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THE TECHNIQUES OF LINEAR MULTIOBJECTIVE PROGRAMMING

par P. L. YU (1) and M. ZELENY (2)

Abstract. — *In this article we derive a generalized version of the simplex method*-Multicriteria Simplex Method, *used to generate the set of all nondominated extreme solutions for linear programming problems with multiple objective functions. A simple nondominance subroutine is developed for testing the nondominance of any extreme solution.*

We discuss an important interaction between Multicriteria Simplex Method and multiparametric linear programming. *In fact we show that the decomposition of a multiparametric space (or a set of weights) into its optimal subsets can be obtained as its natural by-product.*

Theoretical results, numerical examples and flow diagram as well as some computer experience are reported.

1. INTRODUCTION

We are concerned with linear programming problems involving multiple (possibly noncommensurable) objective functions. To resolve this type of decision problems, we could use the concept of domination structures (See Ref. 1-3) or linear multi-parametric programming (See Ref. 4-5). We propose a simple technique, *Multicriteria Simplex Method,* to generate the set of all nondominated extreme point solutions and show how the *direct multiparametric approach* (Ref. 6) turns out to be computationally inefficient. It is also redundant because the decomposition of the parametric space is a by-product of Multicriteria Simplex Method.

Though, in general, the solution to a multicriteria problem does not have to be an extreme point, the entire set of all nondominated solutions can be effectively generated from the set of all nondominated extreme points (see Ref. 3 or 5).

Linear Multiobjective Programming represents a part of a broader field of study, *Multiple Criteria Decision Making.* We refer interested readers to some recent works summarizing up-to-date state of the arts (see for example Ref. 7, 8, 13).

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Before going further, for convenience, let us introduce the following notation. Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$. Then

(i) $x = y$ if and only if $x_j = y_j$ for all $j = 1, ..., n$.

(ii) $x \geqq y$ if and only if $x_j \geqq y_j$ for all $j = 1, ..., n$.

(iii) $x \geqslant y$ if and only if $x_j \geqq y_j$ for all $j = 1, ..., n$ and $x \neq y$.

Usually we shall denote a set or a matrix by a capital character. Given a matrix A, we will find it convenient to use A^i and A_j to denote its ith row and jth column respectively, and a_{ij} its element in the ith row and the jth column.

In order to simplify the presentation, let us assume that we have a compact decision space defined by

$$X = \{ x \in R^n \mid Ax \leqq b, x \geqq 0 \} \quad , \quad A \text{ is of order } m \times n. \tag{1}$$

Let $C = C_{l \times n}$ be a matrix with l rows $(C^1, ..., C^l)^T$ so that $C^k x$, $k = 1, ..., l$, is the kth objective function of our problem. Given a *domination cone* Λ (which is assumed to be convex) and $x^1, x^2 \in X$, we say that x^1 is *dominated* by x^2 if $Cx^1 \in Cx^2 + \Lambda$ and $Cx^1 \neq Cx^2$. A point $x \in X$ is a *N-point* if it is not dominated by any other feasible point of X; otherwise it is a D-point.

For simplicity, the sets of all N-points and all D-points will be denoted by N and D respectively.

If we denote the set of all extreme points of X by $X_{ex} = \{ x^1, ..., x^r \}$, then let $N_{ex} = N \cap X_{ex}$ be the *set of all nondominated extreme points.* We see that N_{ex} is finite because X is compact.

Given $\lambda \in R^l$, let

$$X^0(\lambda) = \{ x^0 \in X \mid \lambda Cx^0 \geqq \lambda Cx, x \in X \}. \tag{2}$$

Thus, $X^0(\lambda)$ is the set of all maximum points of λCx over X.

Note that λCx is *bilinear* in λ and x.

Given a cone Λ, we define its *polar cone*

$$\Lambda^* = \{ \lambda \mid \lambda d \leqq 0, \text{ for all } d \in \Lambda \}.$$

If $\Lambda = \{ x \mid Dx \leqq 0 \}$ is a *polyhedral cone*, we see that $\Lambda^* = \{ y D \mid y \geqq 0 \}$. It can be shown that (Remark 5.9 of Reference 1) *the relative interior of* Λ^* is given by $(\Lambda^*)^I = \{ y D \mid y > 0 \}$.

(It is understood that x is a column vector; y a row vector. Both represent vectors of R^l.)

We present a theorem describing necessary and sufficient conditions for a point to be nondominated. Its proof is given in Ref. 3.

Theorem 1.1. Suppose that Λ is a polyhedral cone. Then

(i) $N \subset \bigcup \{ X^0(\lambda) \mid \lambda \in (\Lambda^*)^I \}$

(ii) if $\operatorname{Int} \Lambda^* \neq \Phi$, then $N = \bigcup \{ X^0(\lambda) \mid \lambda \in \operatorname{Int} \Lambda^* \}$.

The theorem holds even if X is not bounded.

We may and will assume that $\Lambda = \{ d \in R^l \mid d \leqq 0 \} = \Lambda^{\leqq}$ to simplify the presentation.

Using the results of Theorem 1.1, we shall derive *Multicriteria Simplex Method* which may be regarded as a natural generalization of the simplex method. With this method we study the « connectedness » of N_{ex} and derive an algorithm to locate the entire set N_{ex}.

2. SIMPLEX METHOD AND $X^0(\lambda)$

Recall that since we limit ourselves to the domination cone $\Lambda = \Lambda^{\leqq}$, it follows that

$$\operatorname{Int} \Lambda^* = \{ d \in R^l \mid d > 0 \} = \Lambda^{>},$$

where Int stands for an interior.

Recall from (2) that $X^0(\lambda)$ is the set of maximum solutions of λCx over X. Treating λC as a row vector, we see that to find $X^0(\lambda)$ is essentially a series of linear programming problems.

REMARK 2.1. Since X is compact, an optimal solution to λCx exists.

We can generate the entire set of all basic feasible optimal solutions, say $X_{ex}^0 = \{ x^1, ..., x^k \}$. Then the set of all optimal solutions to λCx is given by $X^0(\lambda) = H(X_{ex}^0)$ (the convex hull generated by X_{ex}^0) (See Ref. 10 or 11). By varying λ over $\Lambda^{>}$, we can locate the entire set N via Theorem 1.1.

Although this method seems reasonable, it is by no means the best way to locate N, because how to vary λ over $\Lambda^{>}$ (See Ref. 4 and 6) is still unresolved and the computational work may be quite demanding. Thus, instead of this direct approach we shall use *Multicriteria Simplex Method* to locate N_{ex}. The method also indicates an efficient way to vary λ over $\Lambda^{>}$ in order to get the set N_{ex}.

Without loss of generality we can assume that $b \geqq 0$ in (1). (See Ref. 10 and 11 for the extension to other types of b.)

From (1), by adding slack variables, the decision space could be defined by the set of all $x \in R^{m+n}$, $x \geqq 0$ and

$$(A, I_{m \times m})x = b. \tag{3}$$

Our new C becomes $(C, O_{m \times m})$.

Given a basis B which is associated with columns $J = \{j_1, j_2, ..., j_m\}$, we shall denote the remaining submatrix and columns with respect to (3) by B' and J' respectively.

Let us introduce a simplex tableau corresponding to some basic feasible solution, say $x = (x_B, x_B) = (y_0, 0)$, associated with B (and J) :

r	Basis	$x_1 \dots x_m \quad x_{m+1} \dots x_j \dots x_{m+n}$	x
1	x_1	$1 \dots 0 \quad y_{1m+1} \cdots y_{1j} \cdots y_{1m+n}$	y_{10}
.	.	. . . .	.
.	.	. . . .	.
.	.	. . . .	.
m	x_m	$0 \dots 1 \quad y_{mm+1} \cdots y_{mj} \cdots y_{mm+n}$	y_{m0}
		$0 \dots 0 \quad z_{m+1} \dots z_j \dots z_{m+n}$	v

Tableau 1

By the simplex method, we can systematically change J, one column at each iteration, so that at each iteration $y_0 = B^{-1}b \geqq 0$ is maintained and the value of the objective function is improved until an optimal solution is obtained. For later reference, let us summarize some relevant results of the simplex method as follows (see Ref. 10). Observe in Tableau 1 that

$$Y = \{ y_{ij} \}_{\substack{i=1,\dots,m \\ j=1,\dots,m+n}} = (I, B^{-1}B') \tag{4}$$

$$z = (z_1, ..., z_{m+n}) = \lambda(C_B Y - C) \tag{5}$$

$$y_0 = (y_{10}, ..., y_{m0})^T = B^{-1}b \tag{6}$$

and

$$v = \lambda C_B B^{-1} b, \tag{7}$$

where λC_B are criteria coefficients associated with B.

Lemma 2.1. Given a feasible basis J so that $y_0 = B^{-1}b \geqq 0$, there are two possible cases which can occur in the simplex tableau :

Case 1. Each $z_j \geqq 0$ for all $j \in J'$. Then $x_B = y_0 = B^{-1}b$ and $x'_B = 0$ is a maximum solution. If each $z_j > 0$ for $j \in J'$, then the optimal solution is unique. Otherwise there may exist infinitely many optimal solutions.

Case 2. There is at least one $j \in J'$ so that $z_j < 0$. Let

$$\theta_j = \frac{y_{po}}{y_{pj}} = \min_r \left\{ \frac{y_{ro}}{y_{rj}} \mid y_{rj} > 0 \right\} {}^{(1)}. \tag{8}$$

Then by introducing the jth column into the basis, by Gaussian elimination technique we get the pth column of the identity matrix in the next tableau, (the element y_{pj} is called the *pivot element*), and we obtain a new basic feasible solution with an increase of the value of the objective function by $-\theta_j z_j$.

REMARK 2.2. Given a simplex tableau corresponding to a basis J_1, suppose we introduce the jth column, $j \in J_1'$, into the basis as described in *Case 2* of Lemma 2.1. We thus produce a new basis J_2, $J_2 = J_1 \cup \{j\} - \{j_p\}$ where $j_p \in J_1$ is the column associated with I_p in the simplex tableau of the basis J_1. Observe that there is exactly one element in J_2 which is not in J_1, and vice versa. Two bases such as J_1 and J_2 which enjoy the above property are known as *adjacent* to each other. The corresponding extreme point solutions are called *adjacent extreme points* of X.

3. MULTICRITERIA SIMPLEX METHOD

Observe that given a basis B, the row vector z in Tableau 1 is given by $\lambda(C_B Y - C)$. Let

$$Z = (C_B Y - C) \tag{9}$$

Then

$$z = \lambda Z. \tag{10}$$

From (9) and (10), we see that given λ the corresponding z can be easily computed whenever Z is known.

Observe that $(C_B Y - C) = (C_B^1 Y - C^1, \dots, C_B^l Y - C^l)^T$. Each $C_B^k Y - C^k$, $k = 1, \dots, l$, can be obtained from the last row of the simplex tableau if we replace λCx by $C^k x$ as the objective function.

For a given basis B (or J), let us construct *Multicriteria Simplex Tableau* as Tableau 2 (for simplicity, we have again rearranged the indices so that J appears in the first m columns).

Note that $\{y_{ij}\}$ is defined exactly as in (4) while $V = (v^1, \dots, v^l) = C_B B^{-1} b$. Note that v^k, $k = 1, \dots, l$ is the value of the kth objective function at the current basis.

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(1) Suppose there is no r so that $y_{rj} > 0$. We have an unbounded solution. Since X is assumed compact, this cannot happen. Thus θ_j is well defined.

r	BASIS	x_1	$\dots$	x_m	x_{m+1}	$\dots$	x_j	$\dots$	x_{m+n}	x
1	x_1	1	$\dots$	0	y_{1m+1}	$\dots$	y_{1j}	$\dots$	y_{1m+n}	y_{10}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	x_m	0	$\dots$	1	y_{mm+1}	$\dots$	y_{mj}	$\dots$	y_{mm+n}	y_{m0}
		0	$\dots$	0	$z^1{}_{m+1}$	$\dots$	z^1_j	$\dots$	$z^1{}_{m+n}$	v^1
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
		0	$\dots$	0	$z^l{}_{m+1}$	$\dots$	z^l_j	$\dots$	$z^l{}_{m+n}$	v^l

TABLEAU 2

Then for $k = 1, \dots, l$ we see that (1)

$$(0, \dots, 0, z^k_{m+1}, \dots, z^k_{m+n}) = (C^k_B Y - C^k) = Z^k.$$

Let us define M as

$$M = \begin{bmatrix} Y \\ Z \end{bmatrix}_{(m+l)\times(m+n)} \tag{11}$$

Observe that M enjoys the following properties,

(i) the submatrix $\{ y_j \mid j \in J \}$, when its rows are properly permutated, forms the identity matrix of order $m \times m$. (12)

(ii) The submatrix $\{ Z_j \mid j \in J \}$ is a zero matrix of order $l \times m$. (13)

For each nonbasic column $j \in J'$, we shall define θ_j as in (8).

By introducing the jth column into the basis we convert M_j into E_p in the next tableau, where E_p is the pth column of the identity matrix of order $m + l$ and p is such that y_{pj} is the pivot element. At each such iteration, M can enjoy the properties (12)-(13) and Y, Z can be easily computed.

REMARK 3.1. The row Z^k, $k = 1, \dots l$ is associated with a linear programming problem with objective function $C^k x$. In view of Lemma 2.1. if at a basis J, $Z^k \geqq 0$, then $x(J)$, the basic feasible solution of J, is an optimal basic solution for $C^k x$. If $z^k_j > 0$, for all $j \in J'$, then $x(J)$ is the unique optimal solution for $C_k x$ and clearly is an N_{ex}-point.

(1) Recall that Z^k (or Z_j) denotes the kth row (or the jth column) of matrix Z. Similarly for matrix M in (11).

REMARK 3.2. Given a basis J and $j \in J'$, by introducing jth column into the basis we produce an adjacent basis J_1 (see Remark 2.2). Then the values of the objective functions increase by $-\theta_j Z_j$. That is $V(J_1) - V(J) = -\theta_j Z_j$, where $V(J) = (v^1, ..., v^l)$ at the basis J. This observation yields :

Theorem 3.1. Given a basis J_0

(i) If there is $j \in J_0'$ so that $\theta_j Z_j \leqslant 0$, then $x(J_0) \in D$.

(ii) If there is $j \in J_0'$ so that $\theta_j Z_j \geqslant 0$, then $x(J_1) \in D$, where J_1 is the new basis obtained by introducing the jth column into the basis.

(iii) Let $j, k \in J_0'$ and J_j and J_k be the new bases obtained by introducing respectively the jth and kth columns into the basis. Suppose that $\theta_j Z_j \geqslant \theta_k Z_k$. Then $x(J_j) \in D$.

Theorem 3.1. and Remark 3.2., although obvious, will be useful in our later computation of N_{ex}.

4. OPTIMAL WEIGHTS AND A NONDOMINANCE SUBROUTINE

Now, given a basis J, let Z be the matrix associated with J. We can then uniquely define.

$$\Lambda(J) = \{ \lambda \mid \lambda Z \geqq 0 \}. \tag{14}$$

Note that $\Lambda(J)$ is a polyhedral cone and $0 \in \Lambda(J)$.

In view of Lemma 2.1., (10) and Theorem 1.1. we state

Theorem 4.1.

(i) $x(J)$ maximizes λCx over X for all $\lambda \in \Lambda(J)$.

(ii) $x(J) \in N_{ex}$ if and only if $\Lambda^{>} \cap \Lambda(J) \neq \varnothing$.

REMARK 4.1. Given J, $\Lambda(J)$ is its associated *set of optimal weights*, because whenever our objectives Cx are linearly weighted as λCx for some $\lambda \in \Lambda(J)$, $x(J)$ maximizes λCx. In the final decision-making, this is very valuable information.

REMARK 4.2. Given a basic feasible solution, we could use Remark 3.1, (i) of Theorem 3.1, and (ii) of Theorem 4.1 to detect whether it is an N_{ex}-point or not. However, although the results are useful, they cannot cover all possible cases. In the remaining part of this section, we shall derive a simple algebraic method, called the *nondominance subroutine*, so that we can test whether an extreme point is an N-point for all possible cases.

Let $x^0 = x(J)$ represent a basic feasible solution with basis J. Let $e = (e_1, ..., e_l)$ and

$$w = \max \sum_{i=1}^{l} e_i \tag{15}$$

subject to :

$$\tilde{X} = \{ (x, e) \mid x \in X, Cx - e \geqq Cx^0, e \geqq 0 \}$$

Theorem 4.2.

(i) x^0 is an N-point if and only if $w = 0$.

(ii) x^0 is a D-point if and only if $w > 0$.

Proof. Observe that $(x^0, 0) \in X$. Thus $w \geqq 0$. It suffices to show (i). However (i) is another way to define an N-point with respect to the domination cone $\Lambda^{\leqq}$. Q.E.D.

Corollary 4.1. If $w > 0$, then the corresponding maximum solution $x^1 \in X$, $Cx^1 \geqslant Cx^0$, is an N-point.

Observe that finding whether $w = 0$ or not in Theorem 4.2 usually does not require too much extra work. In order to see this, let B be the basis associated with x^0 or J. The problem of (15) in a block simplex tableau can be written

$$\left[\begin{array}{c|c|c|c} A_{m\times n} & I_{m\times m} & 0_{m\times l} & b_{m\times 1} \\ \hline C_{l\times n} & 0_{l\times m} & -I_{l\times l} & Cx^0 \\ \hline 0_{1\times n} & 0_{1\times m} & -1_{1\times l} & 0 \end{array}\right] \begin{array}{r} (16) \\ (17) \\ (18) \end{array}$$

where $1_{1\times l} = (1, 1, ..., 1)$.

In the above matrix, the first and second columns are the coefficients associated respectively with the original variables and the added slack variables, the third column is the coefficients associated with the new variable e in (15). Note that (16) is the constraint that $x \in X$, (17) is the constraint that $Cx\text{-}e \geqq Cx^0$, and (18) corresponds to the objective of (15).

We could rewrite (16)-(18) as follows :

$$\left[\begin{array}{c|c|c|c} B^{-1}A & B^{-1} & 0_{m\times l} & B^{-1}b \\ \hline C_BB^{-1}A - C & C_BB^{-1} & I_{l\times l} & 0_{l\times 1} \\ \hline 1_{1\times l}[C_BB^{-1}A - C] & 1_{1\times l}[C_BB^{-1}] & 0_{1\times l} & 0 \end{array}\right] \begin{array}{r} (19) \\ (20) \\ (21) \end{array}$$

Note that $(19) = B^{-1} \cdot (16)$, $(20) = C_B(19)\text{-}(17)$ (observe that $C_BB^{-1}b = Cx^0$), and $(21) = 1_{1\times l} \cdot (20) + (18)$.

Observe that (19)-(21) supply a feasible simplex tableau for Problem (15) with the basic feasible solution $(x, e) = (x^0, 0)$.

Comparing (19) and (20) with (4), (9) and (12) we see that [1].

$$\left[\begin{array}{c|c} B^{-1}A & B^{-1} \\ \hline C_B B^{-1}A - C & C_B B^{-1} \end{array}\right] = \begin{bmatrix} Y \\ Z \end{bmatrix} \tag{22}$$

From (19)-(22) we see that to construct a simplex tableau for Problem (15) does not require much extra work. The conditions in Theorem 4.2 could be easily verified. In particular, we have the following sufficiency condition :

Theorem 4.3. Given a basis J, suppose $1_{1\times l}Z \geqq 0$. Then $x(J)$ is an N_{ex}-point.

Proof. Because the first two blocks of (21), given by $1_{1\times l}Z$, $1_{1\times l}Z \geqq 0$ implies that $(x(J), 0)$ is an optimal solution to (15) with value $w = 0$ (see Lemma 2.1). The assertion follows immediately from Theorem 4.2.

REMARK 4.3. Suppose that the condition in Theorem 4.3 is not satisfied. Because of the special structure of (19)-(21), the problem of (15) usually can be simply solved in a few iterations. In order to use the results of (19)-(22) and Theorem 4.2-4.3, one can append an extra row, corresponding to the objective function $1_{1\times l}C$, to the simplex tableau. (See the example discussed in Section 6.)

5. DECOMPOSITION OF $\Lambda^{>}$ AND CONNECTEDNESS OF N_{ex}

Given a basis J, we could define its set of optimal weights $\Lambda(J)$ as in (14). Now suppose that for some $k \in J'$, $Z_k \neq 0$, $\theta_k < \infty$. Let us introduce the k^{th} column into the basis. Suppose that y_{pk} is the pivot element. Then we will produce an adjacent basis K so that

$$K' = J' \cup \{j_p\} - \{k\}$$

Without confusion (rearrange the indices, if necessary), let $p = j_p$. Then

$$K' = J' \cup \{p\} - \{k\} \tag{23}$$

Let W denote $Z(K)$ (the submatrix Z associated with K).

We want to study the relation between $\Lambda(J)$ and $\Lambda(K)$. Toward this end, observe that by Gaussian elimination technique,

$$W_j = \begin{cases} 0 & \text{if } j \in K \\ -Z_k/y_{pk} & \text{if } j = p \in K' \\ Z_j - y_{pj}Z_k/y_{pk} & \text{if } j \in K' - \{p\} \end{cases} \tag{24}$$

---

(1) Note, $C_j = 0$ if j is an index associated with a slack variable.

Since it is the pivot element, $y_{pk} > 0$ (see Lemma 2.1).

$$\text{Let } H_k = \{ \lambda \mid \lambda Z_k = 0 \}. \tag{25}$$

Since $y_{pk} > 0$, $\lambda(- Z_k/y_{pk}) \geqslant 0$ if and only if $\lambda Z_k \leqslant 0$. We see that

$$\Lambda(K) \subset \{ \lambda \mid \lambda Z_k \leqslant 0 \}. \tag{26}$$

But,

$$\Lambda(J) \subset \{ \lambda \mid \lambda Z_k \geqslant 0 \}. \tag{27}$$

We see, from (25)-(27), that H_k is a hyperplane in R^l, which separates the polyhedral cones $\Lambda(K)$ and $\Lambda(J)$.

Next, since $\lambda \in H_k$ implies that $\lambda Z_k = 0$, we have

$$H_k \cap \Lambda(J) = \{ \lambda \mid \lambda Z_k = 0 \quad , \quad \lambda Z_j \geqslant 0 \quad , \quad j \in J' - \{ k \} \} \tag{28}$$

and from (24) we also have

$$H_k \cap \Lambda(K) = \{ \lambda \mid \lambda Z_k = 0 \quad , \quad \lambda Z_j \geqslant 0 \quad , \quad j \in K' - \{ p \} \}. \tag{29}$$

However from (23), we have $K' - \{ p \} = J' - \{ k \}$. Thus (26)-(29) imply that

$$H_k \cap \Lambda(J) = H_k \cap \Lambda(K) = \Lambda(J) \cap \Lambda(K). \tag{30}$$

We summarize the above results into

Theorem 5.1. Given a basis J, suppose that $Z_k \neq 0$ and $\theta_k < \infty$. Let K be the adjacent new basis obtained by introducing the k^{th} column into the basis. Then H_k as defined in (25) separates $\Lambda(J)$ and $\Lambda(K)$. Furthermore, the equalities of (30) hold.

REMARK 5.1. Given $\Lambda(J)$ and $\Lambda(K)$, we say that $\Lambda(J)$ and $\Lambda(K)$ are *adjacent* if (26), (27) and (30) hold. Theorem 5.1 says that by introducing the column k with $Z_k \neq 0$ and $\theta_k < \infty$, into the basis, the new adjacent basis K will produce $\Lambda(K)$ which is adjacent to $\Lambda(J)$. However, it is possible that $\Lambda(J) \cap \Lambda(K) = \{ 0 \}$ and $\Lambda(K) \cap \Lambda^{\geqq} = \{ 0 \}$. If this case occurs, introducing the k^{th} column into the basis does not help solve our problem. This case can be avoided if $H_k \cap \Lambda(J) \geqq \cap \Lambda \neq \{ 0 \}$ (thus the intersection contains more than the zero point). A column $k \in J'$ with this property will be called an effective constraint of $\Lambda(J)$. Note from Theorem 5.1 that by introducing an effective constraint Z_k of $\Lambda(J)$ into the basis, we will produce $\Lambda(K)$ which has a nonempty intersection with $\Lambda^{\geqq} - \{ 0 \}$.

Now observe that for a given λ, λCx will either have an unbounded or optimal solution over X. In either case, by simplex method, λ will be contained by some Λ^j_∞ or $\Lambda(J_k)$. (Observe that Λ^j_∞ identifies $\Lambda(J)$ for $\theta_k = 0$). Since we have a finite number of bases and each basis has only finite number of columns, we know that there are finite number of Λ^j_∞ and $\Lambda(J_k)$ which form a covering of Λ . More precisely.

Theorem 5.2. There are $\{ \Lambda^j_\infty \mid j = 1, ..., p \}$ and $\{ \Lambda(J_k) \mid k = 1, ..., q \}$ so that each Λ^j_∞ or $\Lambda(J_k)$ has a nonempty intersection with $\Lambda^{\geqq}$ and

(i) $$\Lambda^{\geqq} \subset \cup \{ \Lambda^j_\infty \mid j = 1, ..., p \} \cup \{ \Lambda(J_k) \mid k = 1, ..., q \}$$

(ii) $\cup \{ \Lambda(J_k) \mid k = 1, ..., q \} \cap \Lambda^{\geqq}$ is a closed convex polyhedron. In fact it is a polyhedral cone.

Proof. (i) is clear from the previous discussion. In order to see (ii), observe that λCx cannot simultaneously have an optimal solution and an unbounded solution. Thus $\{ \Lambda^j_\infty \}$ and $\{ \Lambda(J_k) \}$ are mutually disjoint, and

$$\begin{aligned} \cup \{ \Lambda(J_k) \mid k = 1, ..., q \} \cap \Lambda^{\geqq} &= \text{Comp} [\cup \{ \Lambda^j_\infty \mid j = 1, ..., p \}] \cap \Lambda^{\geqq} \\ &= \cap \{ \text{Comp}\, \Lambda^j_\infty \mid j = 1, ..., p \} \cap \Lambda^{\geqq}, \end{aligned}$$

where Comp designates a complement to a set. (The first equality comes from (i) and mutual disjointness which implies that $\lambda \in \Lambda^{\geqq}$ is in some $\Lambda(J_k)$ if and only if it is not in some Λ^j_∞.)

We see that each Comp Λ^j_∞ is a closed half space. Our conclusion of (ii) is clear from (30). Q.E.D.

REMARK 5.2. Theorem 5.2 states that there are finite number of Λ^j_∞ and $\Lambda(J_k)$ that will cover $\Lambda^{\geqq}$. Such Λ^j_∞ and $\Lambda(J_k)$ can be located by Multicriteria Simplex Method. Theorem 5.1 and Remark 5.1 provide a way to generate adjacent « nonoverlapping » polyhedral cones in the parametric space. It is not reasonable to adopt a method of direct decomposition of $\Lambda^{\geqq}$ to resolve our problem. The method starts with a $\Lambda(J)$ so that $\Lambda(J) \cap \Lambda^{\geqq} \neq \emptyset$, then uses Theorem 5.1 to generate the adjacent $\Lambda(J_k)$ or Λ^j_∞. The procedure is repeated until $\Lambda^{\geqq}$ is completely covered by $\{ \Lambda(J_k) \}$ and $\{ \Lambda^j_\infty \}$. This method was discussed in [6]. For a detailed discussion of its shortcomings and inefficiency see Ref. 4.

REMARK 5.3. Once J is found to be an N_{ex}-basis (Theorem 4.2), Eq. (14) can be used to find its related set of optimal weights $\Lambda(J)$ at no extra work from the multi-criteria simplex tableau. Thus our remaining crucial task is to find the set N_{ex} by Multicriteria Simplex Method.

Let $E = \{ x(i) \mid i = 1, ..., p \}$ be a set of extreme points of X.

We say that E is *connected* if it contains only one point or if for any two points $x(i)$, $x(k)$ in E, there is a sequence $\{ x(i_1), ..., x(i_r) \}$ in E so that $x(i_l)$ and $x(i_{l+1})$, $l = 1, ..., r - 1$, are adjacent and $x(i_1) = x(j)$, $x(i_r) = x(k)$.

Following a similar proof as in [5], we have

Theorem 5.3. The set N_{ex} is connected.

Proof. Let $x(i), x(j) \in N_{ex}$. Suppose I and J are the bases associated with $x(i)$ and $x(j)$ respectively. Then, by (ii) of Theorem 1.1, both $\Lambda(I) \cap \Lambda^{>}$ and $\Lambda(J) \cap \Lambda^{>}$ are not empty. Let $\lambda_i \in \Lambda(I) \cap \Lambda^{>}$ and $\lambda_j \in \Lambda(J) \cap \Lambda^{>}$. Since $\Lambda^{>}$ is convex the line segment $[\lambda_i, \lambda_j] \subset \Lambda^{>}$. From Theorem 5.2, we can find a finite sequence $\{ \Lambda(J_k) \mid k = 1, ..., r \}$ so that $[\lambda_i, \lambda_j] \cap \Lambda(J_k) \neq \varnothing$ and

$$[\lambda_i, \lambda_j] \subset \cup \{ \Lambda(J_k) \mid k = 1, ..., r \}.$$

In view of (ii) of Lemma 2.1 and Theorem 1.1, we see that we can find a sequence of N_{ex}-points $\{ x_{i1}, ..., x_{ir} \}$ so that x_{il} is adjacent to x_{il+1}, $l = 1, ..., r-1$, and $x_{i1} = x(i)$, $x_{ir} = x(j)$. Q.E.D.

REMARK 5.4. In view of Theorem 5.3 we can construct a connected graph $(\mathcal{A}, \mathcal{V})$ for N_{ex}, where $\mathcal{V}$ is the set of all vertices corresponding to N_{ex}, and $\mathcal{A}$ is the set of all arcs in the graph. Given (1) $x^1, x^2 \in N_{ex}$ the arc $a(x^1, x^2)$ which connects x^1 and x^2 is in $\mathcal{A}$ if and only if x^1 and x^2 are adjacent. With this definition we see that the graph $(\mathcal{A}, \mathcal{V})$ is connected.

6. A METHOD TO GENERATE THE ENTIRE SET N_{ex} AND AN EXAMPLE

In order to generate the set N_{ex}, we can first find a basis J_1 for an N_{ex}-point, if $N_{ex} \neq \varnothing$. In view of Remark 5.4, if there is any other N_{ex}-point, we must have an N_{ex}-basis J_2 adjacent to J_1. Thus we could use results of this section to search for such a J_2. If there is no such J_2, J_1 is the unique N_{ex}-point. Otherwise, we consider all adjacent, but unexplored feasible bases to $\{ J_1, J_2 \}$ to see if there is any other N_{ex}-basis among them. If there is none, $\{ J_1, J_2 \}$ represents the set N_{ex}. Otherwise, we add a new N_{ex}-basis to $\{ J_1, J_2 \}$ and continue with the procedure until the entire set N_{ex} is located.

We shall use Flow Diagram 1 to explain our procedure more precisely. In the diagram, we have used the following notation :

(i) For each basis J, we use $\mathfrak{D}(J)$ to denote the set of all « obviously » dominated bases which are adjacent to J. That is, those dominated adjacent bases which can easily be checked by Theorem 3.1. We also use $\mathcal{A}(J)$ to denote the set of all adjacent bases to J which are not in $\mathfrak{D}(J)$ and their nondominance have not been checked before. Thus $\mathcal{A}(J)$ denotes the set of all adjacent bases to J of which the nondominance must be checked by nondominance subroutine.

(ii) At each step i, N_i and D_i are the sets of all *checked* nondominated and dominated extreme points respectively, while W_i is the set of all possible bases of which the nondominance must be established by the nondominance subroutine.

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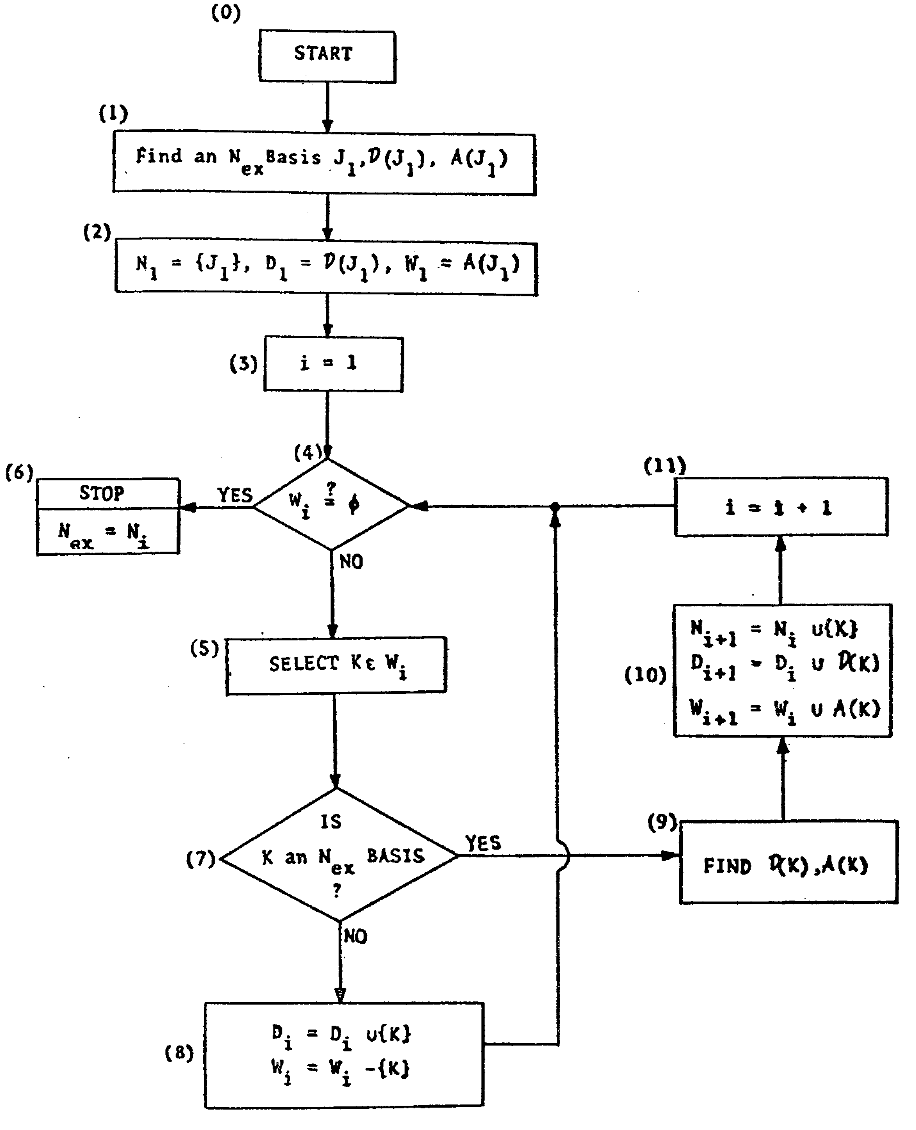
(1) It is convenient, without confusion, for us to use x^1, x^2 to represent their bases J_1, J_2 and the resulting basic feasible solutions $x(J_1)$, $x(J_2)$ as well.


(0)
START
(1)
Find an N_{ex} Basis $J_1, \mathcal{D}(J_1), A(J_1)$
(2)
$N_1 = \{J_1\}, D_1 = \mathcal{D}(J_1), W_1 = A(J_1)$
(3)
$i = 1$
(4)
$W_i \overset{?}{=} \phi$
YES
NO
(6)
STOP
$N_{ex} = N_i$
(5)
SELECT $K \in W_i$
(7)
IS K an N_{ex} BASIS ?
YES
NO
(8)
$D_i = D_i \cup \{K\}$
$W_i = W_i - \{K\}$
(9)
FIND $\mathcal{D}(K), A(K)$
(10)
$N_{i+1} = N_i \cup \{K\}$
$D_{i+1} = D_i \cup \mathcal{D}(K)$
$W_{i+1} = W_i \cup A(K)$
(11)
$i = i + 1$

We briefly describe Flow Diagram 1 as follows :

Box (1). From our assumption that X is compact; we know that $N_{ex} \neq \emptyset$. Since $1_{1\times l} \in \Lambda^{>}$, the maximum solution to $1_{1\times l}Cx$ over X is an N_{ex}-point. We may use this N_{ex}-point te start with.

$\mathfrak{D}(J_1)$ and $\mathcal{A}(J_1)$ are to found by Theorem 3.1.

Box (2) and (3) are clear.

Box (4)-(6). Suppose $W_i = \emptyset$. Since N_{ex} is connected (Theorem 5.3 and Remark 5.3), we know that we have already located all N_{ex}-points.

Thus we stop at Box (6). Otherwise, we go to Box (5).

Observe that if $N_{ex} = N_i$, then there are i N_{ex}-points.

Box (7)-(11). In Box (7) we use nondominance subroutine to verify whether K is an N_{ex}-basis or not. If it is, we get one more N_{ex}-point and go through Box (9)-(11). Note, in Box (9), again we use Theorem 3.1 to find $\mathfrak{D}(K)$. To find $\mathcal{A}(K)$ we need to use the record of N_i and D_i. Once $\mathfrak{D}(K)$ and $\mathcal{A}(K)$ are found, Box (10) and (11) are clear. Suppose that K is not an N_{ex}-basis. We go to Box (8). We see that D_i is increased by one, while W_i is decreased by one.

An Example (Problem 1)

The objective functions :

$$Cx = \begin{bmatrix} 1 & 2 & -1 & 3 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 & 3 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 & -1 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_7 \end{pmatrix}$$

The constraints :

$$Ax = \begin{bmatrix} 1 & 2 & 1 & 1 & 2 & 1 & 2 \\ -2 & -1 & 0 & 1 & 2 & 0 & 1 \\ -1 & 0 & 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & 2 & -1 & 1 & -2 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_7 \end{pmatrix} \leqq \begin{pmatrix} 16 \\ 16 \\ 16 \\ 16 \end{pmatrix}$$

$$x_j \geqq 0 \quad , \quad i = 1, \ldots, 7$$

We set up the initial multicriteria simplex tableau as in Tableau 2. Observe that the last row of the tableau is corresponding to the row of $1_{1\times l}Cx$ (see Remark 4.3).

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	
x_8	1	2	1	1	2	1	2	1	0	0	0	16
x_9	− 2	− 1	0	1	2	0	1	0	1	0	0	16
x_{10}	− 1	0	1	0	2	0	− 2	0	0	1	0	16
x_{11}	0	1	2	− 1	1	− 2	− 1	0	0	0	1	16
v^1	− 1	− 2	1	− 3	− 2	0	− 1	0	0	0	0	0
v^2	0	− 1	− 1	− 2	− 3	− 1	0	0	0	0	0	0
v^3	− 1	0	− 1	1	0	1	1	0	0	0	0	0
w	− 2	− 3	− 1	− 4	− 5	0	0	0	0	0	0	0

TABLEAU 3

Observe that $\theta_5 Z_5 \leqslant 0$. In view of Theorem 3.1, the current basis is dominated. By introducing column 5 into the basis (the circle indicates the pivot element), we get Tableau 4.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	
x_5	1/2	1	1/2	1/2	1	1/2	1	1/2	0	0	0	8
x_9	− 3	− 3	− 1	0	0	− 1	− 1	− 1	1	0	0	0
x_{10}	− 2	− 2	0	− 1	0	− 1	− 4	− 1	0	1	0	0
x_{11}	− 1/2	0	3/2	− 3/2	0	− 5/2	− 2	− 1/2	0	0	1	8
v^1	0	0	2	− 2	0	1	1	1	0	0	0	16
v^2	3/2	2	1/2	− 1/2	0	1/2	3	3/2	0	0	0	24
v^3	− 1	0	− 1	1	0	1	1	0	0	0	0	0
w	1/2	2	3/2	− 3/2	0	5/2	5	5/2	0	0	0	40

TABLEAU 4

Now, Theorem 3.1 could not tell whether the current basis is nondominated or not. We have to use nondominance subroutine. Observe that the last column of (20) and (21) is 0. Thus each $\theta_j = 0$ for the problem (15). To know whether $w > 0$ or not, it may suffice to check only (20) and (21). Also from Tableau 4, we see $Z_j = 0$ for $j = 9, 10, 11$. In checking (20) and (21) these columns will never be changed. Deleting the first four rows and columns 5, 9-11 of Tableau 4, we get the essential part of the simplex tableau associated with (20)-(21) for verifying whether $w > 0$ or not as in Tableau 5. Observe that the last row is the criterion row for the subproblem (15). Going through the usual simplex method, we find an optimal solution for (15) in Tableau 7. Observe that since $w = 0$, the basis associated with Tableau 4 is nondominated. Denote its basic feasible solution by x^1. Then $x^1 = (0, 0, 0, 0, 8, 0, 0)$.

	x_1	x_2	x_3	x_4	x_6	x_7	x_8	e_1	e_2	e_3	
e_1	0	0	2	−2	1	1	1	1	0	0	0
e_2	3/2	2	1/2	−1/2	1/2	3	3/2	0	1	0	0
e_3	−1	0	−1	1	1	1	0	0	0	1	0
w	1/2	2	3/2	−3/2	5/2	5	5/2	0	0	0	0

TABLEAU 5

	x_1	x_2	x_3	x_4	x_6	x_7	x_8	e_1	e_2	e_3	
e_1	−2	0	0	0	3	3	1	1	0	2	0
e_2	1	2	0	0	1	7/2	3/2	0	1	1/2	0
x_3	−1	0	−1	1	1	1	0	0	0	1	0
w	−1	2	0	0	4	13/2	5/2	0	0	3/2	0

TABLEAU 6

	x_1	x_2	x_3	x_4	x_6	x_7	x_8	e_1	e_2	e_3	
e_1	0	4	0	0	5	10	4	1	2	3	0
x_1	1	2	0	0	1	7/2	3/2	0	1	1/2	0
x_4	0	2	−1	1	2	9/2	3/2	0	1	3/2	0
w	0	4	0	0	5	10	4	0	1	2	0

TABLEAU 7

From (14) and Tableau 4, we get $\Lambda(x^1)$ to be the solution set of the following system of linear inequalities

$$(\lambda_1, \lambda_2, \lambda_3) \begin{bmatrix} 0 & 0 & 2 & -2 & 1 & 1 & 1 \\ 3/2 & 2 & 1/2 & -1/2 & 1/2 & 3 & 3/2 \\ -1 & 0 & -1 & 1 & 1 & 1 & 0 \end{bmatrix} \geqq 0 \qquad (31)$$

In order to have a two dimensional graphical representation, observe that each ray of $\Lambda^{\leqq} \subset R^3$ is one-to-one corresponding to a point of the simplex

$$S = \{ (\lambda_1, \lambda_2, \lambda_3) \mid \lambda_1, \lambda_2, \lambda_3 \geqslant 0 \quad , \quad \lambda_1 + \lambda_2 + \lambda_3 = 1 \}$$

Thus, by setting $\lambda_1 = 1 - \lambda_2 - \lambda_3$ and deleting the redundant constraints of (31) we get the set $\Lambda(x^1)$ in S as

$$\Lambda(x^1) = \{ (\lambda_2, \lambda_3) \mid 3\lambda_2/2 - \lambda_3 \geqslant 0 \quad , \quad 3\lambda_2/2 + 3\lambda_3 = 2 \} \qquad (32)$$
$$\lambda_2, \lambda_3 \geqslant 0$$

The set of $\Lambda(x^1) \cap S$ is given in Figure 1.

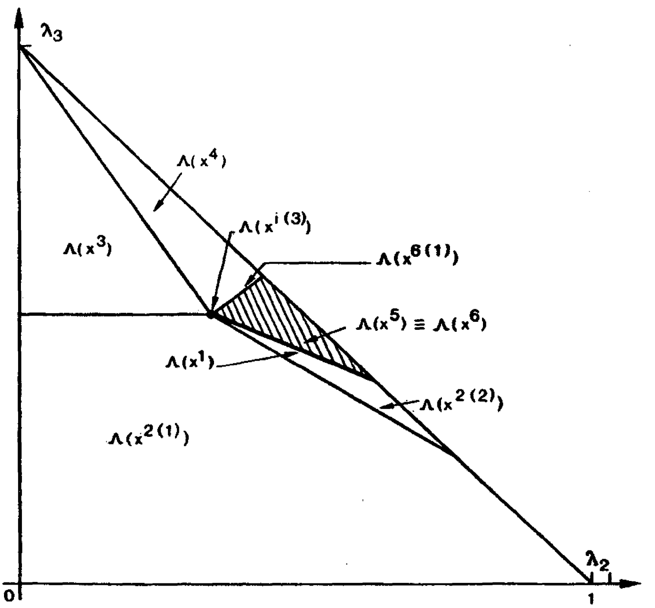


Figure 1

Observe that if we use y_{25} and y_{35} as pivot elements we will get $x^{1(1)}$ and $x^{1(2)}$, both of them are different bases associated with x^1. However,

$$\Lambda(x^{1(j)}) \cap S \neq \varnothing \quad , \quad j = 1, 2.$$

From the basis $x^{1(1)}$ we get another basis for x^1. We call it $x^{1(3)}$. Note, $\Lambda(x^{1(3)}) = \left(\frac{1}{3}, \frac{1}{2}\right) \neq \Lambda(x^1)$.

Following the procedure described in the previous section, we get the set N_{ex} and its related $\{ \Lambda(J_k) \}$ as in Tableau 8,

		Corresponding basis	Values of objective functions C^1x	C^2x	C^3x	Corresponding subset of Λ
Nondominated extreme points	x^1	$\{5, 9, 10, 11\}$	16	24	0	$\Lambda(x^1)$
	x^2	$\{4, 9, 10, 11\}$	48	32	-16	$\Lambda(x^2)$
	x^3	$\{1, 9, 10, 11\}$	16	0	6	$\Lambda(x^3)$
	x^4	$\{1, 3, 9, 10\}$	0	8	16	$\Lambda(x^4)$
	x^5	$\{3, 4, 9, 10\}$	16/3	64/3	16/3	$\Lambda(x^5)$
	x^6	$\{3, 5, 9, 10\}$	16/3	64/3	16/3	$\Lambda(x^6)$
Some other degenerate bases	$x^{1(3)}$	$\{1, 5, 10, 11\}$	16	24	0	$\Lambda(x^{1(3)})$
	$x^{2(1)}$	$\{1, 4, 10, 11\}$	48	32	-16	$\Lambda(x^{2(1)})$
	$x^{2(2)}$	$\{3, 4, 10, 11\}$	48	32	-16	$\Lambda(x^{2(2)})$
	$x^{6(1)}$	$\{1, 3, 5, 9\}$	16/3	64/3	16/3	$\Lambda(x^{6(1)})$

Tableau 8

where $x^1 = (0, 0, 0, 0, 8, 0, 0)$, $x^2 = (0, 0, 0, 16, 0, 0, 0)$,

$x^3 = (16, 0, 0, 0, 0, 0, 0)$, $x^4 = (8, 0, 8, 0, 0, 0, 0)$,

$x^5 = \left(0, 0, \frac{32}{3}, \frac{16}{3}, 0, 0, 0\right)$ and $x^6 = \left(0, 0, \frac{16}{3}, 0, \frac{16}{3}, 0, 0\right)$.

Also, we get the following subsets :

$$\Lambda(x^1) = \left\{ \lambda \,\middle|\, \frac{3}{2}\lambda_2 - \lambda_3 \geqslant 0 ; \frac{3}{2}\lambda_2 + 3\lambda_3 \geqslant 2 ; \frac{3}{2}\lambda_2 + 3\lambda_3 \leqslant 2 \right\}$$

$$\Lambda(x^2) = \{ \lambda \mid 4\lambda_3 \leqslant 2 ; 3\lambda_2 + 6\lambda_3 \leqslant 4 ; 3\lambda_2 + 6\lambda_3 \leqslant 4 \}$$

$$\Lambda(x^3) = \{ \lambda \mid 3\lambda_2 + 2\lambda_3 \leqslant 2 ; 4\lambda_3 \geqslant 2 ; - 3\lambda_2 + 2\lambda_3 \geqslant 0 \}$$

$$\Lambda(x^4) = \left\{ \lambda \,\middle|\, -\frac{3}{2}\lambda_2 + 3\lambda_3 \geqslant 1 ; -\frac{3}{2}\lambda_2 + 3\lambda_3 \geqslant 1 ; \frac{3}{2}\lambda_2 + \lambda_3 \geqslant 1 \right\}$$

$$\Lambda(x^5) = \left\{ \lambda \,\middle|\, - \lambda_2 + 2\lambda_3 \leqslant \frac{2}{3} ; \lambda_2 + 2\lambda_3 \geqslant \frac{4}{3} \right\}$$

$$\Lambda(x^6) = \left\{ \lambda \,\middle|\, - \lambda_2 + 2\lambda_3 \leqslant \frac{2}{3} ; \lambda_2 + 2\lambda_3 \geqslant \frac{4}{3} \right\}$$

and

$$\Lambda(x^{1(3)}) = \left\{ \lambda \,\middle|\, 2\lambda_2 + \frac{8}{3}\lambda_3 \leqslant 2 ; \frac{3}{2}\lambda_2 + \frac{8}{3}\lambda_3 \geqslant 2 ; \frac{1}{2}\lambda_2 - \frac{1}{3}\lambda_3 \geqslant 0 \right\}$$

$$\Lambda(x^{2(1)}) = \{ \lambda \mid 9\lambda_2 + 14\lambda_3 \leqslant 10 ; 4\lambda_3 \leqslant 2 \}$$

$$\Lambda(x^{2(2)}) = \{ \lambda \mid 9\lambda_2 + 14\lambda_3 \geqslant 10 ; 3\lambda_2 + 6\lambda_3 \leqslant 4 ; 3\lambda_2 + 6\lambda_3 \leqslant 4 \}$$

$$\Lambda(x^{6(1)}) = \left\{ \lambda \,\middle|\, -\frac{1}{2}\lambda_2 + \lambda_3 \geqslant \frac{1}{3} ; -\frac{1}{2}\lambda_2 + \lambda_3 \leqslant \frac{1}{3} ; \lambda_2 + 2\lambda_3 \geqslant \frac{4}{3} \right\}$$

7. A COMPUTER EXPERIENCE

We have coded Multicriteria Simplex Program in FORTRAN according to Flow Diagram 1 (for details see Ref. 5). Our examples are executed on *IBM 7040* (1).

It takes a total time of 2.881 minutes to get the set N_{ex} for the problem described in Section 6. Observe that the problem is by no means trivial because of the existence of degeneracies and alternative solutions.

We then try the following problem :

The Objective Functions

$$\begin{bmatrix} 2 & 5 & 1 & -1 & 6 & 8 & 3 & -2 \\ 5 & -2 & 5 & 0 & 6 & 7 & 2 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_8 \end{pmatrix}$$

The Constraints

$$\begin{bmatrix} 1 & 3 & -4 & 1 & -1 & 1 & 1 & 1 \\ 5 & 2 & 4 & -1 & 3 & 7 & 2 & 7 \\ 0 & 4 & -1 & -1 & -3 & 0 & 0 & 1 \\ -3 & -4 & 8 & 2 & 3 & -4 & 5 & -1 \\ 12 & 8 & -1 & 4 & 0 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 8 & -12 & -3 & 4 & -1 & 0 & 0 & 0 \\ -5 & -6 & 12 & 1 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_8 \end{pmatrix} \leqslant \begin{pmatrix} 40 \\ 84 \\ 18 \\ 100 \\ 40 \\ -12 \\ 30 \\ 100 \end{pmatrix}$$

$$x_j \geqslant 0 \quad , \quad j = 1, \ldots, 8.$$

(1) The speed of IBM 7040 is much slower than that of the IBM 360. We will try in the near future to execute the same problems on the IBM 360 and report the experience. A useful computational experience with an alternative algorithm is available in Ref. 15.

Observe that this problem is intentionally complicated. For instance, $C^3 = -A^6$ (note, C^3 is the third row of the objectives, while A^6, is the sixth row for the constraints), also $A^3 = A^2 - C^2$. Such dependencies will certainly make our computation more lengthy. Note that in this problem the upper limit on the number of feasible bases is $\binom{16}{8} = 12{,}870$. However, we get only 3 N_{ex}-points. It takes a total time of 0.814 minutes to execute the problem.

In the next problem we use the same constraints as in the previous one, however we have five objective functions :

$$\begin{bmatrix} 3 & -7 & 4 & 1 & 0 & -1 & -1 & 8 \\ 2 & 5 & 1 & -1 & 6 & 8 & 3 & -2 \\ 5 & -2 & 5 & 0 & 6 & 7 & 2 & 6 \\ 0 & 4 & -1 & -1 & -3 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_8 \end{pmatrix}$$

The set N_{ex} contains 70 points. It takes 15.65 minutes to carry out the computation. For details see reference 5.

Observe that our method for locating N_{ex} indeed is a combination of a modified linear program and an enumeration technique. The time required to locate N_{ex} consequently depends on the size of the problem (the dimensionality of A and C) and the number of total N_{ex}-points (the interrelation among the rows of A and C). One can easily imagine that when the dimensions of A and C get large, it might become very difficult to incorporate the method efficiently. To illustrate how the number of N_{ex}-points can effect the computation time, observe that our first problem has a lower dimensionality than the second problem but it takes more time for locating N_{ex} because its N_{ex} contains more elements. Also, although the third problem has the same A as the second one, it takes much longer to locate its N_{ex} than it does for the second problem because its N_{ex} contains 70 elements while the second one contains only 3 elements.

8. CONCLUSIONS

We have shown that the set of all nondominated extreme points can be generated quite efficiently. In complex multicriteria situations the final solution can be any N-point, not necessarily an N_{ex}-point. Therefore we might be interested in finding N rather than N_{ex}. One method using N_{ex} to generate complete set N is discussed in References 3 and 5. This method produces the set of nondominated faces of a convex polyhedron.

The nondominated solutions are essentially the first step toward good decision making. Actual selection of the final solution from N remains a challenging task, see Ref. 12 and 14. For an excellent review of possible ways of resolution see Reference 8.

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