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DIRECT CALCULATION OF SENSITIVITY TO THE COEFFICIENTS OF THE BASIC SUBMATRIX IN PARAMETRIC LINEAR PROGRAMMING (*)

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Abstract. — *This paper demonstrates a method for direct calculation of sensitivities with respect to coefficients in the constraints of a linear program. Sensitivity coefficients are derived for both marginal changes and finite changes in one or a group of constraint coefficients corresponding to basic variables.*

An example of application is given in the field of power flow analysis with changing admittance in a meshed electric power transmission and generation system. Another application for parametric analysis in the modeling of the entire energy sector is discussed.

Keywords: linear programming, sensitivity, constraint coefficients, direct method.

Résumé. — *Au terme d'un bref développement mathématique, une formule simple est donnée pour le calcul direct du changement de la valeur optimale de la fonction-objectif d'un programme linéaire pour un changement infinitésimal donné à un ou plusieurs coefficients des contraintes, lorsque ces coefficients multiplient des variables appartenant à la base optimale du programme linéaire. Cette formule utilise exclusivement les valeurs optimales des variables primales et duales du problème.*

Une seconde formule est donnée pour le calcul direct du changement de la valeur optimale de la fonction-objectif lorsque les changements aux coefficients des contraintes ont une valeur non infinitésimale mais finie.

Mots clés : programmation linéaire, sensibilité, coefficients des contraintes, méthode directe.

1. INTRODUCTION

The purpose of this paper is to demonstrate a direct way of computing sensitivity coefficients in linear programs when a constraint coefficient corresponding to a basic variable is incurring small changes. These results will generalize the marginal information derived from the duality theory regarding the sensitivity to changes in the cost coefficients and in the right-hand-side terms of the constraints of a linear program.

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The difficulty of computing these sensitivities is due to the intricate interrelationships between all the variables within linear programs when the basic submatrix is modified. Indeed, changing one term of that matrix changes all the terms of its inverse.

Two types of sensitivity coefficients will be determined. The simplest one will characterize the sensitivity of the objective function to infinitesimal change in the modified constraint coefficient (gradient). The second type of sensitivity coefficient will be valid for finite change in the modified constraint coefficient under the assumption that the current basis remains optimal. The problem of determining the validity range of that assumption has already received a satisfactory solution [1].

These results will then be generalized to the case of simultaneous changes of several coefficients in the basic submatrix.

2. BASIC CHARACTERISTICS OF LP PROBLEMS

The "canonical" formulation of linear minimization problems is as follows:

$$\text{Minimize} \quad Z = \sum_{j=1}^{n} c_j x_j, \tag{1}$$

subject to equality and inequality constraints:

$$\sum_{j=1}^{n} a_{ij} x_j \geqq d_i, \tag{2}$$

$$x_j \geqq 0, \tag{3}$$

with $i = 1, 2 \ldots m$ and $j = 1, 2 \ldots n$.

The dual of problem (1) is:

$$\text{Maximize} \quad W = \sum_{i=1}^{m} u_i d_i, \tag{4}$$

subject to:

$$\sum_{i=1}^{m} u_i a_{ij} \leqq c_j, \tag{5}$$

$$u_i \geqq 0, \tag{6}$$

with $i = 1, 2 \ldots m$ and $j = 1, 2 \ldots n$.

Using the following matrix notations:

C, row vector of c_j; X, column vector of x_j; D, column vector of d_i; U, row vector of u_i; A, matrix of coefficients a_{ij}

the primal problem (1) and the dual problem (4) can be written in a compact form:

Primal:

$$\operatorname{Min} Z = C.X, \tag{7}$$

subject to:

$$A.X \geqq D, \tag{8}$$

$$X \geqq 0. \tag{9}$$

Dual:

$$\operatorname{Max} W = U.D, \tag{10}$$

subject to:

$$U.A \leqq C, \tag{11}$$

$$U \geqq 0. \tag{12}$$

The following relationships describe well known properties of the LP solution [1, 2]:

$$Z = C^B.X^B, \tag{13}$$

$$X^B = B^{-1}.D, \tag{14}$$

$$U = C^B.B^{-1}, \tag{15}$$

where B is the non-singular submatrix of A which corresponds to the basic variables X^B as follows:

$$\underset{m,\,n}{A} = (\underset{m,\,m}{B} \mid \underset{m,\,n-m}{H}), \tag{16}$$

$$X = \left(\frac{X^B}{X^H}\right), \tag{17}$$

X^H is the set of out-of-base variables.

Using equations (13) to (15) yields directly the following sensitivity coefficients:

$$\frac{\partial Z}{\partial c_j} = x_j, \tag{18}$$

$$\frac{\partial Z}{\partial d_i} = u_i. \tag{19}$$

The following sections will derive similar formulas for small changes in the coefficients a_{ij} when the variable x_j belongs to the optimal basis.

3. MATRIX INVERSION LEMMA

3.1. General formulation

It is shown in [3] that, if a matrix B is modified into $\tilde{B}$ by:

$$\underset{m,\,m}{\tilde{B}} = \underset{m,\,m}{B} + \underset{m,\,1}{\alpha} \,.\, \underset{1,\,m}{\beta^T} \tag{20}$$

and if the inverse B^{-1} is known, then the inverse $\tilde{B}^{-1}$ can be computed as follows:

$$\boxed{\underset{m,\,m}{\tilde{B}^{-1}} = \underset{m,\,m}{B^{-1}} - \underset{m,\,m}{B^{-1}} \,.\, \underset{m,\,1}{\alpha} \,.\, \underset{1,\,1}{(1+\beta^T . B^{-1} . \alpha)^{-1}} \,.\, \underset{1,\,m}{\beta^T} \,.\, \underset{m,\,m}{B^{-1}}} \tag{21}$$

The two vectors α and β are any two vectors. The proper selection of α and β allow to perform a wide range of modifications to B. Equation (21) is referred to as the Matrix Inversion Lemma for inversion of modified matrices.

3.2. Modification to one element of B

If $\tilde{B}$ is to differ from B by only one element:

$$\tilde{b}_{ij} = b_{ij} + \Delta b_{ij}, \tag{22}$$

then, equation (20) can be advantageously rewritten as:

$$\tilde{B} = B + \Delta b_{ij} . \varepsilon_i . \varepsilon_j^T, \tag{23}$$

where ε_j^T is the row vector in which all entries are zeroes but entry j which is equal to one :

$$\varepsilon_j^T = \underset{\begin{matrix}1 & & j & & m\end{matrix}}{\boxed{\begin{matrix}0 & 0 & 1 & 0 & 0\end{matrix}}}$$

so that $\varepsilon_i . \varepsilon_j^T$ is the null-matrix but in entry ij:

$$\begin{array}{c|ccc|} \multicolumn{1}{c}{} & & j & \multicolumn{1}{c}{} \\ \cline{2-4} & 0 & \vdots & 0 \\ i & \cdots & 1 & \\ & 0 & & \\ \cline{2-4} \end{array}$$

In this particular case, the matrix inversion lemma (21) takes the simple form:

$$\tilde{B}^{-1}=B^{-1}-\Delta b_{ij}.B^{-1}.\varepsilon_i.(1+\varepsilon_j^T.B^{-1}.\varepsilon_i.\Delta b_{ij})^{-1}.\varepsilon_j^T.B^{-1}. \tag{24}$$

Let us denote by b_{ij}^{-1} the term (i, j) of B^{-1}.

Since $\varepsilon_j^T.B^{-1}.\varepsilon_i=b_{ij}^{-1}$, equation (24) can be written:

$$\tilde{B}^{-1}=B^{-1}-\Delta b_{ij}\frac{B^{-1}.\varepsilon_i.\varepsilon_j^T.B^{-1}}{1+b_{ij}^{-1}.\Delta b_{ij}}, \tag{25}$$

which is valid for the modification (22).

4. SENSITIVITY ANALYSIS IN LP PROBLEMS

4.1. Modification of one constraint coefficient of a basic variable

Let X be the optimal solution of problem (7) before modification to matrix A, Z be the corresponding value of the objective function, and U be the vector of dual variables. Our goal is to determine the change in the objective function Z for a given change in the constraints coefficients corresponding to basic variables i.e. in matrix B.

We assume that the change in one of the terms of B does not change the composition of the optimal basis. The validity limits of this assumption are stated in chapter 7 of [1] and will not be discussed here. The non singularity of B is also a classical assumption that does not need further discussion. Under these assumptions, and, after deletion of the superscript B, the new solution is:

$$\tilde{Z}=C.\tilde{X}, \tag{26}$$

$$=Z+\Delta Z. \tag{27}$$

But, equation (14) gives:

$$\tilde{X}=\tilde{B}^{-1}.D, \tag{28}$$

so that:

$$Z+\Delta Z=C.\tilde{B}^{-1}.D. \tag{29}$$

Using the matrix inversion lemma (25) for $\tilde{B}^{-1}$ in (29) yields:

$$Z+\Delta Z=C.B^{-1}.D-\Delta b_{ij}\frac{C.B^{-1}.\varepsilon_i.\varepsilon_j^T.B^{-1}.D}{1+b_{ij}^{-1}.\Delta b_{ij}}, \tag{30}$$

so that:

$$\frac{\Delta Z}{\Delta b_{ij}}=\frac{-U.\varepsilon_i.\varepsilon_j^T.X}{1+b_{ij}^{-1}.\Delta b_{ij}}.$$

Finally, for a finite change Δb_{ij}:

$$\boxed{\frac{\Delta Z}{\Delta b_{ij}}=\frac{-u_i.x_j}{1+b_{ij}^{-1}.\Delta b_{ij}}.} \tag{31}$$

While, for an infinitesimal change, by taking the limit for Δb_{ij} tending to zero:

$$\boxed{\frac{\partial Z}{\partial b_{ij}}=-u_i.x_j.} \tag{32}$$

It is worth noticing that the sensitivity coefficients given by equations (31) and (32) are valid even if the modification Δb_{ij} apply to an element b_{ij} that was equal to zero in the initial basic submatrix $\bar{B}$.

4.2. General perturbation of several constraint coefficients

If the perturbation can be described by:

$$\tilde{B}=B+h.\alpha.\beta^T, \tag{33}$$

then the sensitivity of Z to a finite change magnitude h is given by:

$$\boxed{\frac{\Delta Z}{\Delta h}=\frac{-U.\alpha.\beta^T.X}{1+h\beta^T.B^{-1}.\alpha},} \tag{34}$$

and, for an infinitesimal change:

$$\boxed{\frac{\partial Z}{\partial h}=-U.\alpha.\beta^T.X.} \tag{35}$$

5. APPLICATION

The above sensitivity analysis has been applied with success to various problems of which two examples are outlined hereafter.

5.1. Sensitivity of load curtailment to branch reinforcement in electric power transmission system

The objective in this application was to find a method for evaluating, in an overloaded network, the efficiency of branch reinforcements in reducing the load curtailment.

The problem of satisfying real power demand D_k at node k in a limited capacity transmission network and generation system is formulated as the following linear program (a linear approximation of the real power flow equations is used):

$$\operatorname{Min} Z = \sum_k R_k. \tag{36}$$

Subject to:

– node balance:

$$A.\Theta + G = D - R; \tag{37}$$

– generation limits:

$$\underline{G} \leqq G \leqq \overline{G}; \tag{38}$$

– transit limits:

$$-\overline{\Psi} \leqq S.\Theta \leqq \overline{\Psi}; \tag{39}$$

where $R=(R_k)$, is the vector of load curtailments necessary to satisfy the constraints; $G=(G_k)$, is the generation vector; $D=(D_k)$, is the load vector; $\Theta=(\theta_k)$, is the vector of voltage angles; S, is the network incidence matrix; $\Psi=(\psi_{km})$, is the vector of voltage angle differences across branches: $\psi_{km}=\theta_k-\theta_m$; $A=(a_{km})$, is the matrix of branch admittances (nodal admittance matrix).

The modification of the admittance of branch km entails a modification of 4 terms of matrix A described as follows:

$$\tilde{A} = A + \Delta a_{km}.\alpha.\alpha^T, \tag{40}$$

where $\tilde{A}$, is the modified admittance matrix; α^T, is the row:

$$\begin{array}{|ccccc|} \hline 0 & 1 & 0 & -1 & 0. \\ \hline \end{array}$$
$$\qquad k \qquad m$$

Applying the formula (35) for infinitesimal change of the admittance a_{km} yields the following sensitivity coefficient:

$$\frac{\partial Z}{\partial a_{km}} = (u_m - u_k) . (\theta_k - \theta_m), \qquad (41)$$

where u_k is the dual variable associated with constraint (37) at node k. Note that this particular result (41) was already derived in a different way in [4].

The above sensitivity coefficient was used to rank the reinforcement candidates for an East-European transmission network, and has proved to be a very effective investment selection criterion [5].

5.2. Sensitivity analysis in LP energy models

Many of the energy models developed during the seventies [6], formulate the energy system optimization problem as a linear program. Some approaches use the information contained in the dual variables of the linear program but as far as we know, no successful attempt was made to evaluate the sensitivity with respect to changes in the coefficients of the constraints describing the interactions within the energy system.

However, some coefficients appearing in the matrix of constraints are of prime importance:

- efficiency of the conversion processes;
- availability rates;
- equipment lifetime.

Sometimes, especially for new technologies, these coefficients are not well known or subject to sudden changes in case of a technological breakthrough.

Preliminary investigations have been conducted on the large multiyear, multinational energy model (EFOM-12C) developed for the European Community [7]. Even in such a large LP (6,000 variables, 5,000 constraints) with some degeneracy, the sensitivity analysis using the concepts developed in this paper has shown promising results. A simplified formulation is shown hereafter to illustrate these results.

Let C_t, be the present worth of the operating cost of a process at year t; x_t, be the operating level of the process at year t; I_t, be the present worth of the investment charge at year t for increasing the capacity of the same process; z_t, be the capacity invested in year t; D_t, be the demand for the output of this process at year t; η, be the conversion efficiency of the process; $r = 1/\text{AVR}$, be the reciprocal of the availability rate to indicate the needed capacity reserve for facing forced outages of the equipment; R_t, be the time evolution of the existing capacity at year 0 given the retirement schedule; L, be the lifetime of the equipment; T, be the horizon of the study.

With the above notations, the cost minimization problem is formulated as follows:

$$\text{minimize} \quad Z = \sum_{t=1}^{T} C_t . x_t + I_t . z_t, \tag{42}$$

subject to:

Constraints	*Dual Variable*	
$\eta . x_t \geqq D_t$	u_t	(43)
$\sum_{k=t-L}^{t} z_k - r . x_t \geqq -R_t$	w_t	(44)

Applying formula (35) to the above problem, yields three interesting results:

– Sensitivity to process efficiency.

If the efficiency η is marginally increased from year θ on, the cost will be changed as follows:

$$\frac{\partial Z}{\partial \eta_\theta} = - \sum_{t=\theta}^{T} u_t . x_t, \tag{45}$$

which quantifies the cost reduction in case of efficiency increase.

– Sensitivity to equipment lifetime.

If all equipment installed at year θ and to be normally retired at year $\theta + L$ is prolonged by one year, then the cost will be changed as follows:

$$\frac{\partial Z}{\partial L_\theta} = - w_{\theta+L+1} . z_\theta \tag{46}$$

(assuming that the investment charges are not modified by this increased lifetime).

If the lifetime of all new equipment is prolonged, the sensitivity of the objective function is given by:

$$\frac{\partial Z}{\partial L} = - \sum_{t=1}^{T-L-1} w_{t+L+1} . z_t, \tag{47}$$

– Sensitivity to availability.

If the coefficient r reflecting the availability of the process is changed marginally starting from year θ, the change in the cost is as follows:

$$\frac{\partial Z}{\partial r_\theta} = \sum_{t=\theta}^{T} \dot{w}_t . x_t . \tag{48}$$

Note that an increase of r corresponds to a decrease of the availability rate. Indeed, since $r = 1/\text{AVR}$, one has:

$$\frac{\partial Z}{\partial \text{AVR}} = \frac{\partial Z}{\partial r} \frac{\partial r}{\partial \text{AVR}} = \frac{-1}{\text{AVR}^2} \frac{\partial Z}{\partial r}, \tag{49}$$

this formula provides for quantification of the cost reduction in case of availability increase.

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