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EXTREMUM CONDITIONS FOR NONLINEAR FUNCTIONS ON CONVEX SETS

by Stefan MITITELU (1)

Abstract. — *The paper establishes necessary and sufficient extremum conditions for a function f that is nonlinear, differentiable on a closed convex set generated by quasi-convex and quasi-concave differentiable functions. Conditions of the same type are also established for the case when nonlinear function f is quasi-convex.*

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Let C be a convex set in the real Euclidean space R^n.

Definition 1. Any function $f : C \to R$ is called quasi-convex in C if for any points x^1, $x^2 \in C$ and any number $\lambda \in [0, 1]$ we have

$$f(\lambda x^1 + (1 - \lambda)x^2) \leqslant \max \{f(x^1), f(x^2)\}. \tag{1}$$

If for any points x^1, $x^2 \in C$, $x^1 \neq x^2$, and any number $\lambda \in (0, 1)$ relation (1) exists with the sign of strict inequality, then function f is called strictly quasi-convex on the set C.

Any strictly quasi-convex function is quasi-convex [7].

Definition 2. The function $f : X \subseteq R^n \to R$ is called positive definite in the point $x^0 \in X$ if its hessian matrix in x^0,

$$H_f(x^0) = \left(\frac{\partial^2 f}{\partial x_i^0 \partial x_j^0}\right)_{1 \leqslant i, j \leqslant n}$$

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is positive definite, i.e. if the following conditions are satisffied :

$$\frac{\partial^2 f}{\partial x_1^{02}} > 0, \quad \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^{02}} & \frac{\partial^2 f}{\partial x_1^0 \partial x_2^0} \\ \frac{\partial^2 f}{\partial x_2^0 \partial x_1^0} & \frac{\partial^2 f}{\partial x_2^{02}} \end{vmatrix} > 0, \ldots, \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^{02}} & \frac{\partial^2 f}{\partial x_1^0 \partial x_2^0} & \cdots & \frac{\partial^2 f}{\partial x_1^0 \partial x_n^0} \\ \frac{\partial^2 f}{\partial x_2^0 \partial x_1^0} & \frac{\partial^2 f}{\partial x_2^{02}} & \cdots & \frac{\partial^2 f}{\partial x_2^0 \partial x_n^0} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_n^0 \partial x_1^0} & \frac{\partial^2 f}{\partial x_n^0 \partial x_2^0} & \cdots & \frac{\partial^2 f}{\partial x_n^{02}} \end{vmatrix} > 0.$$

Theorem 1. Let be function f quasi-convex and differentiable on the convex set $C \subseteq R^n$ and a point $x^0 \in C$ where $\nabla f(x^0) = 0$. If function f is positive definite in x^0, then x^0 is the global minimum of the function f on C.

Proof. Since f is positively defined in $x^0 \in C$ it results that x^0 is a point of strict local minimum of the function f in C. However, for a quasi-convex function any strict local minimum is the global minimum of the function [2], [8].

Theorem 2. Let be functions f, $g = (g_1, ..., g_m)$ and $h = (h_1, ..., h_p)$ differentiable on the convex set $K \subseteq R^n$, the convex set $C = \{ x \in K / g(x) \leqslant 0, h(x) = 0 \}$, a point $x^0 \in C$ and the set $\mathfrak{J}^0 = \{ i / g_i(x^0) = 0 \}$.

We assume that function $g_i (i \in \mathfrak{J}^0)$ are quasi-convex on K, while h is quasi-convex and quasi-concave on K.

If $u_i^0 \geqslant 0 (i \in \mathfrak{J}^0)$ and $v^0 = (y_1^0, ..., v_p^0) \in R^p$ exist, such that

(2) $$(x - x^0)' \nabla_x L(x^0, u^0, v_0) \geqslant 0, (\forall) x \in C,$$

where

$$L(x, u, v) = \varepsilon f(x) + \sum_{i \in \mathfrak{J}^0} u_i^0 g_i(x) + v' h(x), \varepsilon = + 1 [\varepsilon = - 1]$$

and if

(*a*) for a $u_{i0}^0 > 0 \quad (i_0 \in \mathfrak{J}^0)$ we have

(3) $$(x - x^0)' \nabla g_{i_0}(x^0) < 0, (\forall) x \in C - \{ x^0 \},$$

or at least for a $v_{j0}^0 > 0$ we have

$$(x - x^0)' \nabla h_{j_0}(x^0) < 0, \quad (\forall) x \in C - \{ x^0 \},$$

or

$$(x - x^0)' \nabla h_{j_0}(x^0) > 0, \quad (\forall) x \in C - \{ x^0 \},$$

then x^0 is a local minimum [maximum] point of the function f on C.

(4) $$(b)\ C \cap \{ x \in R^n/(x - x^0)' \nabla f(x^0) = 0 \} = \{ x^0 \},$$

then x^0 is a local minimum [maximum] of the function f on C.

(c) function f is quasi-convex continuously differentiable in C and $\nabla f(x^0) \neq 0$, then x^0 is the global minimum [maximum] of the function f on C.

Proof. (a) Since relation $h(x) = 0$ is equivalent to relations $h(x) \leqslant 0$ and $-h(x) \leqslant 0$ it is enough to prove the theorem for the case when we have only $g(x) \leqslant 0$. Then only rel. (3) is necessary for the proof.

Since functions g_i are quasi-convex on C, relations

$$g_i(x) \leqslant g_i(x^0) = 0 (i \in \mathfrak{J}^0), \qquad (\forall) x \in C,$$

imply relations [7] :

(5) $$(x - x^0)' \nabla g_i(x^0) \leqslant 0, i \in \mathfrak{J}^0, \qquad (\forall) x \in C,$$

where for $i = i_0$ there exists the sign of strict inequality, according to (3).

For $u_i^0 \geqslant 0, i \in \mathfrak{J}^0$, where $u_{i0}^0 > 0$, from (5) it results

(6) $$(x - x^0)' \sum_{i \in \mathfrak{J}^0} u_i^0 \nabla g_i(x^0) < 0, (\forall)\, x \in C - \{ x^0 \}.$$

From rels. (2) and (6) it results

$$(x - x^0)' \nabla f(x^0) > 0, \qquad (\forall) x \in C - \{ x^0 \},$$

and according to [3], P. 36, point x^0 is a local minimum [maximum] of the function f in the set C.

(b) From relations (2) and (5) where $u_i^0 \geqslant 0, i \in \mathfrak{J}^0$, it results

(7) $$(x - x^0)' \nabla f(x^0) \geqslant 0, \qquad (\forall) x \in C$$

and taking account of (4), from (7) it results

$$(x - x^0)' \nabla f(x^0) > 0, \qquad (\forall) x \in C - \{ x^0 \}.$$

According to point (a) x^0 is a point of local minimum [maximum] of the function f in C.

(c) Relation (7) result, where f is quasi-convex on C. If $\nabla f(x^0) \neq 0$ from (7) :

(8) $$x^{0\prime} \nabla f(x^0) = \min_{x \in C} x' \nabla f(x^0)$$

will result and according to proposition 5 in [5] rel. (8) is equivalent to

$$f(x^0) = \min_{x \in C} f(x)$$

Theorem 3. Let be functions f differentiable and $g = (g_1, ..., g_m)$ differentiable and quasi-convex on the convex set $K \subseteq R^n$, point $x^0 \in C = \{ x \in K / g(x) \leqslant 0, x \geqslant 0 \}$ with nonempty interior and the set

$$\mathfrak{J}^0 = \{ i / g_i(x^0) = 0 \}.$$

(n) A necessary condition that point x^0 should be a local minimum [maximum] of the function f in C is that $u^0 \in R^m$ should exist, so that

$$\begin{aligned} &x^0 \geqslant 0, y^0 = \varepsilon \nabla f(x^0) + \sum_{i=1}^{m} u_i^0 \nabla g_i(x^0) \geqslant 0, x^{0\prime} y^0 = 0 \\ &g(x^0) \leqslant 0, \qquad u^0 \geqslant 0, \qquad u^{0\prime} g(x^0) = 0, \end{aligned} \tag{9}$$

where $\varepsilon = +1 [\varepsilon = -1]$ or equivalently :

$$\begin{aligned} &x^0 \geqslant 0, \quad \nabla_x L(x^0, u^0) \geqslant 0, \quad x^{0\prime} \nabla_x L(x^0, u^0) = 0 \\ &u^0 \geqslant 0, \quad \nabla_u L(x^0, u^0) \leqslant 0, \quad u^{0\prime} \nabla_u L(x^0, u^0) = 0 \end{aligned} \tag{9'}$$

where

$$L(x, u) = \varepsilon f(x) + u' g(x)$$

(S_1) If for a $u_{i_0}^0 > 0 \quad (i_0 \in \mathfrak{J}^0)$ we have

$$(x - x^0)' \nabla g_{i_0}(x^0) < 0, (\forall) x \in C - \{ x^0 \}, \tag{10}$$

then relations (9) or (9′) are also sufficient so that x^0 should be a point of local minimum [maximum] of the function f in C.

$$(S_2) \text{ If } C \cap \{ x \in R^n / (x - x^0)' \nabla f(x^0) = 0 \} = \{ x^0 \},$$

then relations (9) or (9′) are also sufficient so that point x^0 should be a local minimum [maximum] of the function f in C.

(S_3) If function f is quasi-convex in C and $\nabla f(x^0) \neq 0$, then relations (9) or (9′) are also sufficient so that point x^0 should be the global minimum [maximum] of the function f on C.

Proof. (n) If x^0 is a point of local minimum of the function f in C, then functions f, g and $h = -x$ verify the Kuhn-Tucker conditions in x^0 [9] : there are $u^0 \in R^m$ and $y^0 \in R^n$ such that

$$
(11) \qquad \begin{aligned}
&\varepsilon\nabla f(x^0) + u^{0\prime}\nabla g(x^0) + y^{0\prime}\nabla h(x^0) = 0 \ (^1)\\
&u^{0\prime}g(x^0) = 0\\
&y^{0\prime}h(x^0) = 0\\
&u^0 \geqslant 0\\
&y^0 \geqslant 0
\end{aligned}
$$

However $\nabla h(x^0) = \underbrace{(-1, ..., -1)}_{n}$ and relations (11) lead immediately to relations (9).

(S_1) From relations (9′) it results

$$(x - x^0)'\nabla_x L(x^0, u^0) \geqslant 0, \qquad (\forall) x \in C$$

The hypotheses of theorem 2 (a) are verified for $h \equiv 0$ and hence x^0 is a point of local minimum [maximum] of the function f in C.

(S_2) The hypotheses of the theorem 2 (b) with $h \equiv 0$ are verified.

(S_3) The hypotheses of the theorem 2 (c) with $h \equiv 0$ are verified.

From theorem 3 the following corollary will result.

Corollary. Let be functions f, $g = (g_1, ..., g_m)$ and $h = (h_1, ..., h_p)$ differentiable on an open convex set $K \subseteq R^{n_1} x R^{n_2}$.

Besides, we assume that function g is quasi-convex on K, while h is quasi-convex and quasi-concave on K.

(n) A necessary condition so that point $(x^0, y^0) \in K$ should be a point of local minimum [maximum] for the function f on the set

$$C = \{ (x, y) \in K/g(x, y) \leqslant 0, \qquad h(x, y) = 0, \qquad y \geqslant 0 \},$$

with nonempty interior is that $u^0 \in R^m$ and $v^0 \in R^p$ should exist so that

$$
(12) \qquad \begin{aligned}
&x^0 \in R^{n_1}, && \nabla_x L(x^0, y^0, u^0, v^0) = 0 &&\\
&y^0 \geqslant 0, && \nabla_y L(x^0, y^0, u^0, v^0) \geqslant 0, && y^{0\prime}\nabla_y L(x^0, y^0, u^0, v^0) = 0\\
&u^0 \geqslant 0, && \nabla_u L(x^0, y^0, u^0, v^0) \leqslant 0, && u^{0\prime}\nabla_u L(x^0, y^0, u^0, v^0) = 0\\
&v^0 \in R^p, && \nabla_v L(x^0, y^0, u^0, v^0) = 0, &&
\end{aligned}
$$

where

$$L(x, y, u, v) = \varepsilon f(x, y) + u'g(x ,y) + v'h(x, y),$$

(1)
$$\nabla g(x^0) = \begin{pmatrix} \nabla g_1(x^0) \\ \vdots \\ \nabla g_m(x^0) \end{pmatrix}$$

while $\varepsilon = +1[\varepsilon = -1]$.

(s_1) If for a $u^0_{i_0} > 0$ we have

$$(x - x^0)'\nabla_x g_{i_0}(x^0, y^0) + (y - y^0)'\nabla_y g_{i_0}(x^0, y^0) < 0, (\forall)(x, y) \in C - \{(x^0, y^0)\}$$

or at least for a $v^0_{j_0} > 0$ we have

$$(x - x^0)'\nabla_x h_{j_0}(x^0, y^0) + (y - y^0)'\nabla_y h_{j_0}(x^0, y^0) < 0, (\forall)(x, y) \in C - \{(x^0, y^0)\}$$

or

$$(x - x^0)'\nabla_x h_{j_0}(x^0, y^0) + (y - y^0)'\nabla_y h_{j_0}(x^0, y^0) > 0, (\forall), (x, y) \in C - \{(x^0, y^0)\}$$

the relations (12) are also sufficient, such that point (x^0, y^0) should be a local minimum [maximum] of the function f on C.

(s_2) If

$$C \cap \{(x, y) \in R^{n_1} x R^{n_2} / (x - x^0)'\nabla_x f(x^0, y^0) + + (y - y^0)'\nabla_y f(x^0, y^0) = 0\} = \{(x^0, y^0)\},$$

the relations (12) are also sufficient such that point (x^0, y^0) should be a local minimum [maximum] of the function f on C.

(s_3) If function f is quasi-convex in C and $\nabla f(x^0, y^0) \neq 0$ then relations (12) are also sufficient such that point (x^0, y^0) should be the global minimum [maximum] of the function f on C.

REMARK. If the vectorial function $g(x)$ is quasi-concave then there are the theorems dual to theorem 2 and 3.

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