-q)}. \tag{12}$$

After reduction, equation (12) becomes:

$$\frac{1}{1+r} \geqq \frac{1-q}{1+d},$$

which is contrary to the assimption that equation (5) holds.

Thus, when:

$$\frac{1-q}{1+d} \geqq \frac{1}{1+r}, \tag{13}$$

then:

$$E_{n+1} \geqq E_n \quad \text{for all } n. \tag{14}$$

When:

$$\frac{1-q}{1+d} \leqq \frac{1}{1+r}, \tag{15}$$

we have:

$$E_{n+1} \leqq E_n \quad \text{for all } n. \tag{16}$$

2. DISCUSSION

Now that we have derived the results presented in equations (13)-(16), it will be helpful to discuss our findings. When equation (13) holds the optimal decision rule for an investor who seeks to maximize his expected return is to set n as long as possible. For example, the investor might set n equal to his life expectancy. If the investor desires to leave an estate then n can be set equal to a longer horizon. On the other hand, when equation (15) holds, the investor would not undertake the investment at all. Finally, if by chance $(1-q)/(1+d)=1/(1+r)$ then the investor is indifferent to the horizon at which n is set. Since the last possibility would not seem too likely, we conclude that, in general, the optimal policy for an individual who actually invests in a systematic withdrawal plan is to set n equal to his maximum horizon.

In addition to the conclusions presented above, equations (13)-(16) also allow other insights. First, given our assumptions, investors are indifferent between choices of q and d which produce the same ratio of $(1-q)/(1+d)$. Next, if $q=0$ then the choice of horizon depends entirely on the relationship between the discound rate (d) and the rate of return (r). If $d<r$, then the investor chooses a long horizon. If $d>r$, then the investment is not undertaken. Finally, when $d=r$, if $q>0$ the investment will not be undertaken and if $q=0$ the investor will be indifferent between undertaking and not undertaking the investment.

3. SUMMARY

In this paper we have considered optimal strategies for an investor to follow in establishing a systematic withdrawal program. We have shown that given our assumptions the optimal horizon depends on the relationship between the probability of failure, the investor's discount rate and the rate of return on the investment. Generally, an investor who undertakes a systematic withdrawal plan should select a withdrawal period equal to his maximum investment horizon.