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SOME APPROXIMATE EQUILIBRIUM RESULTS FOR THE $E_k/M/r$ NO-QUEUE SYSTEM (*)

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Abstract. — *Approximate formulae for the utilization of a group of servers in the $E_k/M/r$ no-queue system are derived. The degree of approximation achieved depends on the particular formula used and on the values of the various parameters involved; the relative percentage errors incurred seem, however, to be well below 1 per cent in absolute value. The derivation of the formulae is based on the underlying principle that two systems of the same general type $E_k/M/r$ with equal parameter k and equal traffic intensities but with unequal number of servers have a "similar" behaviour.*

Service systems where no queue is formed are quite common in practice, the original physical situation motivating their study, just after the turn of this century, being the simple telephone network [2]. Over the years many models of such systems have been devised, solved and applied in the design of telephone exchanges [3, 5]; and fairly recently these models have been used in the design of file organization systems and in the study of situations where either a queue is not allowed to form because of space limitations (limiting case of balking) or customers are unwilling to join a queue because of long expected queueing time (limiting case of impatience).

The system considered in this paper can be described as follows : there are r servers in parallel each having an exponential service time distribution with mean μ^{-1}. Intervals between successive arrivals have an Erlang $- k$ distribution; the average rate of arrival of customers is λ, so that the traffic intensity $\rho = \lambda/r\mu$. No queue is formed : all customers that arrive and find every server busy are turned away.

When dealing with such a system one is usually faced with the problem of balancing the efficient use of the group of servers against the provision of acceptable service for the customers. One is therefore primarily interested in calculating the utilization factor of the servers (also known as server occupancy) which will be denoted $U(r, k)$ and the probability of a customer being turned away (also known as probability of loss or blocking probability) which will be denoted $B(r, k)$.

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According to Takács [6], who discusses the $GI/M/r$ no-queue system in detail,

$$U(r, k) = \frac{\rho \sum_{j=1}^{r} \binom{r}{j} D_j}{1 + \sum_{j=1}^{r} \binom{r}{j} D_j} \tag{1}$$

with

$$D_j = \sum_{i=1}^{j} \left[\frac{1 - \varphi(i\mu)}{\varphi(i\mu)}\right],$$

where $\varphi(s)$ is the Laplace-Stieltjes transform of the inter-arrival times distribution given, in the case of Erlang inter-arrival times, by

$$\varphi(s) = \left[\frac{1}{1 + \dfrac{s}{k\rho r\mu}}\right]^k$$

The blocking probability can then be calculated as

$$B(r, k) = [\rho - U(r, k)]/\rho \tag{2}$$

It is interesting to note that in a single-server no-queue system the complicated-looking formula (1) is greatly simplified :

$$U(1, k) = \rho\left[1 - \frac{(k\rho)^k}{(1 + k\rho)^k}\right] \tag{3}$$

DERIVATION OF APPROXIMATE FORMULAE FOR $U(r, k)$

With the assumption that inter-arrival times have an Erlang distribution, $U(r, k)$ depends on ρ, r and k only; and, according to Benès [1], for ρ and r fixed, $U(r, k)$ is an increasing function of k with limiting values $U(r, 1)$, corresponding to the system $M/M/r$, and $U(r, \infty)$ if $k \to \infty$.

We now consider two such systems, $E_k/M/r_j$ and $E_k/M/r_i$, both with equal traffic intensities but with $r_j > r_i$ and proceed in writing a relationship of the general form :

$$\frac{U(r_j, k)}{U(r_i, k)} = S\,\frac{U(r_j, 1)}{U(r_i, 1)} + (1 - S)\,\frac{U(r_j, \infty)}{U(r_i, \infty)} \tag{4}$$

where S is a factor, yet to be specified, with limiting values 1 and 0 for $k = 1$ and ∞ respectively, regardless of r_j, r_i or ρ.

(4) is an expression giving $U(r_j, k)$ in terms of $U(r_i, k)$; consequently, if the relative percentage errors incurred are small,

a) the hypothesis that the $E_k/M/r_j$ and $E_k/M/r_i$ no-queue systems with equal traffic intensities have a similar behaviour would be established, the quantity on the right hand side being regarded as the similarity ratio; and

b) (4) could be used as an approximate formula for the evaluation of $U(r_j, k)$ provided that $U(r_i, k)$ with $r_i = r_j - 1, r_j - 2, \ldots$ or even 1 and the similarity ratio were both known.

A method for the determination of S has now to be devised. It was felt initially that solving (4) for S could provide some indication on how sensitive S is to the values of r_i, r_j, k and ρ. A computer programme developed by Sellmeyer [4] was used for the calculation of the six utilization factors in (4) and the computed values of S were then tabulated for selected values of the various parameters involved. Quite unexpectedly, a promising result emerged, namely that S seemed to be rather insensitive to the value of ρ. On the basis of this information, the determination of S can be achieved by expressing (4) in the limiting case of heavy traffic conditions ($\rho \to \infty$). By expanding $[1 - \varphi(i\mu)]/\varphi(i\mu)$ into a power series of ρ, it can be easily verified that :

$$\lim_{\rho \to \infty} U(r, k) = \frac{2\rho r + 2(r - 1) - (1 + k)/k}{2\rho r + 2r}$$

in which case simple algebra yields $S = 1/k$ so that :

$$U(r_j, k) \simeq \left[\frac{1}{k} \cdot \frac{U(r_j, 1)}{U(r_i, 1)} + \frac{k - 1}{k} \cdot \frac{U(r_j, \infty)}{U(r_i, \infty)}\right] U(r_i, k) \tag{5}$$

We evaluated formula (5) for $2 \leq r_j \leq 100$, $1 \leq r_i \leq r_{j-1}$, $k > 1$ and $0.4 \leq \rho \leq 3.0$ and calculated the corresponding theoretical values for $U(r_j, k)$ using the programme in [4]. The results obtained are briefly outlined in the next section.

EVALUATION OF THE APPROXIMATE FORMULAE DERIVED

Evaluation of formula (5) with $r_i = 1$

If we set $r_i = 1$, formula (5) is greatly simplified because the expression in (3) for $U(1, k)$ *can be worked out manually*; some results for $U(r_j, 1)$ and $U(r_j, \infty)$ are given in Appendices 1 and 2 for easy reference.

a) For a given value of k, any value of $U(r_j, k)$ can be approximated with a relative percentage error :

$$e = 100 \text{ (approximate value } - \text{ theoretical value)/(theoretical value)}$$

which (i) is negative and tends to 0 as $\rho \to \infty$; (ii) is not sensitive to the value of r_j; and (iii) is very small in absolute value, not exceeding 1 per cent over the whole range of values for ρ, k and r_j considered.

b) For given values of r_j and ρ, $U(r_j, k)$ can be approximated with a relative percentage error which varies with k as originally expected : it tends to 0 as k approaches 0 or 1 and attains a maximum absolute value somewhere in-between, namely for $k \simeq 2$. Table 1, for example, gives the relative percentage errors incurred if formula (5) with $r_i = 1$ is used for the evaluation of $U(4, k)$ for $\rho = 0.5, 1.0, 1.5, 2.0, 3.0$ and $k = 1, 2, 3, 4, 9, 25, 100$ and ∞.

TABLE 1

Relative percentage errors incurred if formula (5) with $r_i = 1$ is used for the evaluation of $U(4, k)$

ρ \ k	1	2	3	4	9	25	100	∞
0.5	0	− 0.97	− 0.93	− 0.82	− 0.46	− 0.18	− 0.05	0
1.0	0	− 0.50	− 0.48	− 0.41	− 0.23	− 0.09	− 0.02	0
1.5	0	− 0.39	− 0.36	− 0.32	− 0.17	− 0.07	− 0.02	0
2.0	0	− 0.31	− 0.29	− 0.25	− 0.14	− 0.05	− 0.01	0
3.0	0	− 0.19	− 0.18	− 0.15	− 0.08	− 0.03	− 0.01	0

Evaluation of formula (5) with $r_i > 1$

If a value of r_i exceeding unity is selected, formula (5) becomes difficult to apply ; however, the relative percentage errors incurred are, generally, smaller in absolute value. In particular it was found that for given values of ρ and k, the degree of similarity in the behaviour of the no-queue systems $E_k/M/r_j$ and $E_k/M/r_i$, as expressed in (5), tends to increase when $(r_j - r_i)/r_j$ decreases. Table 2 gives the relative percentage errors incurred in evaluating U (10, 2) for $\rho = 0.5, 1.0, 1.5, 2.0, 3.0$ and $r_i = 1, 2, 3, 4, 6$ and 8.

TABLE 2

Relative percentage errors incurred if formula (5) is used for the evaluation of U (10, 2)

ρ \ r_i	1	2	3	4	6	8
0.5	− 0.74	− 0.35	− 0.16	− 0.05	0.04	0.04
1.0	− 0.55	− 0.33	− 0.23	− 0.17	− 0.09	− 0.04
1.5	− 0.43	− 0.28	− 0.20	− 0.15	− 0.08	− 0.03
2.0	− 0.32	− 0.20	− 0.14	− 0.10	− 0.05	− 0.02
3.0	− 0.19	− 0.11	− 0.07	− 0.04	− 0.02	− 0.01

CONCLUSIONS

From a practical viewpoint formulae (5) with $r_i = 1$ and (2) appear to provide a reliable, simple and economical tool for the analysis of $E_k/M/r$ no-queue systems. If interested in redesigning an already existing system, where the arrival pattern may not lend itself to direct measurement, the values of ρ and k are unknown; provided that an Erlang distribution of inter-arrival times seems a reasonable distribution for one to adopt, it might be possible to estimate the values of $U(r, v)$ and $B(r, v)$ by other means and thus obtain a value for ρ by solving (2). Repetitive application of (5) could then provide an estimate for k.

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SERVER UTILIZATION U(r,1) IN M/M/r NO - QUEUE SYSTEMS

ρ \ r	1	2	3	4	6	8
0.4	0.285714	0.339623	0.364090	0.377413	0.390256	0.395528
0.6	0.375000	0.452055	0.491840	0.516776	0.546516	0.563450
0.8	0.444444	0.536082	0.585275	0.617484	0.658576	0.684485
1.0	0.500000	0.600000	0.653846	0.689320	0.735078	0.764430
1.2	0.545455	0.649682	0.705134	0.741353	0.787611	0.816937
1.4	0.583333	0.689119	0.744387	0.779984	0.824695	0.852477
1.6	0.615385	0.721030	0.775117	0.809418	0.851739	0.877493
1.8	0.642857	0.747292	0.799678	0.832398	0.872085	0.895779
2.0	0.666667	0.769231	0.819672	0.850730	0.887822	0.909598
2.2	0.687500	0.787798	0.836214	0.865635	0.900291	0.920342
2.4	0.705882	0.803695	0.850096	0.877956	0.910377	0.928900
2.6	0.722222	0.817444	0.861891	0.888291	0.918682	0.935858
2.8	0.736842	0.829443	0.872024	0.897069	0.925626	0.941614
3.0	0.750000	0.840000	0.880814	0.904608	0.931510	0.946448

ρ \ r	10	15	20	30	50	100
0.4	0.397877	0.399643	0.399936	0.399998	0.400000	0.400000
0.6	0.574115	0.588079	0.594123	0.598427	0.599867	0.600000
0.8	0.702671	0.731417	0.748471	0.767903	0.785047	0.796806
1.0	0.785418	0.819684	0.841108	0.867540	0.895213	0.924300
1.2	0.837690	0.871063	0.891501	0.916067	0.940658	0.964476
1.4	0.871801	0.902126	0.920113	0.940953	0.960676	0.978259
1.6	0.895102	0.922106	0.937676	0.955189	0.971105	0.984593
1.8	0.911735	0.935738	0.949269	0.964156	0.977319	0.988133
2.0	0.924074	0.945512	0.957386	0.970241	0.981391	0.990371
2.2	0.933528	0.952809	0.963343	0.974609	0.984248	0.991907
2.4	0.940971	0.958441	0.967881	0.977884	0.986356	0.993024
2.6	0.946965	0.962905	0.971442	0.980424	0.987973	0.993872
2.8	0.951886	0.966523	0.974307	0.982449	0.989250	0.994538
3.0	0.955992	0.969511	0.976658	0.984099	0.990285	0.995073

SERVER UTILIZATION $U(r,\infty)$ IN D/M/r NO – QUEUE SYSTEMS

ρ \ r	1	2	3	4	6	8
0.4	0.367166	0.388176	0.395246	0.397987	0.399610	0.399920
0.6	0.486675	0.534704	0.557958	0.571419	0.585605	0.592281
0.8	0.570796	0.636692	0.671852	0.694598	0.723069	0.740529
1.0	0.632121	0.706928	0.747212	0.773679	0.807689	0.829411
1.2	0.678482	0.756584	0.797811	0.824518	0.858284	0.879437
1.4	0.714642	0.792904	0.833053	0.858508	0.889898	0.908991
1.6	0.743582	0.820342	0.858572	0.882291	0.910841	0.927744
1.8	0.767244	0.841666	0.877711	0.899636	0.925480	0.940441
2.0	0.786939	0.858646	0.892501	0.912741	0.936181	0.949507
2.2	0.803580	0.872447	0.904226	0.922940	0.944297	0.956260
2.4	0.817822	0.883865	0.913722	0.931077	0.950638	0.961465
2.6	0.830148	0.893454	0.921554	0.937704	0.955717	0.965588
2.8	0.840917	0.901613	0.928115	0.943197	0.959869	0.968930
3.0	0.850406	0.908634	0.933685	0.947818	0.963323	0.971691

ρ \ r	10	15	20	30	50	100
0.4	0.399983	0.400000	0.400000	0.400000	0.400000	0.400000
0.6	0.595713	0.598931	0.599716	0.599978	0.600000	0.600000
0.8	0.752444	0.770395	0.780254	0.790259	0.797151	0.799815
1.0	0.844891	0.870063	0.885733	0.904987	0.925046	0.946017
1.2	0.894258	0.917770	0.931923	0.948608	0.964827	0.979862
1.4	0.922042	0.942058	0.953604	0.966601	0.978431	0.988479
1.6	0.939053	0.955935	0.965370	0.975670	0.984692	0.992034
1.8	0.950284	0.964686	0.972558	0.980984	0.988197	0.993934
2.0	0.958160	0.970633	0.977345	0.984434	0.990416	0.995108
2.2	0.963949	0.974910	0.980741	0.986842	0.991940	0.995904
2.4	0.968367	0.978122	0.983267	0.988613	0.993049	0.996478
2.6	0.971841	0.980617	0.985215	0.989968	0.993892	0.996911
2.8	0.974639	0.982608	0.986761	0.991037	0.994554	0.997250
3.0	0.976939	0.984232	0.988017	0.991902	0.995086	0.997522