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DISCRETE TIME ORDERING POLICIES WITH MINIMAL REPAIR (*)

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Abstract. — *We treat the ordering policies with minimal repair which are operated at discrete times. We obtain the optimum policy which minimizes the expected cost rate in the steady-state by introducing an expedited order, a regular order, a minimal repair, a salvage, and an inventory costs and a lead time. It is shown that there exists a finite and unique optimum policy under certain conditions.*

Résumé. — *Nous traitons les politiques de commande comportant une réparation minimale effectuées à des instants discrets. Nous obtenons la politique optimale qui minimise l'espérance du coût par unité de temps dans le comportement asymptotique en introduisant une commande accélérée, une commande régulière, une réparation minimale, un coût de récupération, un coût de stockage, et un délai. On montre qu'il existe, sous certaines conditions, une politique optimale finie et unique.*

1. INTRODUCTION

In recent years, systems have become more large-scale and complicated, such as cars, airplanes, and missiles. These systems cause great damages once they fail. Thus, the necessity of maintenance policies have increased and many authors have studied such policies. For example, Barlow and Proschan [1] discussed an age replacement, a block replacement, and an inspection policies, and Kaio and Osaki [2] discussed an ordering policy. In the former the spares are provided immediately if necessary, but in the latter the spares are not always provided instantaneously even if necessary and there is usually any delay between ordered time and delivered time.

In this paper we treat the ordering policy with minimal repair which are operated at discrete times (*see* Nakagawa and Osaki [3] and Weiss [5]), and obtain the optimum policy which minimizes the expected cost rate in the steady-state introducing some costs and a lead time. We show that there exists a finite and unique optimum policy under certain conditions.

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2. MODEL AND ASSUMPTIONS

Consider a one-unit system, where each spare is only provided after a lead time by an order. Each failure is detected instantaneously and for each failed unit *minimal repair* [by which system failure rate is undisturbed (*see* Barlow and Proschan [1], p. 96)] is made with negligible time. The original unit starts operating at time point 0. The planning horizon is infinite. If the original unit does not fail up to a prespecified time point N ($N = 0, 1, 2, \ldots$), then the regular order for a spare is made at N and the spare is delivered after a lead time L. If the original unit fails up to the delivery of the spare, then the minimal repair is made for each failure, and immediately the original unit is replaced by the spare as soon as it is delivered. Also, if the original unit does not fail up to the delivery of the spare, the delivered spare is put into inventory until the original unit fails. At failure time point

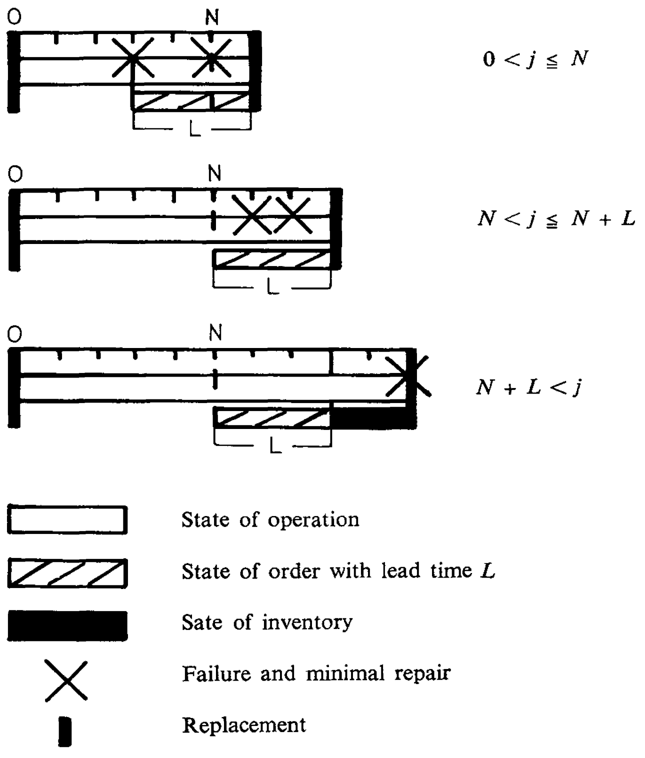

Figure 1. — Possible realizations of one cycle.

the original unit is replaced by that spare and the minimal repair is made for the failed unit simultaneously. The spare in inventory does not fail or deteriorate. On the other hand, if the original unit fails up to N, then at failure time point the minimal repair and the expedited order is made simultaneously, and the original unit is replaced by the spare as soon as it is delivered after a lead time L. If the original unit fails up to the delivery of the spare, then the minimal repair is made for each failure. In this case, the regular order is not made. Each replacement is made instantaneously, and after a replacement the system starts operating immediately. The cycle then repeats.

The lifetime for each unit obeys an arbitrary discrete time distribution $\{p_j\}_{j=1}^{\infty}$, i. e., p_j is the probability that the unit fails at time point j ($j = 1, 2, 3, ...$), with a finite mean $1/\lambda$. Let us introduce the following five costs. The cost c_1 is suffered for each expedited order made up to N, the cost c_2 is suffered for each regular one made at N, and the cost c_3 is suffered for each minimal repair made at each failure time point. The cost k per unit time is suffered for inventory, and furthermore the cost s is suffered as a salvage cost at each replacement time point. We assume that $c_1 > c_2$. These assumptions seem to be reasonable.

Under the above assumptions, we define an interval from one replacement to the following replacement as one cycle, and analyze this model (see *Fig.* 1).

3. ANALYSIS AND THEOREM

Let us define the failure rates as follows;

$$r(N) \equiv p_N \Big/ \sum_{j=N}^{\infty} p_j, \tag{1}$$

$$R(N) \equiv \sum_{j=N}^{N+L-1} p_j \Big/ \sum_{j=N}^{\infty} p_j. \tag{2}$$

Both functions $r(N)$ and $R(N)$ have the same monotone properties with respect to non-negative N, as same as the continuous type.

Next, we obtain the expected cost per one cycle $A(N)$:

(i) the expected order cost is

$$\sum_{j=1}^{N} c_1 p_j + \sum_{j=N+1}^{\infty} c_2 p_j = c_1 \sum_{j=1}^{N} p_j + c_2 \sum_{j=N+1}^{\infty} p_j. \tag{3}$$

(ii) when the original unit fails, the minimal repair is always made for each failure. Then, the expected cost for minimal repairs is

$$c_3\left[\sum_{j=1}^{N}\left\{1+\sum_{i=j+1}^{j+L} r(i)\right\}p_j + \sum_{j=N+1}^{N+L}\left\{1+\sum_{i=j+1}^{N+L} r(i)\right\}p_j + \sum_{j=N+L+1}^{\infty} p_j\right] = c_3\left[1+\sum_{j=1}^{N}\sum_{i=j+1}^{j+L} r(i)\,p_j + \sum_{j=N+1}^{N+L}\sum_{i=j+1}^{N+L} r(i)\,p_j\right]. \quad (4)$$

(iii) when the spare is delivered, if the original unit does not fail, the spare is put into inventory until the original unit fails. Then, the expected cost for inventory is

$$k\sum_{j=N+L+1}^{\infty}(j-N-L)\,p_j = k\sum_{i=N+L+1}^{\infty}\sum_{j=i}^{\infty} p_j. \quad (5)$$

Thus, from the above three expected costs and a salvage cost, $A(N)$ is

$$A(N) = c_1\sum_{j=1}^{N} p_j + c_2\sum_{j=N+1}^{\infty} p_j + c_3\left[1+\sum_{j=1}^{N}\sum_{i=j+1}^{j+L} r(i)\,p_j + \sum_{j=N+1}^{N+L}\sum_{i=j+1}^{N+L} r(i)\,p_j\right] + k\sum_{i=N+L+1}^{\infty}\sum_{j=i}^{\infty} p_j + s. \quad (6)$$

Also, the mean time of one cycle, $B(N)$ is

$$B(N) = \sum_{j=1}^{N}(j+L)\,p_j + \sum_{j=N+1}^{N+L}(N+L)\,p_j + \sum_{j=N+L+1}^{\infty} j p_j = 1/\lambda + \sum_{i=N+1}^{N+L}\sum_{j=1}^{i-1} p_j. \quad (7)$$

Thus, the expected cost rate in the steady-state $C(N)$ is

$$C(N) = A(N)/B(N) \quad (8)$$

(*see* Ross [4], p. 52), and

$$C(0) = \left[c_2 + c_3\left\{1+\sum_{j=1}^{L}\sum_{i=j+1}^{L} r(i)\,p_j\right\} + k\sum_{i=L+1}^{\infty}\sum_{j=i}^{\infty} p_j + s\right]\Bigg/\left[1/\lambda + \sum_{i=1}^{L}\sum_{j=1}^{i-1} p_j\right], \quad (9)$$

$$C(\infty) = \left[c_1 + c_3\left\{1+\sum_{j=1}^{\infty}\sum_{i=j+1}^{j+L} r(i)\,p_j\right\} + s\right]\Big/[1/\lambda + L]. \quad (10)$$

We can assume that $C(N) < k$, i. e., that the cost in the case that any ordering policy is applied is cheaper than the cost in the case that only the minimal repair is made.

Define the following function from the numerator of the difference of $C(N)$:

$$q(N) = \left[(c_1 - c_2)\, r(N+1) + c_3\, r(N+L+1)\, R(N+1) - k\{1 - R(N+1)\}\right] B(N) - R(N+1)\, A(N). \quad (11)$$

Here, we have the following theorem for the optimum ordering time point N^* minimizing the expected cost rate in the steady-state $C(N)$. The proof is given in Appendix.

THEOREM 1: (1) *Suppose that the failure rate is strictly increasing:*

(i) *if* $q(0) < 0$ *and* $q(\infty) > 0$, *then there exists a finite and unique optimum ordering time point* N^* $(0 < N^* < \infty)$ *satisfying*

$$q(N-1) < 0 \quad \text{and} \quad q(N) \geqq 0; \quad (12)$$

(ii) *if* $q(0) \geqq 0$, *then the optimum ordering time point is* $N^* = 0$, *i. e., the order for a spare is made at when the unit is put in service*;

(iii) *il* $q(\infty) \leqq 0$, *then the optimum ordering time point is* $N^* \to \infty$, *i. e., the order for a spare is made at the same time point as the first failure of the original unit.*

(2) *Suppose that the failure rate is decreasing:*

(i) *If*

$$\left[c_2 + c_3\left\{1 + \sum_{j=1}^{L} \sum_{i=j+1}^{L} r(i)\, p_j\right\} + k \sum_{i=L+1}^{\infty} \sum_{j=i}^{\infty} p_j + s\right][1/\lambda + L] < \left[c_1 + c_3\left\{1 + \sum_{j=1}^{\infty} \sum_{i=j+1}^{j+L} r(i)\, p_j\right\} + s\right] \times \left[1/\lambda + \sum_{i=1}^{L} \sum_{j=1}^{i-1} p_j\right], \quad (13)$$

then $N^* = 0$;

(ii) *If*

$$\left[c_2 + c_3\left\{1 + \sum_{j=1}^{L} \sum_{i=j+1}^{L} r(i)\, p_j\right\} + k \sum_{i=L+1}^{\infty} \sum_{j=i}^{\infty} p_j + s\right][1/\lambda + L] \geqq \left[c_1 + c_3\left\{1 + \sum_{j=1}^{\infty} \sum_{i=j+1}^{j+L} r(i)\, p_j\right\} + s\right] \times \left[1/\lambda + \sum_{i=1}^{L} \sum_{j=1}^{i-1} p_j\right], \quad (14)$$

then $N^* \to \infty$.

4. REMARKS

We have treated the optimum ordering policies with minimal repair which are operated at the discrete times. We have obtained a theorem on the optimum policy minimizing the expected cost rate in the steady-state, and shown that there exists a finite and unique optimum policy under certain conditions.

In particular, we obtain the following results without assuming the monotone properties of the failure rate. That is, if $q(0) < 0$, then there exists at least one optimum ordering time point N^* $(0 < N^* \leqq \infty)$, and if $q(\infty) > 0$, then there exists at least one optimum ordering time point N^* $(0 \leqq N^* < \infty)$. These facts can be verified from the asymptotic behavior of the difference of $\log C(N)$ as N tends to 0 or ∞.

APPENDIX

THE PROOF OF THEOREM 1

From the equation (11):

$$\begin{aligned} q(N+1)-q(N) = {} & [(c_1-c_2)\{r(N+2)-r(N+1)\} \\ & +c_3 R(N+2)\{r(N+L+2)-r(N+L+1)\}]\, B(N+1) \\ & +[R(N+2)-R(N+1)]\,[\{c_3\, r(N+L+1) \\ & +k\}\, B(N+1)-D(N+1)]. \quad (\mathrm{A.1}) \end{aligned}$$

First, we assume the case that tha failure rate is strictly increasing. Thus, we have $q(N+1)-q(N) > 0$, i. e., $q(N)$ is strictly increasing.

If $q(0) < 0$ and $q(\infty) > 0$, then there exists a finite and unique N^* $(0 < N^* < \infty)$ which minimizes the expected cost rate $C(N)$ satisfying the equation (12) since $q(N)$ is strictly increasing.

If $q(0) \geqq 0$, then $C(N+1)-C(N) \geqq 0$ for any non-negative N. Thus, $N^* = 0$.

If $q(\infty) \leqq 0$, then $C(N+1)-C(N) \leqq 0$ for any non-negative N. Thus, $N^* \to \infty$.

Secondly, we assume the case that the failure rate is decreasing. Thus, we have $q(N+1)-q(N) \leqq 0$, i. e., $q(N)$ is decreasing. If $q(0) > 0$, then $q(\infty) \geqq 0$ or $q(\infty) < 0$, if $q(0) = 0$, then $q(\infty) \leqq 0$, and if $q(0) < 0$, then $q(\infty) < 0$. Thus, $C(0)$ or $C(\infty)$ is not greater than $C(N)$ for any N.

Thus, if $C(0) < C(\infty)$, i. e., the equation (13) holds, then $N^* = 0$, and if $C(0) \geqq C(\infty)$, i. e., the equation (14) holds, then $N^* \to \infty$.

Q. E. D.

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