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RAIRO. Recherche opérationnelle, tome 13, n° 1 (1979), p. 3-12

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ADJACENT VERTICES OF THE ALL 0-1 PROGRAMMING POLYTOPE (*) (1)

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Abstract. — *We give a constructive characterization of adjacency relations between vertices of the convex hull of feasible 0-1 points of an all 0-1 program. This characterization can be used, for instance, to generate all vertices of the convex hull, adjacent to a given vertex. As a by-product, we establish a strong bound on the diameter of the convex hull of feasible 0-1 points.*

Any linear 0-1 programming problem can be brought (by using binary expansion on the slack variables, when necessary, or other devices) to the form of an equality-constrained all 0-1 program:

$$(P) \quad \min \{ cx \mid x \in X, x \text{ integer} \}$$

where

$$X = \{ x \in \mathbb{R}^n \mid Ax = b, x \geq 0 \}$$

and where A is $m \times n$, and $Ax = b$ implies $x_j \leq 1, \forall j \in N = \{ 1, \dots, n \}$.

We will assume, without loss of generality, that A has no identical columns or zero columns, and is of full row rank. The j -th column of A will be denoted a_j .

Let (P') denote the linear program associated with (P) , i. e.,

$$(P') \quad \max \{ cx \mid x \in X \}.$$

Further, let X_I be the convex hull of the feasible 0-1 points, i. e.,

$$X_I = \text{conv} \{ x \in X \mid x \text{ integer} \}$$

and let $\text{vert } X$ ($\text{vert } X_I$) denote the set of vertices of X (of X_I). X_I is the all 0-1 programming polytope referred to in the title.

(*) Manuscrit reçu Octobre 1977, révisé Janvier 1978.

(1) The research underlying this paper was supported by the National Science Foundation by grant MPS73-08534 A02.

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It is a well-known property of 0-1 programs that every feasible integer solution is basic. Hence, $\text{vert } X_I \subseteq \text{vert } X$. Further, a solution associated with a feasible basis B , whose columns are indexed by I , is integer if and only if $\sum_{i \in Q} a_i = b$ for some $Q \subseteq I$, and Q is unique whenever it exists.

Finally, if B is a feasible basis, I and J are the associated basic and nonbasic index sets, and $\bar{a}_j = B^{-1}a_j$. To simplify notation, we assume the components of x to have been ordered so that $I = \{1, \dots, m\}$; thus the components of \bar{a}_j are \bar{a}_{ij} , $i = 1, \dots, m$. Observe that $\bar{a}_{ij} > 0$ for at least one $i \in I$ and every $j \in J$, since X is bounded. Further, we denote

$$\bar{a}^j = \begin{pmatrix} \bar{a}_j \\ -e_j \end{pmatrix} \quad (1)$$

where e_j is the $(n - m)$ -dimensional unit vector whose j -th component is 1; i. e., the n -vector \bar{a}^j is the j -th column of the Tucker-tableau. The k -th component of \bar{a}^j is denoted by \bar{a}_k^j .

Given a linear program, two bases are called *adjacent* if they differ in exactly one column. Two *basic feasible solutions* are called *adjacent* if they are adjacent vertices of the feasible set (i. e., distinct vertices contained in an edge, or 1-dimensional face). Two adjacent bases may be associated with the same solution; while two adjacent basic feasible solutions may be associated with two bases that are not adjacent to each other.

The results of this paper were first shown in [1], [2] to hold for the set partitioning problem, a special case of the problem considered here. Most of the proofs given in [2] carry over to the general case with only minor changes, but the main result (the sufficiency part of Theorem 3) requires a different approach. For the sake of completeness, we give all the proofs.

THEOREM 1 : Let x^1 and x^2 be two feasible integer solutions to (P') . Let B be a basis matrix associated with x^1 , let I and J be the index sets for the basic and nonbasic variables respectively, and for $i = 1, 2$, let

$$Q_i = \{j \in N \mid x_j^i = 1\}, \quad \bar{Q}_i = N - Q_i.$$

Then, denoting $\bar{a}_j = B^{-1}a_j$, $j \in J$,

$$\sum_{j \in J \cap Q_2} \bar{a}_{kj} = \begin{cases} 1 & k \in Q_1 \cap \bar{Q}_2 \\ -1 & k \in Q_2 \cap \bar{Q}_1 \cap I \\ 0 & k \in (Q_1 \cap Q_2) \cup (\bar{Q}_1 \cap \bar{Q}_2 \cap I). \end{cases} \quad (2)$$

Proof : From the definition of Q_i , $i = 1, 2$, we have

$$\sum_{k \in Q_1} a_k = \sum_{k \in Q_2} a_k,$$

which implies

$$\begin{aligned} \sum_{j \in J_1 \cap Q_2} a_j &= \sum_{k \in Q_2} a_k - \sum_{k \in Q_2 \cap I} a_k \\ &= \sum_{k \in Q_1} a_k - \sum_{k \in Q_2 \cap I} a_k \\ &= \sum_{k \in Q_1 \cap \bar{Q}_2} a_k - \sum_{k \in Q_2 \cap \bar{Q}_1 \cap I} a_k \end{aligned}$$

(by subtracting and adding $\sum_{k \in Q_1 \cap Q_2} a_k$).

Premultiplying the last equation by B^{-1} then produces (2), since the vectors a_k , $k \in Q_1$ and $k \in I$, are columns of B . Q. E. D.

Next we state the converse of Theorem 1.

THEOREM 2: Let x^1 be a feasible integer solution to (P') , let B, I, J, Q_1 and \bar{a}_j , $j \in J$, be defined as in Theorem 1. Further, let the index set $Q \subseteq J$ satisfy

$$\sum_{j \in Q} \bar{a}_{kj} = \begin{cases} 0 & \text{or} & 1 & k \in Q_1 \\ 0 & \text{or} & -1 & k \in I_1 \cap \bar{Q}_1. \end{cases} \quad (3)$$

Then x^2 defined by

$$x^2 = x^1 - \sum_{j \in Q} \bar{a}^j \quad (4)$$

is a basic feasible solution to (P') , and

$$x_j^2 = \begin{cases} 1 & j \in Q_2 = Q \cup S \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

where

$$S = \{ k \in Q_1 \mid \sum_{j \in Q} \bar{a}_{kj} = 0 \} \cup \{ k \in \bar{Q}_1 \cap I \mid \sum_{j \in Q} \bar{a}_{kj} = -1 \}.$$

Proof: Consider the problem (\bar{P}') in $(n + 1)$ -space, obtained from (P') by augmenting A with the composite column $a_{j_*} = \sum_{j \in Q} a_j$. The transformed column $\bar{a}_{j_*} = B^{-1}a_{j_*}$ has an entry $\bar{a}_{kj_*} = 1$ for some $k \in Q_1$, for otherwise (3) implies $\bar{a}_{kj_*} \leq 0, \forall k \in I$, which is impossible in view of the boundedness of the solution set. Pivoting on $\bar{a}_{kj_*} = 1$ yields a feasible solution \tilde{x}^2 to (\bar{P}') , defined by

$$\tilde{x}_j^2 = \begin{cases} 1 & j \in \{j_*\} \cup S \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\sum_{j \in S} a_j + a_{j_*} = \sum_{j \in S \cup Q} a_j = e$$

it follows that x^2 as defined by (5) is feasible for (P') . Since x^2 is integer, it is also basic. From Theorem 1, relation (4) follows with $Q = J_1 \cap Q_2$. Q. E. D.

A set $Q \subset J$ for which (3) holds will be called *decomposable* if it can be

partitioned into two subsets, Q^* and Q^{**} , such that (3) remains true when Q is replaced by Q^* and Q^{**} respectively.

We now give a necessary and sufficient condition for two integer vertices of X to be adjacent on X_I .

THEOREM 3: Let x^1 and x^2 be two feasible integer solutions to (P') , with B , I , J , Q_1 and \bar{a}_j , $j \in J$, defined as in Theorem 1, and $Q_2 = \{j \in N \mid x_j^2 = 1\}$. Then x^2 is adjacent to x^1 on X_I (i. e., x^1 and x^2 lie on an edge of X_I) if and only if $Q = J \cap Q_2$ is not decomposable.

Proof: (i) Necessity. Suppose Q is decomposable into Q^* and Q^{**} . Then the vectors

$$x^i = x^1 - \sum_{j \in S_i} \bar{a}^j, \quad i = 2, 3, 4 \quad (6)$$

where $S_2 = Q$, $S_3 = Q^*$, $S_4 = Q^{**}$, and \bar{a}^j is defined by (1), are all feasible integer solutions to (P') , hence vertices of X_I . Let $\pi x = \pi_0$ be a supporting hyperplane for X_I , such that $\pi x^i = \pi_0$ for $i = 1, 2$ and $\pi x \leq \pi_0$, $\forall x \in X_I$. (If no such hyperplane exists, then x^1 and x^2 are not adjacent on X_I , and the statement is proved.) Then from (6)

$$\begin{aligned} \pi x^1 &= \pi x^2 \\ &= \pi x^1 - \pi \left(\sum_{j \in Q} \bar{a}^j \right) = \pi_0 \end{aligned}$$

or

$$\pi \left(\sum_{j \in Q} \bar{a}^j \right) = 0, \quad (7)$$

whereas

$$\pi x^3 = \pi x^1 - \pi \left(\sum_{j \in Q^*} \bar{a}^j \right) \leq \pi_0 = \pi x^1$$

$$\pi x^4 = \pi x^1 - \pi \left(\sum_{j \in Q^{**}} \bar{a}^j \right) \leq \pi_0 = \pi x^1$$

or

$$\pi \left(\sum_{j \in Q^*} \bar{a}^j \right) \geq 0, \quad \pi \left(\sum_{j \in Q^{**}} \bar{a}^j \right) \geq 0. \quad (8)$$

Then from (7) and (8) we have

$$\pi \left(\sum_{j \in Q^*} \bar{a}^j \right) = 0, \quad \pi \left(\sum_{j \in Q^{**}} \bar{a}^j \right) = 0$$

or $\pi x^3 = \pi x^4 = \pi_0$. Hence any supporting hyperplane for X_I that contains x^1 and x^2 , also contains x^3 and x^4 ; i. e., x^1 and x^2 cannot lie on an edge of, or be adjacent on, X_I .

(ii) Sufficiency. Suppose x^1 and x^2 are not adjacent on X_I . Let F be the face of minimal dimension of X_I , which contains both x^1 and x^2 (F is clearly

unique), and let x^{11}, \dots, x^{1p} be the vertices of F adjacent to x^1 on F . The (translated) convex polyhedral cone

$$C(x^1) = \{ x \mid x = x^1 + (x^{1i} - x^1)\lambda_i, \lambda_i \geq 0, i = 1, \dots, p \}$$

is known (see for instance [3]) to be the intersection of those halfspaces H_i^+ , $i = 1, \dots, p$, such that $x^1 = \bigcap_{i=1}^p H_i$, where $H_i = bdH_i^+$. Since $\{ H_i^+ \}_{i=1}^p$ is a subset of the set of halfspaces whose intersection is X_I , clearly $X_I \subseteq C(x^1)$, and therefore every vertex x of F can be expressed as

$$x = x^1 + \sum_{i=1}^p (x^{1i} - x^1)\lambda_i, \quad \lambda_i \geq 0, \quad i = 1, \dots, p. \tag{9}$$

Since x^2 is not adjacent to x^1 , $p \geq 2$. From Theorems 1 and 2,

$$x^{1i} = x^1 - \sum_{j \in Q_{1i}} \bar{a}^j, \quad i = 1, \dots, p \tag{10}$$

and

$$x^2 = x^1 - \sum_{j \in Q} \bar{a}^j \tag{11}$$

where $Q_{1i} \subseteq J$, $i = 1, \dots, p$, and $Q \subseteq J$.

Since F is the lowest-dimensional face of X_I containing both x^1 and x^2 , there exist $\lambda_i > 0$ for $i = 1, \dots, p$, such that (9) holds with $x = x^2$. For if not, then x^2 is contained in a face F' of $C(x^1)$ such that

$$\dim F' < \dim C(x^1) = \dim F.$$

But $F'' = \text{aff } F' \cap X_I$ is a face of X_I that contains both x^1 and x^2 and

$$\dim F'' = \dim F' < \dim F,$$

which contradicts the assumption that F is the lowest-dimensional face of X_I containing both x^1 and x^2 . Using (10) and the fact that (9) holds with $x = x^2$ for some $\lambda_i > 0$, $i = 1, \dots, p$, we have

$$\begin{aligned} x^2 &= x^1 + \sum_{i=1}^p (x^{1i} - x^1)\lambda_i \\ &= x^1 - \sum_{i=1}^p \left(\sum_{j \in Q_{1i}} \bar{a}^j \right) \lambda_i \end{aligned}$$

and from (11)

$$\sum_{j \in Q} \bar{a}^j = \sum_{i=1}^p \left(\sum_{j \in Q_{1i}} \bar{a}^j \right) \lambda_i, \quad \text{with } \lambda_i > 0, \quad i = 1, \dots, p,$$

which implies $Q = \bigcup_{i=1}^p Q_{1i}$.

We now partition Q into two subsets $Q^* = Q_{11}$ and $Q^{**} = Q - Q^*$. To complete the proof, we will show that (3) holds when Q is replaced by Q^{**} (for Q^* this follows from Theorem 1). This will be done by showing that x^{**} is a feasible integer solution to (P') , where

$$\begin{aligned} x^{**} &= x^1 - \sum_{j \in Q^{**}} \bar{a}^j \\ &= x^2 + \sum_{j \in Q^*} \bar{a}^j \end{aligned} \quad (12)$$

Theorem 1 then implies that (3) holds with Q replaced by Q^{**} .

First, from Theorem 1 and the definition of Q^* , x^{**} is integer. Next we show by contradiction that $x^{**} \geq 0$. Suppose $x_k^{**} < 0$. Then from (12), $x_k^2 = 0$ and $\sum_{j \in Q_{1i}} \bar{a}_k^j = -1$ (since $Q^* = Q_{11}$). But from (10), this implies (for $i = 1$) $x_k^1 = 0$, and hence

$$\sum_{j \in Q_{1i}} \bar{a}_k^j \leq 0, \quad \forall i \in \{1, \dots, p\} \quad (13)$$

But

$$x_k^2 = x_k^1 - \sum_{i=1}^p \left(\sum_{j \in Q_{1i}} \bar{a}_k^j \right) \lambda_i, \quad \lambda_i > 0, \quad i = 1, \dots, p; \quad (14)$$

hence $x_k^2 > 0$, contradicting our earlier finding that $x_k^2 = 0$. Hence, $x^{**} \geq 0$.

Suppose on the other hand that $x_k^{**} > 1$. By (12), $x_k^2 = 1$ and $\sum_{j \in Q_{11}} \bar{a}_k^j = 1$ (since $Q_{11} = Q^*$). But from (10), this implies (for $i = 1$) that $x_k^1 = 1$, and hence that (13) holds with reversed inequality. Again from (14) we conclude that $x_k^2 < 1$, contradicting our earlier finding that $x_k^2 = 1$. Consequently, $0 \leq x_k^{**} \leq 1$ for all $k \in N$. Finally, $Ax^{**} = b$, since

$$\begin{aligned} A\bar{a}^j &= (B, R) \begin{pmatrix} B^{-1}a_j \\ -e_j \end{pmatrix} \\ &= a_j - a_j = 0, \quad \forall j \in J \end{aligned}$$

where R is the submatrix of A consisting of the columns $a_j, j \in J$. Hence x^{**} is a feasible 0-1 point. $Q. E. D.$

COROLLARY 3.1 : Let x^1 and x^2 be two vertices of X_I , and let B, I, J and $\bar{a}^j, j \in J$, be defined as above. Then x^2 is not adjacent to x^1 on X_I , if and only if there exists a family of p sets $Q_{1i} \subseteq J, i = 1, \dots, p$, such that

- (i) $p \geq 2$;
- (ii) $Q_{1i} \cap Q_{1k} = \emptyset, \quad \forall i \neq k$;
- (iii) the points

$$x^{1i} = x^1 - \sum_{j \in Q_{1i}} \bar{a}^j, \quad i = 1, \dots, p$$

are vertices of X_I , adjacent to x^1 ; and

$$(iv) \quad \begin{aligned} x^2 &= x^1 - \sum_{i=1}^p \sum_{j \in Q_{1i}} \bar{a}^j \\ &= x^1 + \sum_{i=1}^p (x^{1i} - x^1). \end{aligned}$$

Proof: (α) Necessity. If x^1 and x^2 are not adjacent on X_I , then by Theorem 3 $Q = J \cap Q_2$ can be partitioned into two subsets Q^* and Q^{**} such that (3) holds with Q replaced by Q^* and Q^{**} . If

$$x^* = x^1 - \sum_{j \in Q^*} \bar{a}^j$$

and

$$x^{**} = x^1 - \sum_{j \in Q^{**}} \bar{a}^j$$

are both adjacent to x^1 , the statement is proved; otherwise the reasoning can be applied to Q^* and/or Q^{**} , and can be repeated as many times as needed to obtain pairwise disjoint sets Q_{1i} , $i = 1, \dots, p$, with $p \geq 2$, which are not decomposable.

(β) Sufficiency. If the condition holds, then $Q = \bigcup_{i=1}^p Q_{1i} = J \cap Q_2$. Furthermore, (7) is satisfied when Q is replaced by Q_{1i} for $i = 1, \dots, p$.

From (iii) it follows that the vectors $\sum_{j \in Q_{1i}} \bar{a}^j$ and $\sum_{j \in Q_{1h}} \bar{a}^j$ are mutually orthogonal for all $i \neq h$, $i, h \in \{1, \dots, p\}$. Consequently, (3) also holds when Q is replaced by $\bigcup_{i=2}^p Q_{1i}$. Thus Q is decomposable into Q_{11} and $\bigcup_{i=2}^p Q_{1i}$, hence x^1 and x^2 are not adjacent. *Q. E. D.*

COROLLARY 3.2: If x^1 and x^2 are two non-adjacent vertices of X_I related to each other by (iv), then for any subset H of $\{1, \dots, p\}$,

$$\begin{aligned} x^* &= x^1 - \sum_{i \in H} \sum_{j \in Q_{1i}} \bar{a}^j \\ &= x^1 + \sum_{i \in H} (x^{1i} - x^1) \end{aligned}$$

is a vertex of X_I .

Proof: From (iii), the vectors $\sum_{j \in Q_{1i}} \bar{a}^j$ and $\sum_{j \in Q_{1h}} \bar{a}^j$ are pairwise orthogonal for all $i, h \in \{1, \dots, p\}$, $i \neq h$; hence if (3) holds for $Q = \bigcup_{i=1}^p Q_{1i}$, then it also holds when Q is replaced by $\bigcup_{i \in H} Q_{1i}$. *Q. E. D.*

Corollary 3.2 can be given the following geometric interpretation. A *path* on X_I between two vertices x, y is a sequence of vertices (x^1, x^2, \dots, x^k) , with $x^1 = x, x^k = y$, such that every pair of vertices $x^i, x^{i+1}, i = 1, \dots, k - 1$, is connected by an edge of X_I ; the length of the path being $k - 1$. The *edge-distance* $d(x, y)$ between x and y is the length of a shortest path between x and y . The *diameter* $\delta(X_I)$ of X_I is the longest edge-distance between any two vertices of X_I .

Let $[a]$ denote the largest integer less than or equal to the real number a . For the next corollary, we assume that the matrix A defining X_I has no identical columns.

COROLLARY 3.3 : $\delta(X_I) \leq \left\lceil \frac{z^*}{2} \right\rceil$ where

$$z^* = \max \left\{ \sum_{j=1}^n x_j \mid x \in X_I \right\}.$$

Proof : Let x^1, x^2 be a pair of vertices of X_I which are at maximal edge-distance from each other, i. e., for which

$$d(x^1, x^2) = \delta(X_I).$$

Further, let B be a basis associated with x^1 ; let I, J, Q_1 and $\bar{a}^j, j \in J$, be defined as above.

From Corollary 3.1,

$$x^2 = x^1 - \sum_{i=1}^p \sum_{j \in Q_{1i}} \bar{a}^j \tag{15}$$

and from Corollary 3.2, (15) holds with $p \geq \delta(X_I)$, since the sequence of vertices $\{x^{10}, x^{11}, \dots, x^{1p}\}$, of X_I , where $x^{10} = x^1$ and $x^{1p} = x^2$, with

$$x^{1k} = x^1 - \sum_{i=1}^k \sum_{j \in Q_{1i}} \bar{a}^j \quad k = 1, \dots, p,$$

defines a path of length p between x^1 and x^2 .

Now let $P = \{1, \dots, p\}$, and let

$$P_1 = \left\{ i \in P \mid \sum_{j \in Q_{1i}} \bar{a}^j = 1 \text{ for exactly one } k \in N \right\}.$$

If $P_1 = \emptyset$, then from (15) and the definition of z^* ,

$$p \leq \left\lceil \frac{|Q_1|}{2} \right\rceil \leq \left\lceil \frac{z^*}{2} \right\rceil$$

which, together with $\delta(X_I) \leq p$, proves the corollary. Suppose now that $P_1 \neq \emptyset$. Then for each $i \in P_1$, the vector $\sum_{j \in Q_{1i}} \bar{a}^j$ has at least two negative components.

For otherwise Q_{1i} is a singleton, say $Q_{1i} = \{ h \}$, and \bar{a}^h is of the form

$$\bar{a}^h = \begin{pmatrix} e_i^m \\ -e_h^{n-m} \end{pmatrix}$$

(where e_j^k is the k -dimensional unit vector whose j -th entry is 1); which implies that the nonbasic column a_h of A is identical to a basic column, contrary to our assumption. Now let

$$\begin{aligned} x^3 &= x^1 - \sum_{i \in P_1} \sum_{j \in Q_{1i}} \bar{a}^j, \\ x^4 &= x^1 - \sum_{i \in P-P_1} \sum_{j \in Q_{1i}} \bar{a}^j, \end{aligned}$$

where both x^3 and x^4 are vertices of X_I (Corollary 3.2). Then

$$x^4 = x^3 - \sum_{i \in P_1} \left(- \sum_{i \in Q_{1i}} \bar{a}^j \right) - \sum_{i \in P-P_1} \sum_{j \in Q_{1i}} \bar{a}^j; \tag{16}$$

but in view of

$$\sum_{j \in Q_{1i}} \bar{a}_k^j \neq 0 \Rightarrow \sum_{j \in Q_{1h}} \bar{a}_k^j = 0, \quad \forall k \in N, \quad \forall i, h \in P, \quad i \neq h,$$

(16) implies that $p \leq \left\lceil \frac{|Q_3|}{2} \right\rceil$, where $Q_3 = \{ j \in N \mid x_j^3 = 1 \}$. Hence, in view of $\delta(X_I) \leq p$ and $|Q_3| \leq z^*$, the corollary follows. Q. E. D.

REMARK: If in the definition of X_I , $A = (A_G, I_m)$ and $b = (e^m)$, where I_m is the identity matrix of order m , $e^m = (1, \dots, 1) \in R^m$, and A_G is the $m \times \left(\frac{m}{2}\right)$ incidence matrix of the complete undirected graph with m vertices, then $\delta(X_I) = \left\lceil \frac{z^*}{1} \right\rceil$, since $\delta(X_I)$ is achieved by the minimum distance between the empty matching and any maximum matching on the matching polytope. In this sense the upper bound on $\delta(X_I)$ given in the above Corollary is a strongest possible one.

The property stated in the next Theorem, which does not hold for arbitrary integer programs, has some interesting algorithmic implications.

THEOREM 4: Let x^1 be a non-optimal vertex of X_I , let x^{1i} , $i = 1, \dots, k$, be those vertices of X_I adjacent to x^1 , and such that $cx^{1i} < cx^1$, $i = 1, \dots, k$. Then the convex polyhedral cone

$$C = \left\{ x \mid x = x^1 + \sum_{i=1}^k (x^{1i} - x^1)\lambda_i, \lambda_i \geq 0, i = 1, \dots, k \right\}$$

contains an optimal vertex of X_I .

Proof: Let \bar{x} be an optimal vertex of X_I . If \bar{x} is adjacent to x^1 , then $\bar{x} \in C$. Otherwise, \bar{x} can be expressed (Corollary 3.1) as

$$\begin{aligned}\bar{x} &= x^1 - \sum_{i=1}^p \sum_{j \in Q_{1i}} \bar{a}^j \\ &= x^1 + \sum_{i=1}^p (x^{1i} - x^1)\end{aligned}$$

where x^{1i} , $i = 1, \dots, p$, are vertices of X_I adjacent to x^1 . Then

$$0 < cx^1 - c\bar{x} = \sum_{i=1}^p \sum_{j \in Q_{1i}} c\bar{a}^j$$

Let $\{1, \dots, p\} = P$, and let $P^+ = \{1, \dots, k\}$. Since $c\bar{x} < cx^1$, $P^+ \neq \emptyset$. From Corollary 3.2, the point

$$\begin{aligned}x^* &= x^1 - \sum_{i \in P^+} \sum_{j \in Q_{1i}} \bar{a}^j \\ &= x^1 + \sum_{i \in P^+} (x^{1i} - x^1)\end{aligned}$$

is a vertex of X_I , and from the definition of P^+ ,

$$\begin{aligned}cx^* &= cx^1 + \sum_{i \in P^+} c(x^{1i} - x^1) \\ &\leq cx^1 + \sum_{i=1}^p c(x^{1i} - x^1) = c\bar{x}.\end{aligned}$$

Thus, since \bar{x} is optimal, so is x^* ; and since the vertices x^{1i} , $i \in P^+$ are among those that generate C , clearly $x^* \in C$. *Q. E. D.*

The above results can be used to generate integer vertices of the feasible set X , adjacent to a given integer vertex x^1 . Namely, by systematically generating composite columns of the form $\bar{a}^{j*} = \sum_{j \in Q} \bar{a}^j$, where Q satisfies the requirements for $x^1 - \bar{a}^{j*}$ to be a vertex of X_I adjacent to x^1 , one can obtain all such vertices. The efficiency of a procedure based on these results will of course be highly dependent on the specific way in which they are used; and in view of the many options that are available, this topic requires further investigation.

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